

***Finite Element Method for Boundary Value Problems
Mathematics and Computations 1st edition Surana
Solutions Manual***

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Chapter 2

Solutions of Problems

Prob. 2.1

$$\frac{d\phi}{dx} - \frac{1}{Pe} \frac{d^2\phi}{dx^2} = 0 \quad \forall \quad x \in \Omega = (0, 1) \subset \mathbb{R}^1 \quad \text{--- (1)}$$

$$\phi(0) = 1, \quad \phi(1) = 0 \quad \text{--- (2)}$$

Solution:

a) Differential operator in (1)

$$A = \frac{d}{dx} - \frac{1}{Pe} \frac{d^2}{dx^2} \quad \text{--- (3)}$$

$\therefore$ (1) can be written as

$$A\phi = 0; \quad x \in \Omega = (0, 1) \subset \mathbb{R}^1 \quad \text{--- (4)}$$

b) Linearity of A:

$\forall u, v \in D_A$, domain of definition of A
and $\forall \alpha, \beta \in \mathbb{R}$

If we can show that

$$A(\alpha u + \beta v) = \alpha Au + \beta Av \quad \text{Then A is linear --- (5)}$$

Consider $A(\alpha u + \beta v)$

$$A(\alpha u + \beta v) = \left(\frac{d}{dx} - \frac{1}{Pe} \frac{d^2}{dx^2} \right) (\alpha u + \beta v)$$

$$= \left(\frac{d}{dx} - \frac{1}{Pe} \frac{d^2}{dx^2} \right) (\alpha u) + \left(\frac{d}{dx} - \frac{1}{Pe} \frac{d^2}{dx^2} \right) (\beta v)$$

using product rule of differentiation for αu and βv .

$$A(\alpha u + \beta v) = \alpha \left(\frac{d}{dx} - \frac{1}{Pe} \frac{d^2}{dx^2} \right) u + \beta \left(\frac{d}{dx} - \frac{1}{Pe} \frac{d^2}{dx^2} \right) v$$

$$= \alpha Au + \beta Av \quad \text{--- (6)}$$

Hence in this case

$$A(\alpha u + \beta v) = \alpha Au + \beta Av \quad \text{holds.}$$

$\therefore$ The differential operator A is linear.

Prob 2.2

$$\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) = 0 \quad \forall x \in \Omega = (0, 1) \subset \mathbb{R}^1 \quad \text{--- (1)}$$

$$\phi(0) = 0 \quad \left(a(x) \frac{d\phi}{dx} \right) \Big|_1 = p_1 \quad \text{--- (2)}$$

Solution:

(a) differential operator in (1)

$$A = \frac{d}{dx} \left(a(x) \frac{d}{dx} \right) \quad \text{--- (3)}$$

 $\therefore$ (1) can be written as

$$A\phi = 0, \quad \forall x \in \Omega = (0, 1) \subset \mathbb{R}^1 \quad \text{--- (4)}$$

(b) Linearity of A:

$$\forall u, v \in D_A$$

$$\text{and } \alpha, \beta \in \mathbb{R}$$

If we can show that

$$A(\alpha u + \beta v) = \alpha Au + \beta Av \quad \text{then } A \text{ is linear} \quad \text{--- (5)}$$

Consider $A(\alpha u + \beta v)$

$$\begin{aligned} A(\alpha u + \beta v) &= \frac{d}{dx} \left(a(x) \frac{d}{dx} \right) (\alpha u + \beta v) \\ &= \left(\frac{d}{dx} \left(a(x) \frac{d}{dx} \right) \right) (\alpha u) + \left(\frac{d}{dx} \left(a(x) \frac{d}{dx} \right) \right) (\beta v) \end{aligned}$$

using product rule of differentiation for αu and βv

$$\begin{aligned} A(\alpha u + \beta v) &= \alpha \left(\frac{d}{dx} \left(a(x) \frac{d}{dx} \right) \right) u + \beta \left(\frac{d}{dx} \left(a(x) \frac{d}{dx} \right) \right) v \\ &= \alpha Au + \beta Av \quad \text{--- (6)} \end{aligned}$$

Hence, in this case

$$A(\alpha u + \beta v) = \alpha Au + \beta Av \quad \text{holds}$$

 $\therefore$ The differential operator A is linear

Prob. 2.3

$$\phi \frac{d\phi}{dx} - \frac{1}{Re} \frac{d^2\phi}{dx^2} = 0 \quad \forall x \in \Omega = (0, 1) \subset \mathbb{R}^1 \quad (1)$$

$$\phi(0) = 1; \quad \phi(1) = 0 \quad (2)$$

Solution:(a) differential operator A :

$$A = \phi \frac{d}{dx} - \frac{1}{Re} \frac{d^2}{dx^2} \quad (3)$$

 $\therefore$ (1) can be written as

$$A\phi = 0 \quad \forall x \in \Omega = (0, 1) \subset \mathbb{R}^1 \quad (4)$$

(b) linearity of A :

$$\forall u, v \in D_A$$

$$\text{and } \forall \alpha, \beta \in \mathbb{R}$$

If we can show that

$$A(\alpha u + \beta v) = \alpha Au + \beta Av \quad \text{then } A \text{ is linear} \quad (5)$$

Consider $A(\alpha u + \beta v)$; substitute $\alpha u + \beta v$ for ϕ in (4) and (3)

$$\begin{aligned} A(\alpha u + \beta v) &= \left((\alpha u + \beta v) \frac{d}{dx} - \frac{1}{Re} \frac{d^2}{dx^2} \right) (\alpha u + \beta v) \\ &= \alpha (\alpha u + \beta v) \frac{du}{dx} + \beta (\alpha u + \beta v) \frac{dv}{dx} \\ &\quad - \frac{\alpha}{Re} \frac{d^2 u}{dx^2} - \frac{\beta}{Re} \frac{d^2 v}{dx^2} \quad (6) \end{aligned}$$

on the other hand (right side of (5))

$$\begin{aligned} \alpha Au + \beta Av &= \alpha \left(u \frac{du}{dx} - \frac{1}{Re} \frac{d^2 u}{dx^2} \right) + \beta \left(v \frac{dv}{dx} - \frac{1}{Re} \frac{d^2 v}{dx^2} \right) \\ &= \alpha u \frac{du}{dx} + \beta v \frac{dv}{dx} - \frac{\alpha}{Re} \frac{d^2 u}{dx^2} - \frac{\beta}{Re} \frac{d^2 v}{dx^2} \quad (7) \end{aligned}$$

Even though the last two terms in (6) and (7) are exactly same, but the first two are not, hence

$A(\alpha u + \beta v) = \alpha Au + \beta Av$ does not hold in this case. Thus, the differential operator A in (3) is not linear.

Prob. 2.41

$$B(\phi, v) = \int_1^2 a(x) \frac{d\phi}{dx} \frac{dv}{dx} dx + b(x) \phi(x) v(x) \quad (1)$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments; i.e. linear in ϕ as well as linear in v .

I. Linearity in ϕ

Let $u, w \in D_B$

and $\alpha, \beta \in \mathbb{R}$

Then we must show that

$$B(\alpha u + \beta w, v) = \alpha B(u, v) + \beta B(w, v) \quad \text{holds} \quad (2)$$

if $B(\phi, v)$ is linear in ϕ .

Consider

$$B(\alpha u + \beta w, v) = \int_1^2 a(x) \frac{d(\alpha u + \beta w)}{dx} \frac{dv}{dx} dx + b(x) (\alpha u(x) + \beta w(x)) v(x) \quad (3)$$

Expanding right side and regrouping the terms.

$$\begin{aligned}
 B(u + \beta w, v) &= \alpha \left(\int_1^2 a(x) \frac{du}{dx} \frac{dv}{dx} dx + b(2) u(2) v(2) \right) + \\
 &\quad \beta \left(\int_1^2 a(x) \frac{dw}{dx} \frac{dv}{dx} dx + b(2) w(2) v(2) \right) \\
 &= \alpha B(u, v) + \beta B(w, v) \quad \text{--- (4)}
 \end{aligned}$$

Hence, (2) holds, thus $B(\phi, v)$ is linear in ϕ .

II. Linearity in v

Let $v_1, v_2 \in D_B$

and $d_1, d_2 \in \mathbb{R}$

Then we must show that

$$B(\phi, d_1 v_1 + d_2 v_2) = d_1 B(\phi, v_1) + d_2 B(\phi, v_2) \text{ holds --- (5)}$$

if $B(\phi, v)$ is linear in v .

Consider

$$\begin{aligned}
 B(\phi, d_1 v_1 + d_2 v_2) &= \int_1^2 a(x) \frac{d\phi}{dx} \frac{d}{dx} (d_1 v_1 + d_2 v_2) dx + b(2) \phi(2) (d_1 v_1(2) + \\
 &\quad d_2 v_2(2)) \quad \text{--- (6)}
 \end{aligned}$$

Expanding the right side of (6) and rearranging the terms

$$\begin{aligned}
 B(\phi, d_1 v_1 + d_2 v_2) &= d_1 \left(\int_1^2 a(x) \frac{d\phi}{dx} \frac{dv_1}{dx} dx + b(2) \phi(2) v_1(2) \right) + \\
 &\quad d_2 \left(\int_1^2 a(x) \frac{d\phi}{dx} \frac{dv_2}{dx} dx + b(2) \phi(2) v_2(2) \right) \\
 &= d_1 B(\phi, v_1) + d_2 B(\phi, v_2) \quad \text{--- (7)}
 \end{aligned}$$

Hence, (5) holds, thus $B(\phi, v)$ is linear in v .

Thus, $B(\phi, v)$ is linear in ϕ as well v , hence

$B(\phi, v)$ is bilinear.

(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric then

$$B(\phi, v) = B(v, \phi) \text{ must hold.} \quad \text{--- (8)}$$

$$B(\phi, v) = \int_1^2 a(x) \frac{d\phi}{dx} \frac{dv}{dx} dx + b(2) \phi(2) v(2) \quad \text{--- (9)}$$

$$B(v, \phi) = \int_1^2 a(x) \frac{dv}{dx} \frac{d\phi}{dx} dx + b(2) v(2) \phi(2) \quad \text{--- (10)}$$

$$= \int_1^2 a(x) \frac{d\phi}{dx} \frac{dv}{dx} dx + b(2) \phi(2) v(2) \quad \text{--- (11)}$$

Comparing (9) and (11) it is straight forward to conclude that

$$B(\phi, v) = B(v, \phi) \text{ holds.}$$

Hence $B(\phi, v)$ is symmetric.

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Prob. 2.5

$$B(\phi, v) = \int_{-1}^1 a(x) \frac{d^2 \phi}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^1 c(x) \phi v dx \quad (1)$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments, i.e. linear in ϕ as well as linear in v .

I. Linearity in ϕ

$$\text{Let } \phi_1, \phi_2 \in D_B \\ \text{and } d_1, d_2 \in \mathbb{R}$$

Then we must show that

$$B(d_1 \phi_1 + d_2 \phi_2, v) = d_1 B(\phi_1, v) + d_2 B(\phi_2, v) \text{ holds} \quad (2)$$

if $B(\phi, v)$ is linear in ϕ .

Consider

$$B(d_1 \phi_1 + d_2 \phi_2, v) = \int_{-1}^1 a(x) \frac{d^2 (d_1 \phi_1 + d_2 \phi_2)}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^1 c(x) (d_1 \phi_1 + d_2 \phi_2) v dx \quad (3)$$

Expanding right side and rearranging the terms

$$\begin{aligned} B(d_1 \phi_1 + d_2 \phi_2, v) &= d_1 \left(\int_{-1}^1 a(x) \frac{d^2 \phi_1}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^1 c(x) \phi_1 v dx \right) + \\ &\quad d_2 \left(\int_{-1}^1 a(x) \frac{d^2 \phi_2}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^1 c(x) \phi_2 v dx \right) \\ &= d_1 B(\phi_1, v) + d_2 B(\phi_2, v) \quad (4) \end{aligned}$$

Hence, (2) holds, thus $B(\phi, v)$ is linear in ϕ .

II Linearity in v

Let $v_1, v_2 \in D_B$

and $\beta_1, \beta_2 \in \mathbb{R}$

Then we must show that

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \text{ holds} \quad (5)$$

if $B(\phi, v)$ is linear in v .

Consider

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \int_{-1}^1 a(x) \frac{d^2 \phi}{dx^2} \frac{d^2 (\beta_1 v_1 + \beta_2 v_2)}{dx^2} dx + \int_{-1}^1 c(x) \phi (\beta_1 v_1 + \beta_2 v_2) dx \quad (6)$$

Expanding the right side of (6) and rearranging the terms.

$$\begin{aligned} B(\phi, \beta_1 v_1 + \beta_2 v_2) &= \beta_1 \left(\int_{-1}^1 a(x) \frac{d^2 \phi}{dx^2} \frac{d^2 v_1}{dx^2} dx + \int_{-1}^1 c(x) \phi v_1 dx \right) + \\ &\quad \beta_2 \left(\int_{-1}^1 a(x) \frac{d^2 \phi}{dx^2} \frac{d^2 v_2}{dx^2} dx + \int_{-1}^1 c(x) \phi v_2 dx \right) \\ &= \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad (7) \end{aligned}$$

Hence, (5) holds, thus $B(\phi, v)$ is linear in v .

Thus, $B(\phi, v)$ is linear in ϕ as well as v , hence

$B(\phi, v)$ is bilinear.

(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric, then

$$B(\phi, v) = B(v, \phi) \text{ must hold.} \quad \text{--- (8)}$$

$$B(\phi, v) = \int_{-1}^1 a(x) \frac{d^2 \phi}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^1 c(x) \phi v dx \quad \text{--- (9)}$$

$$B(v, \phi) = \int_{-1}^1 a(x) \frac{d^2 v}{dx^2} \frac{d^2 \phi}{dx^2} dx + \int_{-1}^1 c(x) v \phi dx \quad \text{--- (10)}$$

$$= \int_{-1}^1 a(x) \frac{d^2 \phi}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^1 c(x) \phi v dx \quad \text{--- (11)}$$

Comparing (9) and (11) we conclude that

$$B(\phi, v) = B(v, \phi) \text{ holds}$$

Hence $B(\phi, v)$ is symmetric.

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Prob. 2.6

$$B(\phi, v) = \int_{-1}^1 \phi \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{--- (1)}$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments i.e. linear in ϕ as well as linear in v .

I. Linearity in ϕ

Let $\phi_1, \phi_2 \in D_B$

and $\alpha_1, \alpha_2 \in \mathbb{R}$

then we must show that

$$B(\alpha_1 \phi_1 + \alpha_2 \phi_2, v) = \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v) \text{ holds --- (2)}$$

if $B(\phi, v)$ is linear in ϕ .

Consider

$$B(\alpha_1 \phi_1 + \alpha_2 \phi_2, v) = \int_{-1}^1 (\alpha_1 \phi_1 + \alpha_2 \phi_2) \frac{d}{dx} (\alpha_1 \phi_1 + \alpha_2 \phi_2) \frac{dv}{dx} dx \quad \text{--- (3)}$$

$$= \alpha_1 \int_{-1}^1 \phi_1 \frac{d}{dx} (\alpha_1 \phi_1 + \alpha_2 \phi_2) \frac{dv}{dx} dx +$$

$$\alpha_2 \int_{-1}^1 \phi_2 \frac{d}{dx} (\alpha_1 \phi_1 + \alpha_2 \phi_2) \frac{dv}{dx} dx$$

$$= \alpha_1^2 \int_{-1}^1 \phi_1 \frac{d\phi_1}{dx} \frac{dv}{dx} dx + \alpha_1 \alpha_2 \int_{-1}^1 \phi_1 \frac{d\phi_2}{dx} \frac{dv}{dx} dx +$$

$$\alpha_1 \alpha_2 \int_{-1}^1 \phi_2 \frac{d\phi_1}{dx} \frac{dv}{dx} dx + \alpha_2^2 \int_{-1}^1 \phi_2 \frac{d\phi_2}{dx} \frac{dv}{dx} dx \quad \text{--- (4)}$$

Consider

$$\alpha_1 B(\phi_1, v) = \alpha_1 \int_{-1}^1 \phi_1 \frac{d\phi_1}{dx} \frac{dv}{dx} dx \quad \text{--- (5)}$$

$$\text{and } \alpha_2 B(\phi_2, v) = \alpha_2 \int_{-1}^1 \phi_2 \frac{d\phi_2}{dx} \frac{dv}{dx} dx \quad \text{--- (6)}$$

From (4), (5) and (6) we note that (2) does not hold, thus $B(\phi, v)$ is not linear in ϕ .

II. Linearity in v

Let $v_1, v_2 \in D_B$

and $\beta_1, \beta_2 \in \mathbb{R}$

Then, we must show that

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{holds --- (7)}$$

if $B(\phi, v)$ is linear in v .

Consider

$$\begin{aligned} B(\phi, \beta_1 v_1 + \beta_2 v_2) &= \int_{-1}^1 \phi \frac{d\phi}{dx} \frac{d(\beta_1 v_1 + \beta_2 v_2)}{dx} dx \\ &= \beta_1 \int_{-1}^1 \phi \frac{d\phi}{dx} \frac{dv_1}{dx} dx + \beta_2 \int_{-1}^1 \phi \frac{d\phi}{dx} \frac{dv_2}{dx} dx \\ &= \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{--- (8)} \end{aligned}$$

Hence, (7) holds, thus $B(\phi, v)$ is linear in v .

From I. and II. we note that $B(\phi, v)$ is linear in v but not linear in ϕ , hence $B(\phi, v)$ is not bilinear.

(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric then

$$B(\phi, v) = B(v, \phi) \text{ must hold.} \quad \text{--- (9)}$$

- Since $B(\phi, v)$ is not bilinear, it can not be symmetric, but we consider details in the following.

$$B(\phi, v) = \int_{-1}^1 \phi \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{--- (10)}$$

$$B(v, \phi) = \int_{-1}^1 v \frac{dv}{dx} \frac{d\phi}{dx} dx \quad \text{--- (11)}$$

Clearly (10) and (11) are not the same, thus $B(\phi, v)$ is not symmetric

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Prob. 2.7

$$B(\phi, v) = \int_{-1}^1 \frac{d\phi}{dx} v dx + \frac{1}{P_0} \int_{-1}^1 \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{--- (1)}$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments i.e. linear in ϕ as well as linear in v .

I. Linearity in ϕ

$$\text{Let } \phi_1, \phi_2 \in D_B$$

and $\alpha_1, \alpha_2 \in \mathbb{R}$

then we must show that

$B(\alpha_1 \phi_1 + \alpha_2 \phi_2, v) = \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v)$ holds — (2)
if $B(\phi, v)$ is linear in ϕ .

Consider

$$\begin{aligned} B(\alpha_1 \phi_1 + \alpha_2 \phi_2, v) &= \int_{-1}^1 \frac{d}{dx}(\alpha_1 \phi_1 + \alpha_2 \phi_2) v dx + \frac{1}{p_e} \int_{-1}^1 \frac{d}{dx}(\alpha_1 \phi_1 + \alpha_2 \phi_2) \frac{dv}{dx} dx \\ &= \alpha_1 \left(\int_{-1}^1 \frac{d\phi_1}{dx} v dx + \frac{1}{p_e} \int_{-1}^1 \frac{d\phi_1}{dx} \frac{dv}{dx} dx \right) + \\ &\quad \alpha_2 \left(\int_{-1}^1 \frac{d\phi_2}{dx} v dx + \frac{1}{p_e} \int_{-1}^1 \frac{d\phi_2}{dx} \frac{dv}{dx} dx \right) \\ &= \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v) \quad \text{--- (3)} \end{aligned}$$

Hence (2) holds, thus $B(\phi, v)$ is linear in ϕ .

(b) Linearity in v .

Let $v_1, v_2 \in D_B$

and $\beta_1, \beta_2 \in \mathbb{R}$

Then we must show that

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{--- (4)}$$

must hold if $B(\phi, v)$ is linear in v .

Consider

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \int_{-1}^1 \frac{d\phi}{dx} (\beta_1 v_1 + \beta_2 v_2) dx + \frac{1}{p_e} \int_{-1}^1 \frac{d\phi}{dx} \frac{d}{dx} (\beta_1 v_1 + \beta_2 v_2) dx \quad \text{--- (5)}$$

or

$$\begin{aligned}
 B(\phi, \beta_1 v_1 + \beta_2 v_2) &= \beta_1 \left(\int_{-1}^1 \frac{d\phi}{dx} v_1 dx + \frac{1}{p_0} \int_{-1}^1 \frac{d\phi}{dx} \frac{dv_1}{dx} dx \right) + \\
 &\quad \beta_2 \left(\int_{-1}^1 \frac{d\phi}{dx} v_2 dx + \frac{1}{p_0} \int_{-1}^1 \frac{d\phi}{dx} \frac{dv_2}{dx} dx \right) \\
 &= \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{--- (6)}
 \end{aligned}$$

Hence, (4) holds, thus $B(\phi, v)$ is linear in v .

Thus, $B(\phi, v)$ is linear in ϕ as well as in v , hence $B(\phi, v)$ is bilinear.

(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric, then

$$B(\phi, v) = B(v, \phi) \quad \text{must hold} \quad \text{--- (7)}$$

$$B(\phi, v) = \int_{-1}^1 \frac{d\phi}{dx} v dx + \frac{1}{p_0} \int_{-1}^1 \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{--- (8)}$$

$$\begin{aligned}
 B(v, \phi) &= \int_{-1}^1 \frac{dv}{dx} \phi dx + \frac{1}{p_0} \int_{-1}^1 \frac{dv}{dx} \frac{d\phi}{dx} dx \\
 &= - \int_{-1}^1 \phi \frac{dv}{dx} dx + \frac{1}{p_0} \int_{-1}^1 \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{--- (9)}
 \end{aligned}$$

From (8) and (9), we conclude that

$$B(\phi, v) \neq B(v, \phi) \quad \text{--- (10)}$$

Hence $B(\phi, v)$ is not symmetric.

Prob 2.8

$$-\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) + cu = f(x) \quad \forall x \in \Omega = (0,1) \subset \mathbb{R}^1 \quad (1)$$

$$\text{BCs: } u(0) = u_0; \quad \left(a(x) \frac{du}{dx} + \beta(u - u_\infty) \right) \Big|_{x=1} = Q_1 \quad (2)$$

(a) Differential operator

$$A = -\frac{d}{dx} \left(a(x) \frac{d}{dx} \right) + c \quad (3)$$

$$\therefore Au - f = 0 \quad \forall x \in \Omega = (0,1) \subset \mathbb{R}^1 \quad (4)$$

where $f = f(x)$ (b) linearity of A :Let $u_1, u_2 \in D_A$ and $\alpha_1, \alpha_2 \in \mathbb{R}$

Then

$$A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 A u_1 + \alpha_2 A u_2 \quad \text{must hold} \quad (5)$$

If the operator A is linear

$$\begin{aligned} \text{Consider } A(\alpha_1 u_1 + \alpha_2 u_2) &= \left(-\frac{d}{dx} \left(a(x) \frac{d}{dx} \right) + c \right) (\alpha_1 u_1 + \alpha_2 u_2) \\ &= \left(-\frac{d}{dx} \left(a(x) \frac{d}{dx} \right) + c \right) (\alpha_1 u_1) + \\ &\quad \left(-\frac{d}{dx} \left(a(x) \frac{d}{dx} \right) + c \right) (\alpha_2 u_2) \\ &= \alpha_1 \left(-\frac{d}{dx} \left(a(x) \frac{du_1}{dx} \right) + c u_1 \right) + \\ &\quad \alpha_2 \left(-\frac{d}{dx} \left(a(x) \frac{du_2}{dx} \right) + c u_2 \right) \end{aligned}$$

$$\therefore A(d_1 u_1 + d_2 u_2) = d_1 A u_1 + d_2 A u_2 \quad \text{--- (6)}$$

Based on (4), (5) holds, hence the operator A is linear.

(c) Symmetry of A .

$$\forall u, v \in D_A \quad \text{with } v = \delta u$$

We must show that

$$(A u, v) = (u, A v) \quad \text{--- (8)}$$

$\forall A$ is symmetric.

Consider $(A u, v)$

$$\begin{aligned} (A u, v) &= \int_0^1 \left(-\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) + c u \right) v dx \\ &= \int_0^1 -\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) v dx + \int_0^1 c u v dx \quad \text{--- (7)} \end{aligned}$$

Perform integration by parts twice in the first term to transfer all differentiation from u to v .

IBP once give:

$$(A u, v) = \int_0^1 \left(\frac{dv}{dx} a(x) \right) \frac{du}{dx} dx - \left(\left(a(x) \frac{du}{dx} \right) v \right) \Big|_0^1 + \int_0^1 c u v dx \quad \text{--- (8)}$$

IBP once more (first term) and regroup terms.

$$(A u, v) = \int_0^1 \left(-\frac{d}{dx} \left(a(x) \frac{dv}{dx} \right) + c v \right) u dx + \left(\left(a(x) \frac{dv}{dx} \right) u \right) \Big|_0^1 -$$

$$\left(\left(a(x) \frac{du}{dx} \right) v \right) \Big|_0^1 \quad \text{--- (9)}$$

or

$$(Au, v) = (u, A^*v) + \langle Au, v \rangle_H \quad \text{--- (10)}$$

where $A^* = -\frac{d}{dx} \left(a(x) \frac{d}{dx} \right) + c \quad \text{--- (11)}$

and $\langle Au, v \rangle_H = \left(a(x) \frac{dv}{dx} u \right) \Big|_0^1 - \left(a(x) \frac{du}{dx} v \right) \Big|_0^1 \quad \text{--- (12)}$

Simplification of Concomitant:

Consider Concomitant $\langle Au, v \rangle$ in (12). Expanding terms

$$\begin{aligned} \langle Au, v \rangle_H &= a(1) u(1) \frac{dv}{dx} \Big|_1 - a(0) u(0) \frac{dv}{dx} \Big|_0 \\ &\quad - a(1) v(1) \frac{du}{dx} \Big|_1 + a(0) v(0) \frac{du}{dx} \Big|_0 \quad \text{--- (13)} \end{aligned}$$

Recall BCP in (2):

$$v(0) = 0 \Rightarrow v(0) = 0 \quad \text{--- (14)}$$

$$\left(a(x) \frac{du}{dx} \right) \Big|_1 = Q_1 - \beta(1) (u(1) - u_\infty(1)) \quad \text{--- (15)}$$

$$\therefore a(1) \frac{dv}{dx} \Big|_1 = -\beta(1) v(1) \quad \text{--- (16)}$$

using (14), (15) and (16) in (13)

$$\langle Au, v \rangle_H = u(1) (-\beta(1) v(1)) - a(0) u_0 \frac{du}{dx} \Big|_0$$

$$-v(1) \left(Q_1 - \beta(1)(u(1) - u_\infty(1)) \right) + a(0) \cancel{v(0)} \frac{dy}{dx} \Big|_0$$

$$\therefore \langle Au, v \rangle_H = -\cancel{\beta(1)u(1)v(1)} - \left(a(0) \frac{dy}{dx} \Big|_0 \right) u_0 - v(1) Q_1 + \cancel{\beta(1)u(1)v(1)} - \beta(1)v(1)u_\infty(1)$$

$$\therefore \langle Au, v \rangle_H = - \left(a(0) \frac{dy}{dx} \Big|_0 \right) u_0 - v(1) Q_1 - (\beta(1)v(1)) u_\infty(1) \quad (17)$$

Hence, finally we have

$$(Au, v) = (u, A^*v) - \underbrace{\left(a(0) \frac{dy}{dx} \Big|_0 \right) u_0 - v(1) Q_1 - \beta(1)v(1) u_\infty(1)}_{\langle Au, v \rangle_H} \quad (18)$$

where

$$A^* = -\frac{d}{dx} \left(a(x) \frac{d}{dx} \right) + c = A \quad (19)$$

Concomitant $\langle Au, v \rangle$ in (17) is not zero if u_0 , Q_1 and $u_\infty(1)$ are not zero i.e. if the BCs are not homogeneous.

- Thus if u_0 , Q_1 and $u_\infty(1)$ are not zero, the differential operator A is not symmetric
- A is symmetric if $u_0 = 0$, $Q_1 = 0$ and $u_\infty(1) = 0$ i.e. if the BCs are homogeneous.

Prob. 2.9

$$-\frac{d}{dx} \left(b(x) \frac{d\phi}{dx} \right) = q(x) \quad \forall x \in \Omega = (0, L) \subset \mathbb{R}^1 \quad (1)$$

$$\text{BCP: } \phi(0) = 0, \quad \left(b(x) \frac{d\phi}{dx} + k\phi \right) \Big|_{x=L} = P_L \quad (2)$$

(a) Differential operator

$$A = -\frac{d}{dx} \left(b(x) \frac{d}{dx} \right) \quad (3)$$

$$A\phi - f = 0, \quad f = q \quad \forall x \in (0, L) \subset \mathbb{R}^1 \quad (4)$$

(b) Linearity of A

Let $\phi_1, \phi_2 \in D_A$

and $\alpha_1, \alpha_2 \in \mathbb{R}$

Then $A(\alpha_1 \phi_1 + \alpha_2 \phi_2) = \alpha_1 A\phi_1 + \alpha_2 A\phi_2$ must hold — (5)

iff The operator A is linear.

$$\text{Consider } A(\alpha_1 \phi_1 + \alpha_2 \phi_2) = \left(-\frac{d}{dx} \left(b(x) \frac{d}{dx} \right) \right) (\alpha_1 \phi_1 + \alpha_2 \phi_2)$$

$$= \left(-\frac{d}{dx} \left(b(x) \frac{d}{dx} \right) \right) (\alpha_1 \phi_1) +$$

$$\left(-\frac{d}{dx} \left(b(x) \frac{d}{dx} \right) \right) (\alpha_2 \phi_2)$$

$$= \alpha_1 \left(-\frac{d}{dx} \left(b(x) \frac{d\phi_1}{dx} \right) \right) +$$

$$\alpha_2 \left(-\frac{d}{dx} \left(b(x) \frac{d\phi_2}{dx} \right) \right)$$

$$= \alpha_1 A\phi_1 + \alpha_2 A\phi_2 \quad (6)$$

Based on (6), (5) holds, hence the operator A is linear.

(c) Symmetry of A

$\forall \phi, v \in D_A$ with $v = \delta\phi$

we must show that

$$(A\phi, v) = (\phi, Av) \quad \text{--- (7)}$$

if A is symmetric

Consider $(A\phi, v)$

$$(A\phi, v) = \int_0^L \left(-\frac{d}{dx} \left(b(x) \frac{d\phi}{dx} \right) \right) v \, dx \quad \text{--- (8)}$$

IBP once: transfer one order of differentiation from $b(x) \frac{d\phi}{dx}$ to v

$$(A\phi, v) = \int_0^L \left(\frac{dv}{dx} b(x) \right) \frac{d\phi}{dx} \, dx - \left(b(x) \frac{d\phi}{dx} v \right) \Big|_0^L \quad \text{--- (9)}$$

IBP once more: transfer one order of differentiation from ϕ to $b(x) \frac{dv}{dx}$

$$(A\phi, v) = \int_0^L \left(-\frac{d}{dx} \left(b(x) \frac{dv}{dx} \right) \right) \phi \, dx + \left(\frac{dv}{dx} b(x) \phi \right) \Big|_0^L - \left(b(x) \frac{d\phi}{dx} v \right) \Big|_0^L \quad \text{--- (10)}$$

$$\therefore (A\phi, v) = (\phi, A^*v) + \langle A\phi, v \rangle_{\mu} \quad \text{--- (11)}$$

In which

$$A^* = -\frac{d}{dx} \left(b(x) \frac{d}{dx} \right) \quad \text{--- (12)}$$

$$\langle A\phi, v \rangle_{\mu} = \left(\frac{dv}{dx} b(x) \phi \right) \Big|_0^L - \left(b(x) \frac{d\phi}{dx} v \right) \Big|_0^L \quad \text{--- (13)}$$

Simplification of Constant

Expanding the boundary terms in (13)

$$\begin{aligned} \langle A\phi, v \rangle_{\mu} &= \left(\frac{dv}{dx} \Big|_L b(L) \right) \phi(L) - \frac{dv}{dx} \Big|_0 b(0) \phi(0) - b(L) \frac{d\phi}{dx} \Big|_L v(L) \\ &\quad + b(0) \frac{d\phi}{dx} \Big|_0 v(0) \quad \text{--- (14)} \end{aligned}$$

Recall BCS in (2):

$$\phi(0) = 0 \Rightarrow v(0) = 0$$

$$\left(b(x) \frac{d\phi}{dx} \right) \Big|_L = P_L - (k\phi) \Big|_L \Rightarrow b(L) \frac{dv}{dx} \Big|_L = -k v(L) \quad \left. \vphantom{\left(b(x) \frac{d\phi}{dx} \right)} \right\} \text{---(15)}$$

Consider Concomitant (14) and substitute from

$$\begin{aligned} \langle A\phi, v \rangle_{\mu} &= (-k v(L)) \phi(L) - \cancel{\frac{dv}{dx} \Big|_0 b(0) \phi(0)} \\ &\quad - (P_L - k\phi(L)) v(L) + \cancel{b(0) \frac{d\phi}{dx} \Big|_0 v(0)} \\ &\quad - k v(L) \phi(L) + k v(L) \phi(L) - P_L v(L) \quad \text{---(16)} \end{aligned}$$

$$\therefore \langle A\phi, v \rangle_{\mu} = -P_L v(L) \quad \text{---(17)}$$

i.e. Concomitant $\langle A\phi, v \rangle_{\mu} \neq 0$ if $P_L \neq 0$ i.e.
 $\langle A\phi, v \rangle_{\mu}$ is only zero if the BCS are homogeneous

Adjoint of A

$$A^* = -\frac{d}{dx} \left(b(x) \frac{d}{dx} \right) = A \quad \text{---(18)}$$

based on (17) and (18) The differential operator A
 is symmetric only if $P_L = 0$ i.e. The BCS are homogeneous.

Prob 2.10

$$\frac{d^2}{dx^2} \left(b(x) \frac{d^2 y}{dx^2} \right) + \frac{d}{dx} \left(c(x) \frac{dy}{dx} \right) + d(x) y = f(x) \quad (1)$$

$$\forall x \in \Omega = (0, L) \subset \mathbb{R}^1$$

$$\text{BCA: } y(0) = y_0$$

$$y(L) = y_L$$

$$\left(b(x) \frac{d^2 y}{dx^2} \right) \Big|_{x=0} = m_0$$

$$\left(b(x) \frac{d^2 y}{dx^2} \right) \Big|_{x=L} = m_L$$

} (2)

(a) differential operator A

$$A = \frac{d^2}{dx^2} \left(b(x) \frac{d^2}{dx^2} \right) + \frac{d}{dx} \left(c(x) \frac{d}{dx} \right) + d(x) \quad (3)$$

$$\therefore Ay - f = 0 \quad \forall x \in (0, L) \subset \mathbb{R}^1 \quad (4)$$

(b) Linearity of A

$$\text{let } y_1, y_2 \in D_A$$

$$\text{and } \alpha_1, \alpha_2 \in \mathbb{R}$$

Then

$$A(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 A y_1 + \alpha_2 A y_2 \quad \text{must hold} \quad (5)$$

If the operator A is linear.

$$\text{Consider } A(\alpha_1 y_1 + \alpha_2 y_2) = \left(\frac{d^2}{dx^2} \left(b(x) \frac{d^2}{dx^2} \right) + \frac{d}{dx} \left(c(x) \frac{d}{dx} \right) + d(x) \right) (\alpha_1 y_1 + \alpha_2 y_2) \quad (5)$$

or

$$\begin{aligned}
A(\alpha_1 \psi_1 + \alpha_2 \psi_2) &= \left(\frac{d^2}{dx^2} \left(b(x) \frac{d^2}{dx^2} \right) + \frac{d}{dx} \left(c(x) \frac{d}{dx} \right) + \lambda(x) \right) (\alpha_1 \psi_1) + \\
&\quad \left(\frac{d^2}{dx^2} \left(b(x) \frac{d^2}{dx^2} \right) + \frac{d}{dx} \left(c(x) \frac{d}{dx} \right) + \lambda(x) \right) (\alpha_2 \psi_2) \\
&= \alpha_1 \left(\frac{d^2}{dx^2} \left(b(x) \frac{d^2 \psi_1}{dx^2} \right) + \frac{d}{dx} \left(c(x) \frac{d \psi_1}{dx} \right) + \lambda(x) \psi_1 \right) + \\
&\quad \alpha_2 \left(\frac{d^2}{dx^2} \left(b(x) \frac{d^2 \psi_2}{dx^2} \right) + \frac{d}{dx} \left(c(x) \frac{d \psi_2}{dx} \right) + \lambda(x) \psi_2 \right) \\
&= \alpha_1 A \psi_1 + \alpha_2 A \psi_2 \quad \text{--- (6)}
\end{aligned}$$

Based on (6), (5) holds, hence the operator A is linear.

(c) Symmetry of operator A

$\forall \psi, v \in D_A$ with $v = \delta \psi$

We must show that

$$(A \psi, v) = (\psi, A v) \quad \text{--- (7)}$$

if A is symmetric

Consider $(A \psi, v)$

$$\begin{aligned}
(A \psi, v) &= \int_0^L \left(\frac{d^2}{dx^2} \left(b(x) \frac{d^2 \psi}{dx^2} \right) + \frac{d}{dx} \left(c(x) \frac{d \psi}{dx} \right) + \lambda(x) \psi \right) v \, dx \\
&= \int_0^L \frac{d^2}{dx^2} \left(b(x) \frac{d^2 \psi}{dx^2} \right) v \, dx + \int_0^L \frac{d}{dx} \left(c(x) \frac{d \psi}{dx} \right) v \, dx + \\
&\quad \int_0^L \lambda(x) \psi v \, dx \quad \text{--- (8)}
\end{aligned}$$

Consider the first term on the right side of (8).

$$\int_0^L \frac{d^2}{dx^2} \left(b(x) \frac{d^2 \psi}{dx^2} \right) v \, dx = \int_0^L \left(\frac{d^2 v}{dx^2} b(x) \right) \frac{d^2 \psi}{dx^2} \, dx + \left(\frac{d}{dx} \left(b(x) \frac{d^2 \psi}{dx^2} \right) v \right) \Big|_0^L - \left(b(x) \frac{d^2 \psi}{dx^2} \frac{dv}{dx} \right) \Big|_0^L \quad (9)$$

(IBP twice)

Transfer two orders of differentiation from ψ to $\frac{d^2 v}{dx^2} b(x)$ in the first term on right side of (9).

$$\therefore \int_0^L \frac{d^2}{dx^2} \left(b(x) \frac{d^2 \psi}{dx^2} \right) v \, dx = \int_0^L \frac{d^2}{dx^2} \left(b(x) \frac{d^2 v}{dx^2} \right) \psi \, dx + \left(\frac{d^2 v}{dx^2} b(x) \frac{d\psi}{dx} \right) \Big|_0^L - \left(\frac{d}{dx} \left(b(x) \frac{d^2 v}{dx^2} \right) \psi \right) \Big|_0^L + \left(\frac{d}{dx} \left(b(x) \frac{d^2 \psi}{dx^2} \right) v \right) \Big|_0^L - \left(b(x) \frac{d^2 \psi}{dx^2} \frac{dv}{dx} \right) \Big|_0^L \quad (10)$$

Consider second term on right side of (8)

$$\int_0^L \frac{d}{dx} \left(c(x) \frac{d\psi}{dx} \right) v \, dx = \int_0^L \frac{d}{dx} \left(c(x) \frac{dv}{dx} \right) \psi \, dx + \left(c(x) \frac{d\psi}{dx} v \right) \Big|_0^L - \left(c(x) \frac{dv}{dx} \psi \right) \Big|_0^L \quad (11)$$

(IBP twice)

using (10) and (11) in (8)

$$\langle A \psi, v \rangle = \int_0^L \left(\frac{d^2}{dx^2} \left(b(x) \frac{d^2 v}{dx^2} \right) + \frac{d}{dx} \left(c(x) \frac{dv}{dx} \right) + 2v \right) \psi \, dx + \langle A \psi, v \rangle_{\Gamma} \quad (12)$$

$${}^N(A\psi, v) = (\psi, A^*v) + \langle A\psi, v \rangle_H \quad \text{--- (13)}$$

in which

$$A^* = \frac{d^2}{dx^2} \left(b(x) \frac{d^2}{dx^2} \right) + \frac{d}{dx} \left(c(x) \frac{d}{dx} \right) + d(x) = A$$

Simplification of concomitant $\langle A\psi, v \rangle_H$

$$\begin{aligned} \langle A\psi, v \rangle_H &= \left(\frac{d^2 v}{dx^2} b(x) \frac{d\psi}{dx} \right) \Big|_0^L - \left(\frac{d}{dx} \left(b(x) \frac{d^2 v}{dx^2} \right) \psi \right) \Big|_0^L \\ &\quad + \left(\frac{d}{dx} \left(b(x) \frac{d^2 \psi}{dx^2} \right) v \right) \Big|_0^L - \left(\left(b(x) \frac{d^2 \psi}{dx^2} \right) \frac{dv}{dx} \right) \Big|_0^L \\ &\quad + \left(c(x) \frac{d\psi}{dx} v \right) \Big|_0^L - \left(c(x) \frac{dv}{dx} \psi \right) \Big|_0^L \quad \text{--- (14)} \end{aligned}$$

Recall BCs in (2).

$$\psi(0) = \psi_0 \quad \Rightarrow \quad v(0) = 0$$

$$\psi(L) = \psi_L \quad \Rightarrow \quad v(L) = 0$$

$$\left(b(x) \frac{d^2 \psi}{dx^2} \right) \Big|_{x=0} = M_0 \quad \Rightarrow \quad b(0) \frac{d^2 v}{dx^2} \Big|_0 = 0$$

$$\left(b(x) \frac{d^2 \psi}{dx^2} \right) \Big|_{x=L} = M_L \quad \Rightarrow \quad b(L) \frac{d^2 v}{dx^2} \Big|_L = 0$$

we expand boundary terms in (14) and then use (15)

$$\begin{aligned}
 \langle A\psi, v \rangle_{\Gamma} = & \left(\frac{d^2 v}{dx^2} b(x) \frac{d\psi}{dx} \right) \Big|_L^0 - \left(\frac{d^2 v}{dx^2} b(x) \frac{d\psi}{dx} \right) \Big|_0^0 \\
 & - \left(\frac{d}{dx} \left(b(x) \frac{d^2 v}{dx^2} \right) \psi \right) \Big|_L^{\psi_L} + \left(\frac{d}{dx} \left(b(x) \frac{d^2 v}{dx^2} \right) \psi \right) \Big|_0^{\psi_0} \\
 & + \left(\frac{d}{dx} \left(b(x) \frac{d^2 \psi}{dx^2} \right) v \right) \Big|_L^0 - \left(\frac{d}{dx} \left(b(x) \frac{d^2 \psi}{dx^2} \right) v \right) \Big|_0^0 \\
 & - \left(b(x) \frac{d^2 \psi}{dx^2} \frac{dv}{dx} \right) \Big|_L^0 + \left(b(x) \frac{d^2 \psi}{dx^2} \frac{dv}{dx} \right) \Big|_0^0 \\
 & \left(c(x) \frac{d\psi}{dx} v \right) \Big|_L^0 - \left(c(x) \frac{d\psi}{dx} v \right) \Big|_0^0 \\
 & - \left(c(x) \frac{dv}{dx} \psi \right) \Big|_L^{\psi_L} + \left(c(x) \frac{dv}{dx} \psi \right) \Big|_0^{\psi_0} \quad \text{--- (16)}
 \end{aligned}$$

∴

$$\begin{aligned}
 \langle A\psi, v \rangle_{\Gamma} = & - \left(\frac{d}{dx} \left(b(x) \frac{d^2 v}{dx^2} \right) \right) \Big|_L \psi_L + \left(\frac{d}{dx} \left(b(x) \frac{d^2 v}{dx^2} \right) \right) \Big|_0 \psi_0 \\
 & - \left(\frac{dv}{dx} \right) \Big|_L M_L + \left(\frac{dv}{dx} \right) \Big|_0 M_0 \\
 & - \left(c(x) \frac{dv}{dx} \right) \Big|_L \psi_L + \left(c(x) \frac{dv}{dx} \right) \Big|_0 \psi_0 \quad \text{--- (17)}
 \end{aligned}$$

This is the final simplified form of the concomitant $\langle A\psi, v \rangle_{\Gamma}$.

1. $\langle A\psi, \psi \rangle_\pi$ goes to zero if $\psi_0 = \psi_L = m_0 = m_L = 0$
 ψ if the boundary conditions are homogeneous otherwise it is nonzero.
2. $A^* = A$ ψ is the adjoint of the operator is same as the operator
3. Based on 1 and 2 The differential operator A is not symmetric with the BCs in (2)
4. If all BCs are homogeneous, then A is symmetric

_____ 0 _____

Prob. 2.11

$$-\frac{d}{dx} \left(\phi \frac{d\phi}{dx} \right) + f(x) = 0 \quad \forall x \in \Omega = (0, 1) \subset \mathbb{R}^1 \quad (1)$$

$$\text{BCs: } \left. \frac{d\phi}{dx} \right|_{x=0} = q_0 \quad \text{and} \quad \phi(1) = \sqrt{2} \quad (2)$$

(a) Differential operator A

$$A = -\frac{d}{dx} \left(\phi \frac{d}{dx} \right) \quad (3)$$

$$\therefore A\phi - f = 0 \quad \forall x \in (0, 1) \subset \mathbb{R}^1 \quad (4)$$

(b) Linearity of A

$$\text{Let } \phi_1, \phi_2 \in D_A$$

$$\text{and } \alpha_1, \alpha_2 \in \mathbb{R}$$

$$\text{Then } A(\alpha_1 \phi_1 + \alpha_2 \phi_2) = \alpha_1 A\phi_1 + \alpha_2 A\phi_2 \quad \text{must hold} \quad (5)$$

if the operator A is linear

$$\text{Consider } A(\alpha_1 \phi_1 + \alpha_2 \phi_2) = -\frac{d}{dx} \left((\alpha_1 \phi_1 + \alpha_2 \phi_2) \frac{d}{dx} (\alpha_1 \phi_1 + \alpha_2 \phi_2) \right) \quad (6)$$

and

$$\alpha_1 A\phi_1 = -\alpha_1 \frac{d}{dx} \left(\phi_1 \frac{d\phi_1}{dx} \right) \quad (7)$$

$$\alpha_2 A\phi_2 = -\alpha_2 \frac{d}{dx} \left(\phi_2 \frac{d\phi_2}{dx} \right) \quad (8)$$

Clearly sum of (7) and (8) is not equal to (6),
hence for this operator A (5) does not hold.

Therefore the differential operator A in (3) is not linear.

- Since the differential operator is not linear it cannot be symmetric.
- In this case A^* is not meaningful.

————— 0 —————