

***Polarized Light and Optical Systems 1st edition
Chipman Solutions Manual***

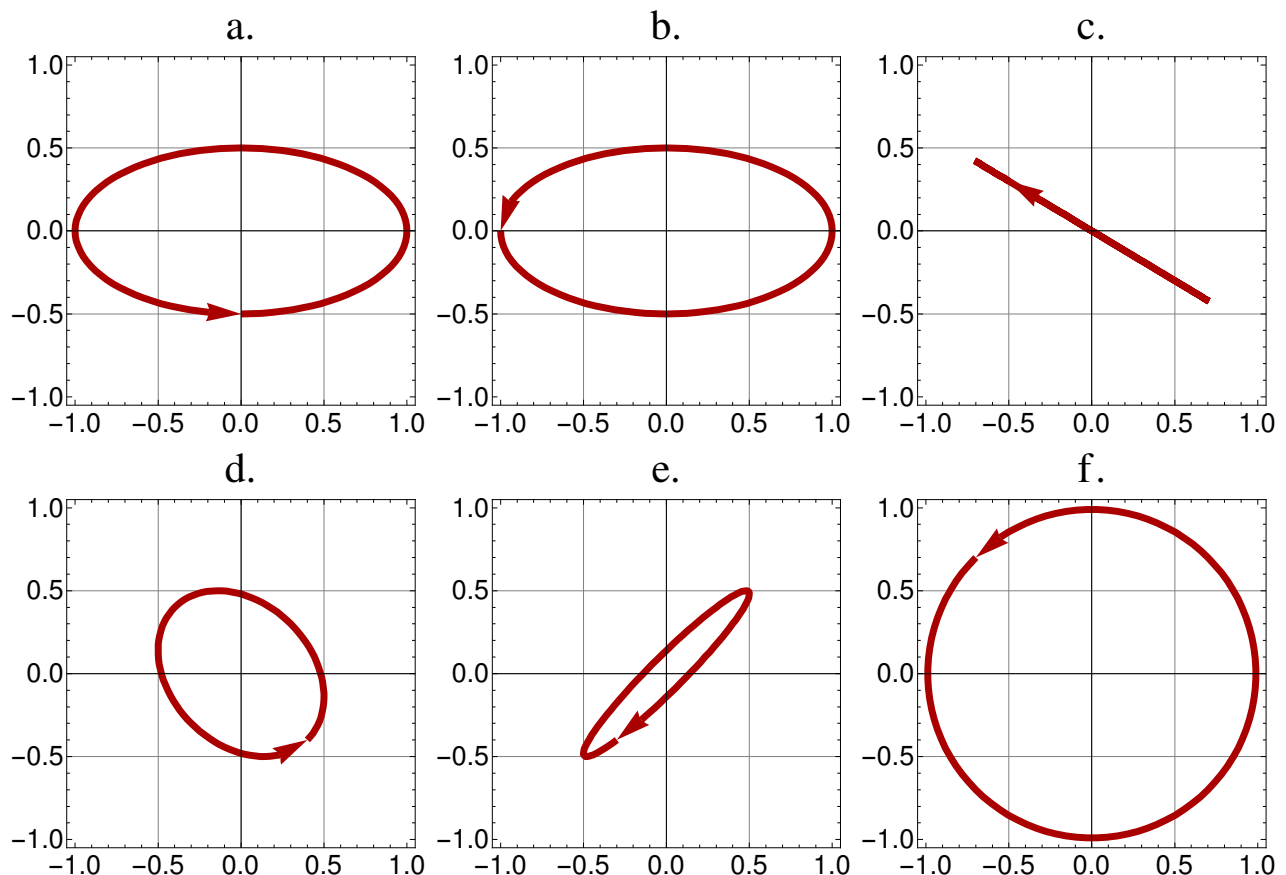
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Polarized Light & Optical Systems

Chapter 2 Polarized Light Problem Sets and Solutions

2.1 Polarization Ellipses

Estimate the Jones vectors for the following polarization ellipses. Find the phases to place the arrow correctly at $t = 0$.



Solution

$$\mathbf{E}(t) = \text{Re} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} e^{-i\frac{2\pi}{T}t} \right], \text{ where}$$

Jones vector: $\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix}$ or $\begin{pmatrix} a_x + i b_x \\ a_y + i b_y \end{pmatrix}$, where $e^{-i\phi} = \cos(\phi) - i\sin(\phi)$

the phases are ϕ_x, ϕ_y

Solving the phase for complex number $a + i b$:

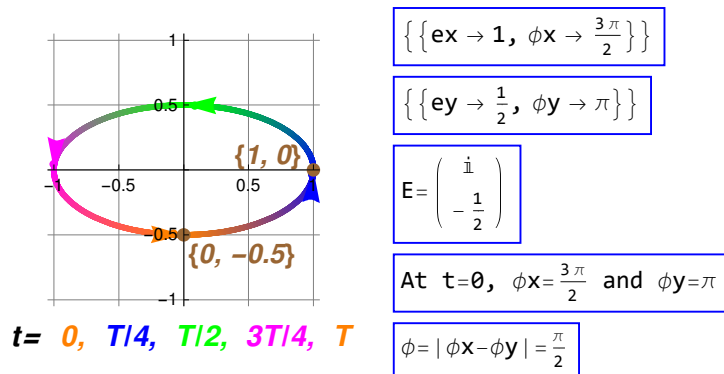
$$\phi = \begin{cases} \tan^{-1}(b/a) & \text{when } a > 0 \\ \tan^{-1}(b/a) + \pi & \text{when } a < 0, b \geq 0 \\ \tan^{-1}(b/a) - \pi & \text{when } a < 0, b < 0 \\ \pi/2 & \text{when } a = 0, b > 0 \\ -\pi/2 & \text{when } a = 0, b < 0 \\ \text{indeterminate} & \text{when } a = 0, b = 0 \end{cases}$$

$$\mathbf{E}(t=0) = \text{Re} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} \right] \text{ and } \mathbf{E}(t=T/4) = \text{Im} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} \right]$$

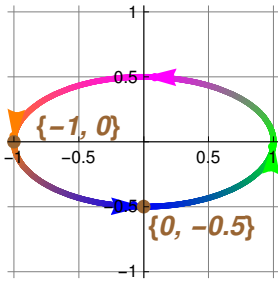
t	t=0	t=T/4	t=T/2	t=3T/4	t=T
$e^{-i \frac{2\pi}{T} t}$	1	-i	-1	i	1
$\mathbf{E}(t)$	$\text{Re} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} \right]$	$\text{Re} \left[\begin{pmatrix} E_x e^{-i(\phi_x + \frac{\pi}{2})} \\ E_y e^{-i(\phi_y + \frac{\pi}{2})} \end{pmatrix} \right]$	$\text{Re} \left[\begin{pmatrix} E_x e^{-i(\phi_x + \pi)} \\ E_y e^{-i(\phi_y + \pi)} \end{pmatrix} \right]$	$\text{Re} \left[\begin{pmatrix} E_x e^{-i(\phi_x + \frac{3\pi}{2})} \\ E_y e^{-i(\phi_y + \frac{3\pi}{2})} \end{pmatrix} \right]$	$\text{Re} \left[\begin{pmatrix} E_x e^{-i(\phi_x + 2\pi)} \\ E_y e^{-i(\phi_y + 2\pi)} \end{pmatrix} \right]$
$\mathbf{E}(t)$	$\text{Re} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} \right]$	$\text{Im} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} \right]$	$-\text{Re} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} \right]$	$-\text{Im} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} \right]$	$\text{Re} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} \right]$

The real and imaginary parts of the Jones vector are calculated at where the arrow starts (t=0) and where the arrow at after a quarter period (t=T/4).

a. $t=0, \mathbf{E} = \text{Re} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ -0.5 \end{pmatrix}$ $t=T/4, \mathbf{E} = \text{Re} \left[\begin{pmatrix} E_x e^{-i(\phi_x + \frac{\pi}{2})} \\ E_y e^{-i(\phi_y + \frac{\pi}{2})} \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



b. When $t=0$, $\mathbf{E} = \{-1, 0\}$
 When $t=T/4$, $\mathbf{E} = \{0, -0.5\}$



$t = 0, T/4, T/2, 3T/4, T$

$$\{ \{ \mathbf{ex} \rightarrow 1, \phi_x \rightarrow \pi \} \}$$

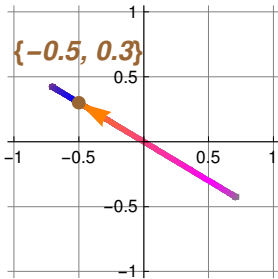
$$\{ \{ \mathbf{ey} \rightarrow \frac{1}{2}, \phi_y \rightarrow \frac{\pi}{2} \} \}$$

$$\mathbf{E} = \begin{pmatrix} -1 \\ -\frac{i}{2} \end{pmatrix}$$

$$\text{At } t=0, \phi_x=\pi \text{ and } \phi_y=\frac{\pi}{2}$$

$$\phi = |\phi_x - \phi_y| = \frac{\pi}{2}$$

- c. When $t=0$, $\mathbf{E}=\{-0.5, 0.3\}$
When $t=T/4$, $\mathbf{E}=\{-0.5, 0.3\}$



$t = 0, T/4, T/2, 3T/4, T$

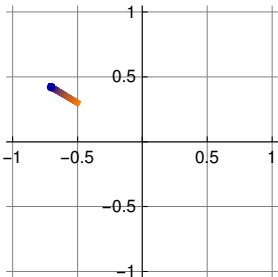
$$\{ \{ \mathbf{ex} \rightarrow 0.707, \phi_x \rightarrow 2.35619 \} \}$$

$$\{ \{ \mathbf{ey} \rightarrow 0.424, \phi_y \rightarrow -0.785398 \} \}$$

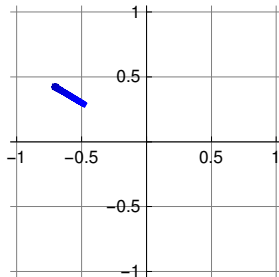
$$\mathbf{E} = \begin{pmatrix} -0.5 - 0.5i \\ 0.3 + 0.3i \end{pmatrix}$$

$$\text{At } t=0, \phi_x=2.35619 \text{ and } \phi_y=-0.785398$$

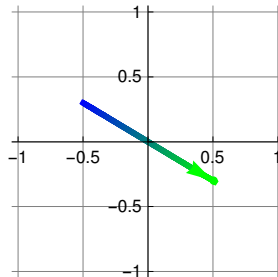
$$\phi = |\phi_x - \phi_y| = 3.14159$$



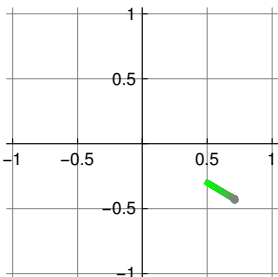
t from 0 to $T/8$



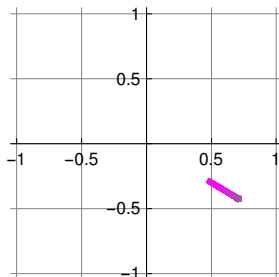
t from $T/8$ to $T/4$



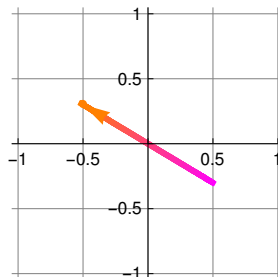
t from $T/4$ to $T/2$



t from $T/2$ to $5T/8$

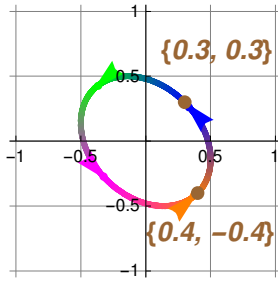


t from $5T/8$ to $3T/4$



t from $3T/4$ to T

- d. When $t=0$, $\mathbf{E}=\{0.4, -0.4\}$
When $t=T/4$, $\mathbf{E}=\{0.3, 0.3\}$



$t = 0, T/4, T/2, 3T/4, T$

$$\{ \{ex \rightarrow 0.5, \phi_x \rightarrow -0.643501\} \}$$

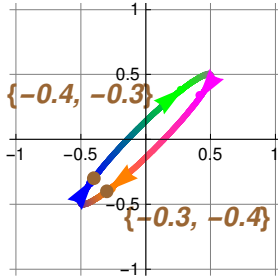
$$\{ \{ey \rightarrow 0.5, \phi_y \rightarrow -2.49809\} \}$$

$$E = \begin{pmatrix} 0.4 + 0.3i \\ -0.4 + 0.3i \end{pmatrix}$$

$$\text{At } t=0, \phi_x = -0.643501 \text{ and } \phi_y = -2.49809$$

$$\phi = |\phi_x - \phi_y| = 1.85459$$

- e. When $t=0$, $E = \{-0.3, -0.4\}$
When $t=T/4$, $E = \{-0.4, -0.3\}$



$t = 0, T/4, T/2, 3T/4, T$

$$\{ \{ex \rightarrow 0.5, \phi_x \rightarrow 2.2143\} \}$$

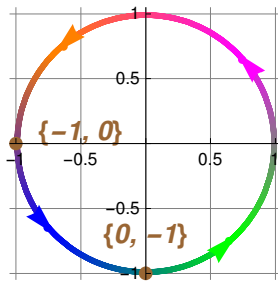
$$\{ \{ey \rightarrow 0.5, \phi_y \rightarrow 2.49809\} \}$$

$$E = \begin{pmatrix} -0.3 - 0.4i \\ -0.4 - 0.3i \end{pmatrix}$$

$$\text{At } t=0, \phi_x = 2.2143 \text{ and } \phi_y = 2.49809$$

$$\phi = |\phi_x - \phi_y| = 0.283794$$

- f. When $t=T/8$, $E = \{-1, 0\}$
When $t=3T/8$, $E = \{0, -1\}$



$t = 0, T/4, T/2, 3T/4, T$

$$\{ \{ex \rightarrow 1., \phi_x \rightarrow 2.35619\} \}$$

$$\{ \{ey \rightarrow 1., \phi_y \rightarrow 0.785398\} \}$$

$$E = \begin{pmatrix} -0.707107 - 0.707107i \\ 0.707107 - 0.707107i \end{pmatrix}$$

$$\text{At } t=0, \phi_x = 2.35619 \text{ and } \phi_y = 0.785398$$

$$\phi = |\phi_x - \phi_y| = 1.5708$$

2.2 Linear, circular, and elliptical Jones vectors

Which of the following Jones vectors are (a) linearly polarized, (b) circularly polarized, 90° or $\pi/2$ out of phase with equal amplitudes, (c) elliptically polarized, with arbitrary phase relationship.

- $(2, 2)$
- $(i/2, 1)$
- $(i, -i)$
- $(1, -4)$

- e. $(2+2i, -2+2i)$
- f. $(2+2i, -2-2i)$
- g. $(0, 1+i)$
- h. $(3, -6i)$
- i. $(2+3i, -3+2i)$
- j. $(2, -2i)$

Solution

Conditions for determination:

Linearly polarized: the x and y components are in phase or 180° out of phase.

Circularly polarized: the x and y components are 90° ($\pi/2$) out of phase with equal amplitudes, i. e. real and imaginary.

Elliptically polarized: the x and y components have arbitrary phases or different amplitudes

Therefore,

Linearly polarized: a, c, d, f, g

Circularly polarized: e, j

Elliptically polarized: b, h, i

2.3 Polarization Vector

Consider the plane wave

$$\mathbf{E}(r, t) = \text{Re} \left\{ e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \begin{pmatrix} a_x + i b_x \\ a_y + i b_y \\ a_z + i b_z \end{pmatrix} \right\}$$

Find the electric field and the Poynting vector at the following times and locations.

- a. $t=0, \mathbf{r} = (0, 0, 0)$
- b. $t=0, \mathbf{r} = \lambda^2 \mathbf{k}/4\pi,$
- c. $t = \pi / \omega, \mathbf{r} = (0, 0, 0)$
- d. $t = 4\pi / \omega, \mathbf{r} = 8 \lambda^2 \mathbf{k}/\pi$

Solution

In a, b, c, and d, first calculate $e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$, then multiply it to $\begin{pmatrix} a_x + i b_x \\ a_y + i b_y \\ a_z + i b_z \end{pmatrix}$ and take the real part to get

electric field.

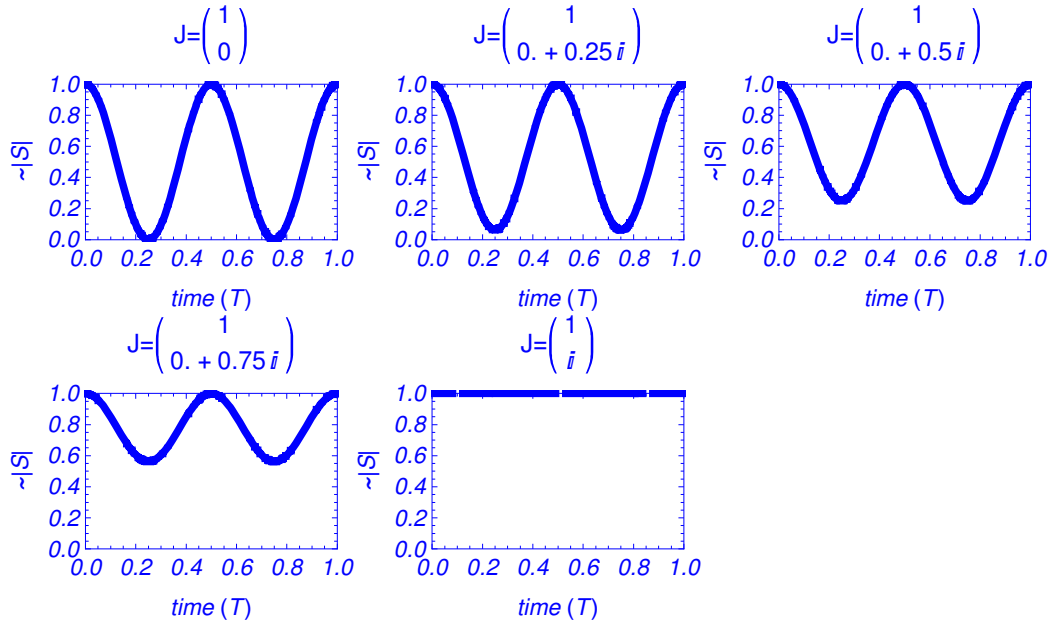
Poynting vector: $\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)$

Since $\mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{H}(\mathbf{r}, t)$, $\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \frac{1}{\mu_0} \mathbf{B}(\mathbf{r}, t) = \epsilon_0 (c^2) \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)$.

Also, $\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)$, then

$$\begin{aligned}
\mathbf{S}(\mathbf{r}, t) &= \epsilon_0 c \mathbf{E}(\mathbf{r}, t) \times \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t) \\
&= \epsilon_0 c \{ \hat{\mathbf{k}} [\mathbf{E}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t)] - \mathbf{E}(\mathbf{r}, t) [\mathbf{E}(\mathbf{r}, t) \cdot \hat{\mathbf{k}}] \} \\
&= \epsilon_0 c \hat{\mathbf{k}} [\mathbf{E}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t)] \\
&= \epsilon_0 c \hat{\mathbf{k}} \left[\operatorname{Re} \left\{ e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \begin{pmatrix} a_x + i b_x \\ a_y + i b_y \\ a_z + i b_z \end{pmatrix} \right\} \cdot \operatorname{Re} \left\{ e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \begin{pmatrix} a_x + i b_x \\ a_y + i b_y \\ a_z + i b_z \end{pmatrix} \right\} \right]
\end{aligned}$$

Its oscillates twice per period for linearly polarized light, and stays constant for circularly polarized light.



a.

$$\mathbf{r} = \{0, 0, 0\}, t=0$$

$$e^{-i\mathbf{k} \cdot \mathbf{r}} = e^{-i0} e^{-i k \cdot \{0, 0, 0\}} = 1$$

$$e^{i\omega t} = e^{i0} = 1$$

$$e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} = 1$$

$$\mathbf{E} = \operatorname{Re} \left\{ \begin{pmatrix} a_x + i b_x \\ a_y + i b_y \\ a_z + i b_z \end{pmatrix} \right\} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

$$\mathbf{S} = c \epsilon_0 (a_x^2 + a_y^2 + a_z^2) \hat{\mathbf{k}}$$

b.

$$\mathbf{r} = \lambda^2 \mathbf{k} / 4\pi, \quad \mathbf{t} = \theta$$

$$e^{-i\mathbf{k} \cdot \mathbf{r}} = e^{-i\lambda^2 \mathbf{k} \cdot \mathbf{k} / 4\pi} = e^{-i(\lambda^2 / 4\pi)(4\pi^2 / \lambda^2)} = e^{-i\pi} = -1$$

$$e^{i\omega \mathbf{t}} = e^{i\theta} 1$$

$$e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega \mathbf{t})} = -1$$

$$\mathbf{E} = \text{Re} \left\{ - \begin{pmatrix} a_x + i b_x \\ a_y + i b_y \\ a_z + i b_z \end{pmatrix} \right\} = \begin{pmatrix} -a_x \\ -a_y \\ -a_z \end{pmatrix}$$

$$S = c \epsilon_0 (a_x^2 + a_y^2 + a_z^2) \hat{\mathbf{k}}$$

c.

$$\mathbf{r} = \{0, 0, 0\}, \quad \mathbf{t} = \frac{\pi}{\omega}$$

$$e^{-i\mathbf{k} \cdot \mathbf{r}} = e^{-i\theta} = 1$$

$$e^{i\omega \mathbf{t}} = e^{i\omega (\pi / \omega)} = e^{i\pi} = -1$$

$$e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega \mathbf{t})} = -1$$

$$\mathbf{E} = \text{Re} \left\{ - \begin{pmatrix} a_x + i b_x \\ a_y + i b_y \\ a_z + i b_z \end{pmatrix} \right\} = \begin{pmatrix} -a_x \\ -a_y \\ -a_z \end{pmatrix}$$

$$S = c \epsilon_0 (a_x^2 + a_y^2 + a_z^2) \hat{\mathbf{k}}$$

d.

$$\mathbf{r} = 8 \lambda^2 \mathbf{k} / \pi, \quad \mathbf{t} = 4\pi / \omega$$

$$e^{-i\mathbf{k} \cdot \mathbf{r}} = e^{-i(8 \lambda^2 / \pi) \mathbf{k} \cdot \mathbf{k}} = e^{-i(8 \lambda^2 / \pi)(4\pi / \lambda^2)} = e^{-i32\pi} = -1$$

$$e^{i\omega \mathbf{t}} = e^{i\omega (4\pi / \omega)} = e^{i4\pi} = 1$$

$$e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega \mathbf{t})} = 1$$

$$\mathbf{E} = \text{Re} \left\{ \begin{pmatrix} a_x + i b_x \\ a_y + i b_y \\ a_z + i b_z \end{pmatrix} \right\} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

$$S = c \epsilon_0 (a_x^2 + a_y^2 + a_z^2) \hat{\mathbf{k}}$$

2.4 Orthogonal Polarization States

a. Find the equation for the normalized Jones vector $\mathbf{f}$ orthogonal to $\mathbf{e}$

$$\mathbf{e} = \{A_x e^{-i\phi_x}, A_y e^{-i\phi_y}\}$$

- b. Verify the equation with right circularly polarized light.
- c. Why is the phase of the orthogonal Jones vector a free parameter which can be chosen arbitrarily?
- d. Given a propagation direction $\mathbf{k}$ and a polarization vector $\mathbf{F}$

$$\mathbf{k} = \{k_x, k_y, k_z\}$$

$$\mathbf{F} = \{A_x e^{-i\phi_x}, A_y e^{-i\phi_y}, A_z e^{-i\phi_z}\}$$

find the polarization vector $\mathbf{h}$ orthogonal to $\mathbf{F}$, normalized or unnormalized.

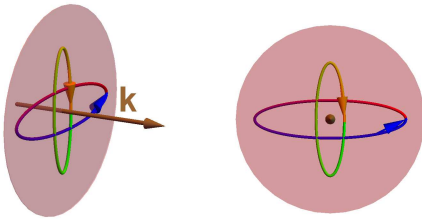
Solution

- a. Find the equation for the normalized Jones vector $\mathbf{f}$ orthogonal to $\mathbf{e}$

$$\mathbf{e} = \{A_x e^{-i\phi_x}, A_y e^{-i\phi_y}\}$$

$$\mathbf{f} = e^{-i\xi} \{-A_y e^{i\phi_y}, A_x e^{i\phi_x}\}$$

$$\mathbf{e} \text{ and } \mathbf{f} \text{ are orthogonal, because } \mathbf{e} \cdot \mathbf{f}^* = e^{-i\xi} (-A_x e^{-i\phi_x} A_y e^{-i\phi_y} + A_y e^{-i\phi_y} A_x e^{-i\phi_x}) = 0$$



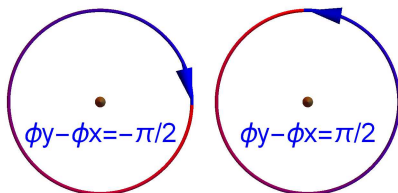
- b. Verify the equation with right circularly polarized light.

Set $\mathbf{e} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ right circularly polarized in decreasing phase convention,

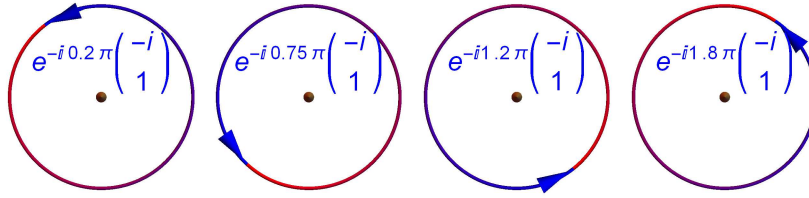
then $A_x=1, \phi_x=0, A_y=1, \phi_y=\pi/2$.

So $\mathbf{f} = e^{-i\xi} \{-1 e^{i\pi/2}, 1 e^{i0}\} = e^{-i\xi} \begin{pmatrix} -i \\ 1 \end{pmatrix} = e^{-i(\xi+\pi/2)} \begin{pmatrix} 1 \\ i \end{pmatrix}$ which is left circularly polarized.

$$\text{RCP: } \mathbf{e} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{LCP: } \mathbf{f} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$



$$e^{-i\xi} \begin{pmatrix} -i \\ 1 \end{pmatrix} :$$



c. Why is the phase of the orthogonal Jones vector a free parameter which can be chosen arbitrarily?

Orthogonality does not depend on the phases.

$\mathbf{e} \cdot \mathbf{f}^* = 0$ regardless of the absolute phase: $e^{-i\xi}$ in $\mathbf{f}$.

d. Given a propagation direction $\mathbf{k}$ and a polarization vector $\mathbf{F}$

$$\mathbf{k} = \{k_x, k_y, k_z\}$$

$$\mathbf{F} = \{A_x e^{-i\phi_x}, A_y e^{-i\phi_y}, A_z e^{-i\phi_z}\}$$

find the polarization vector $\mathbf{h}$ orthogonal to $\mathbf{F}$, normalized or unnormalized.

Polarization vector $\mathbf{h}$ is orthogonal to propagation vector $\mathbf{k}$, which means $\mathbf{h} \cdot \mathbf{k} = 0$.

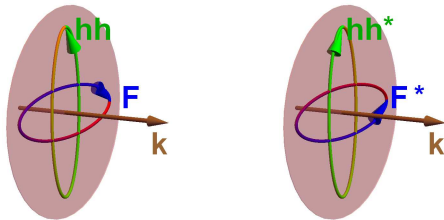
Polarization vector $\mathbf{F}$ is orthogonal to polarization vector $\mathbf{h}$, which means $\mathbf{F} \cdot \mathbf{h} = 0$.

Therefore $\mathbf{h}$ is $\mathbf{k} \times \mathbf{F}^* =$

$$\begin{pmatrix} -e^{i\phi_y} k_z A_y + e^{i\phi_z} k_y A_z \\ e^{i\phi_x} k_z A_x - e^{i\phi_z} k_x A_z \\ -e^{i\phi_x} k_y A_x + e^{i\phi_y} k_x A_y \end{pmatrix}$$

This can be verified by $\{\mathbf{h} \cdot \mathbf{k}, \mathbf{F} \cdot \mathbf{h}\} =$

$$\{0, 0\}$$



For example, similar to part a and b, but now in 3D: say $\mathbf{k} = \{0, 0, 1\}$ along z ,

$\mathbf{F} = \{1, -i, 0\}$ right circularly polarized,

$$\mathbf{k} = \{0, 0, 1\} \text{ along } z$$

$$\mathbf{F} = \{1, -i, 0\} \text{ right circularly polarized}$$

$$\mathbf{h} = \{-i, 1, 0\} \text{ is left circularly polarized}$$

2.5 Rotate Jones vectors

Rotate the following Jones vectors 45° counterclockwise (+x toward +y)

- a. $v1 = (1, i)$
- b. $v2 = (3, 3)$
- c. $v3 = (0, -2)$
- d. $v4 = (1+i, 1-i)$

Solution

Rotation matrix for 45° counterclockwise $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

- a. $\left\{ \frac{1-i}{\sqrt{2}}, \frac{1+i}{\sqrt{2}} \right\}$
- b. $\{0, 3\sqrt{2}\}$
- c. $\{\sqrt{2}, -\sqrt{2}\}$
- d. $\{i\sqrt{2}, \sqrt{2}\}$

2.6 Flux of Jones vectors

- a. What is the normalized flux of the Jones vector $E3 = \begin{pmatrix} w + i x \\ y + i z \end{pmatrix}$?
- b. What is the flux of E3 in W/m^2 , (Watts per meter squared)?
- c. What are the units of the Jones vector elements $w+i x$ and $y+i z$?

Solution

- a. The normalized flux is $E3^\dagger \cdot E3 = (w - i x, y - i z) \cdot \begin{pmatrix} w + i x \\ y + i z \end{pmatrix} = w^2 + x^2 + y^2 + z^2$.
- b. The flux is $\frac{\epsilon_0 c}{2}(w^2 + x^2 + y^2 + z^2)$
- c. Jones vector elements are expressed in volts/meter.

2.7 Circularly polarized basis

- a. Find the matrix for a change of basis from the left and right circularly polarized basis states to the xy-basis.
- b. Convert the Jones vectors $(E_L, E_R) = (e^{i\eta}, e^{-i\eta}) / \sqrt{2}$ from the circular basis into the xy-basis and identify the type of polarization states.

Solution

To change the basis from Cartesian xy coordinates into LR, the rows of the change of basis matrix $R_{xy \rightarrow LR}$ are the conjugates of the states

$$R_{xy \rightarrow LR} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

To convert in the reverse direction, the change of basis matrix is $R_{LR \rightarrow xy}$ is the matrix inverse

$$R_{LR \rightarrow xy} = (R_{xy \rightarrow LR})^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$a. \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \cdot \begin{pmatrix} E_L \\ E_R \end{pmatrix}$$

$$b. \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \cos \eta \\ \sin \eta \end{pmatrix} \text{ which is linearly polarized at angle } \eta.$$

2.8 Basis conversion

Convert the six basis polarization states Table 2.2 from Jones vectors in the ordinary linear xy-basis into the LR circular basis

Solution

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix};$$

$$R \cdot \{1, 0\}$$

$$\left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$$

$$R \cdot \{0, 1\}$$

$$\left\{ -\frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}} \right\}$$

$$R \cdot \{1, 1\} / \sqrt{2}$$

$$\left\{ \frac{1}{2} - \frac{i}{2}, \frac{1}{2} + \frac{i}{2} \right\}$$

$$R \cdot \{1, -1\} / \sqrt{2}$$

$$\left\{ \frac{1}{2} + \frac{i}{2}, \frac{1}{2} - \frac{i}{2} \right\}$$

$$R \cdot \{1, i\} / \sqrt{2}$$

$$\{1, 0\}$$

$$R \cdot \{1, -i\} / \sqrt{2}$$

$$\{0, 1\}$$

2.10 Polarization vector

What are the two directions that light propagating with polarization vector

$$\mathbf{E} = \begin{pmatrix} 4j \\ 6 \\ 4j \end{pmatrix}$$

might be propagating? This can be determined by taking the cross product between the E-field at two different times using the real field representation, not the exponential form.

Solution

$$\frac{\text{Re}[\text{polV}] \times \text{Im}[\text{polV}]}{|\text{Re}[\text{polV}] \times \text{Im}[\text{polV}]|} = \left\{ \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\} \text{ or}$$

$$\frac{\text{Im}[\text{polV}] \times \text{Re}[\text{polV}]}{|\text{Im}[\text{polV}] \times \text{Re}[\text{polV}]|} = \left\{ -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\}$$

2.11 Circularly Polarized Light

$$\mathbf{e}_2 = \{6j, 10, -8j\}$$

- Show that the state $\mathbf{e}_2$ is circularly polarized.
- Find the axis of light propagation. The direction along the axis is undetermined.
- Which direction would the light be propagating to be left circularly polarized?

Solution

- Electric Field for decreasing phase convention: $\mathbf{E}(t) = \text{Re} \left[\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} e^{-j \frac{2\pi}{T} t} \right]$

$$\mathbf{E}(t) = \text{Re} \left[\begin{pmatrix} 6j \\ 10 \\ -8j \end{pmatrix} e^{-j \frac{2\pi}{T} t} \right]$$

$$= \text{Re} \left\{ \begin{pmatrix} 6j \\ 10 \\ -8j \end{pmatrix} \left[\cos \left(\frac{2\pi}{T} t \right) - j \sin \left(\frac{2\pi}{T} t \right) \right] \right\}$$

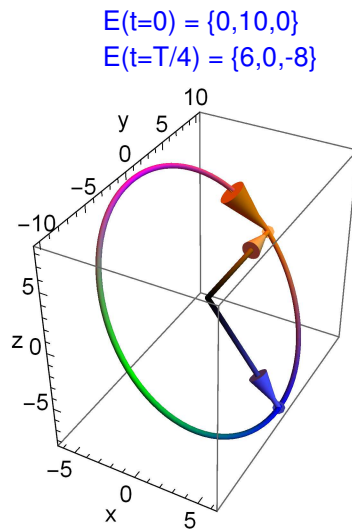
$$= \begin{pmatrix} -6 \sin \left(\frac{2\pi}{T} t \right) \\ 10 \cos \left(\frac{2\pi}{T} t \right) \\ 8 \sin \left(\frac{2\pi}{T} t \right) \end{pmatrix}$$

$$|\mathbf{E}(t)| = \sqrt{6^2 \sin^2 \left(\frac{2\pi}{T} t \right) + 10^2 \cos^2 \left(\frac{2\pi}{T} t \right) + 8^2 \sin^2 \left(\frac{2\pi}{T} t \right)}$$

$$= 10$$

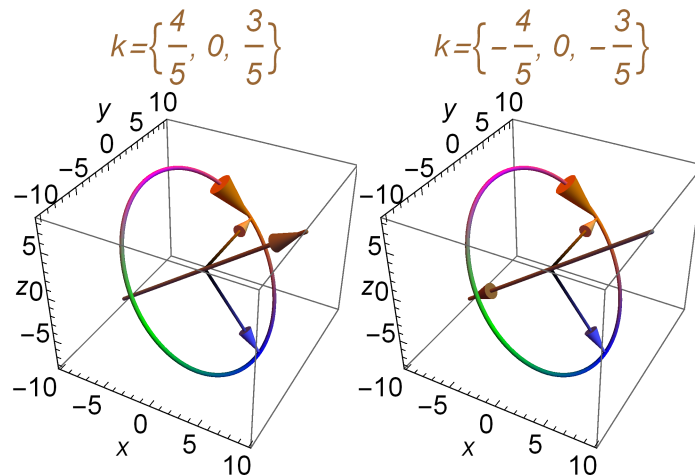
$|\mathbf{E}|$ is constant at all time, so $\mathbf{e}_2$ is circularly polarized.

- Consider the $\mathbf{E}$ vector at two arbitrary times



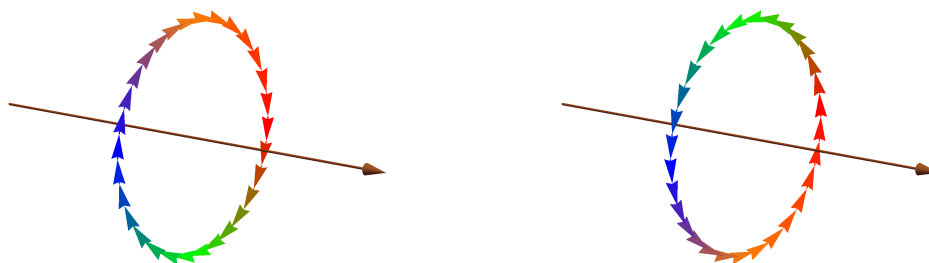
The cross product of the two vectors gives the propagation axis, direction not specified:

$$\left\{ -\frac{4}{5}, 0, -\frac{3}{5} \right\} \text{ or } \left\{ \frac{4}{5}, 0, \frac{3}{5} \right\}$$



- c. Here right and left circular polarizations with the same propagation directions are plotting in time.
As time progresses, $\mathbf{E}$ evolves following the arrows:

Right Circularly Polarized through Time *Left Circularly Polarized through Time*
 (Left Hand Rule) (Right Hand Rule)



From part b, $\mathbf{k} = \left\{ -\frac{4}{5}, 0, -\frac{3}{5} \right\}$ follows the *Right hand rule*, and therefore is left circular polarized

Decreasing Phase Convention	Right Circularly Polarized	Left Circularly Polarized
In time: $\mathbf{E}(t) = \text{Re} \left[\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} e^{-i \frac{2\pi}{T} t} \right]$	LHR	RHR
In space: $\mathbf{E}(\mathbf{r}) = \text{Re} \left[\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} e^{+i \vec{k} \cdot \vec{r}} \right]$	RHR	LHR

The propagation direction: $\mathbf{E}(t=0) \times \mathbf{E}(t=T/4) = \text{Re} \left[\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} e^{-i \theta} \right] \times \text{Re} \left[\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} e^{-i \frac{\pi}{2}} \right] = \text{Re} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \times \text{Im} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$ obeys

Right hand rule,
and therefore corresponds to Left circularly polarization.

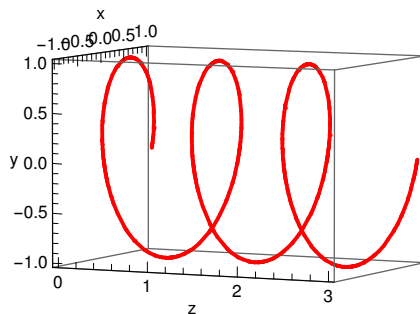
2.12 3-D Polarization Helices

The equation for a circularly polarized monochromatic plane wave electric field traces a helix for a given point in space or time. Graph this helix for right circularly polarized light with amplitude 1, $E(z, t) = \text{Re} \left[e^{i \left(\frac{2\pi}{\lambda} kz - \omega t - \phi \right)} \frac{A}{\sqrt{2}} (1, -i, 0) \right]$.

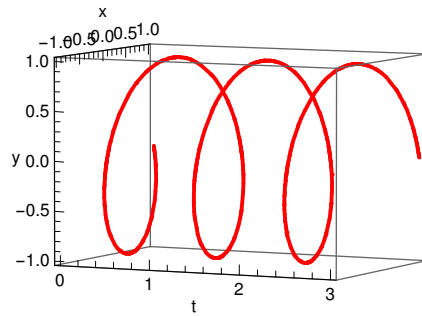
- Generate a three dimensional view of the helix generated in space by the tip of a right circularly polarized vector field (time is fixed).
- What is the sign of the helicity, left or right handed?
- Generate a three dimensional view of the helix generated in x , y , and t by the tip of a right circularly polarized vector field.
- What is the sign of the helicity, left or right handed? Consider the plane wave

Solution

- Space helix is right handed



- Time helix is left handed



2.13

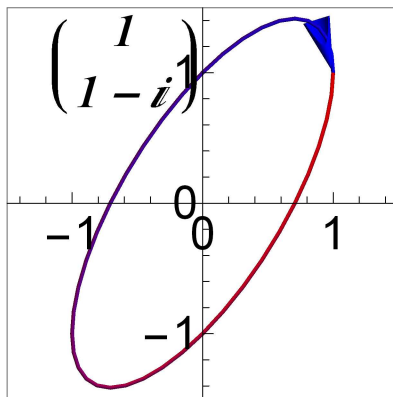
For each of the following Jones vectors

$$\mathbf{E1} = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}; \quad \mathbf{E2} = \begin{pmatrix} -i \\ -i \end{pmatrix}; \quad \mathbf{E3} = \begin{pmatrix} -i \\ 1 \end{pmatrix}; \quad \mathbf{E4} = \frac{1}{4} \begin{pmatrix} 5 \\ 3 + i \end{pmatrix}; \quad \mathbf{E5} = \frac{1}{2} \begin{pmatrix} i \\ 1 + i\sqrt{2} \end{pmatrix}; \quad \mathbf{E6} = \begin{pmatrix} -i \\ -\pi/3 \end{pmatrix};$$

- Plot the polarization ellipse and indicate the direction the electric field is rotating.
- Calculate the phase difference between the x- and y-components, $\delta(\phi) = \phi_x - \phi_y$, the orientation of the major ellipse, ψ , and the normalized flux, P . For circularly polarized light, the orientation may be undefined.
- Calculate the degree of circular polarization (DoCP) defined as $|\text{PL} - \text{PR}| / (\text{PL} + \text{PR})$.

Solution

1.



Orientation $\frac{1}{2} (\pi - \text{ArcTan}[2]) = 58.2825 \text{ degrees}$

$\phi_x - \phi_y = \frac{\pi}{4}$

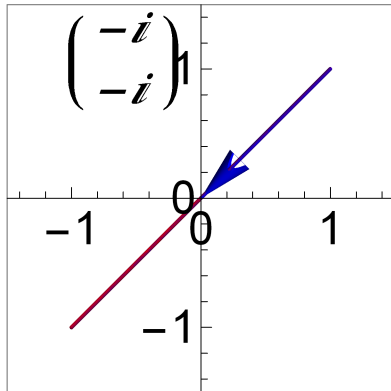
Flux $\frac{3}{2}$

$P_L = \frac{1}{2}$

$P_R = \frac{5}{2}$

DoCP $\frac{2}{3}$

2.



Orientation $\frac{\pi}{4} = 45.$ degrees

$\phi_x - \phi_y$ 0

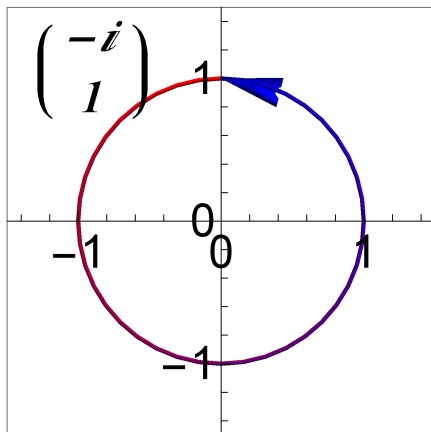
Flux 2

P_L 1

P_R 1

DoCP 0

3



Orientation $\psi = 0 = 0$ degrees

$\phi_x - \phi_y$ $-\frac{\pi}{2}$

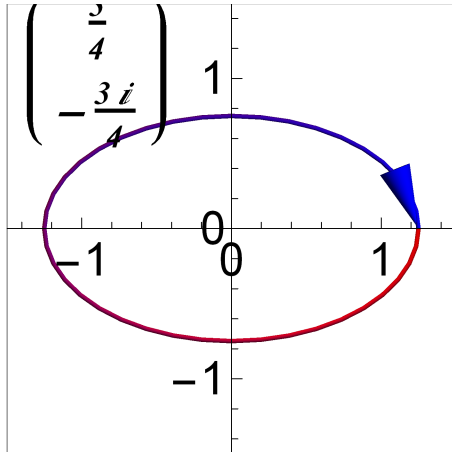
Flux 2

P_L 2

P_R 0

DoCP 1

4



Orientation $\theta = 0$. degrees

$\phi_x - \phi_y = \frac{\pi}{2}$

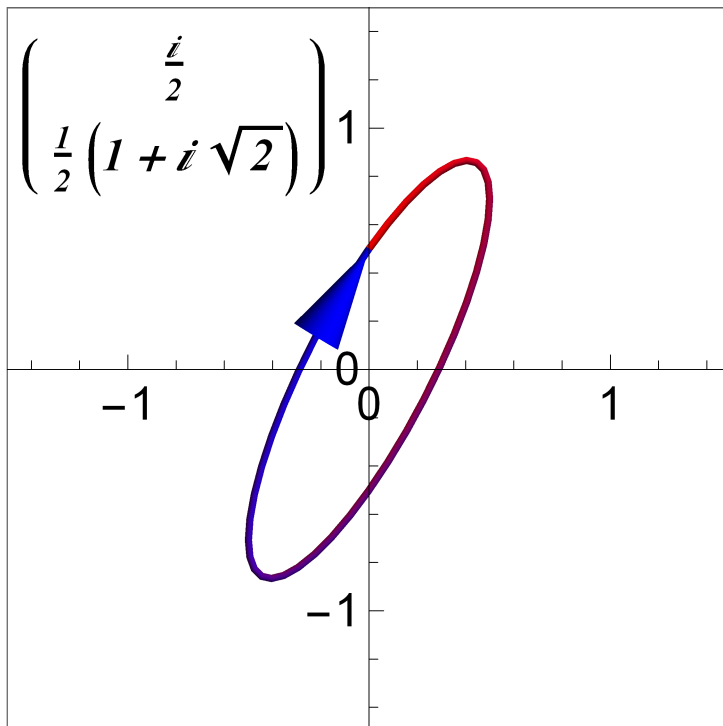
Flux $\frac{17}{8}$

$P_L = \frac{1}{8}$

$P_R = 2$

DoCP $\frac{15}{17}$

5



$$\phi_x - \phi_y = \frac{\pi}{2} - \text{ArcTan}[\sqrt{2}] = 0.61548$$

$$\text{Orientation} = \frac{1}{2} (\pi - \text{ArcTan}[\sqrt{2}]) = 62.6322 \text{ degrees}$$

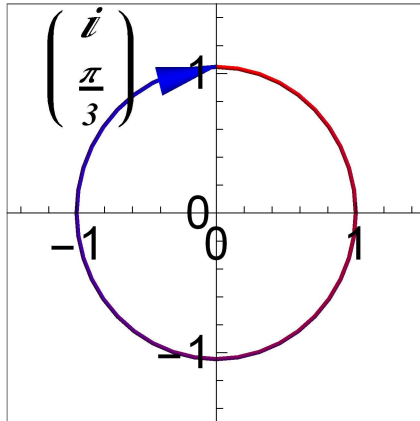
$$\text{Flux} = 1 = 1.$$

$$P_L = \frac{1}{4} = 0.25$$

$$P_R = \frac{3}{4} = 0.75$$

$$\text{DoCP} = \frac{1}{2} = 0.5$$

6



$$\phi_x - \phi_y = \frac{\pi}{2} = 1.5708$$

$$\text{Orientation} = \frac{\pi}{2} = 90. \text{ degrees}$$

$$\text{Flux} = \frac{1}{9} (9 + \pi^2) = 2.09662$$

$$P_L = \frac{1}{18} (-3 + \pi)^2 = 0.0011138$$

$$P_R = \frac{1}{18} (3 + \pi)^2 = 2.09551$$

$$\text{DoCP} = \frac{6\pi}{9 + \pi^2} = 0.998938$$

2.14

a. Compute the normal vector $\hat{\mathbf{w}}$ to complete the right handed orthonormal basis set $(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}})$ where

$$\mathbf{u} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} ; \quad \mathbf{v} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}$$

b. Write the expression for a rotating unit vector $\hat{\mathbf{s}}(t)$ perpendicular to $\hat{\mathbf{u}}$, rotating clockwise about the $\hat{\mathbf{u}}$ -axis when looking into $\hat{\mathbf{u}}$, at an angular velocity of ω rad/sec, such

that $\hat{\mathbf{s}}(0) = \mathbf{v}$. This is an expression for the electric field at a point associated with right circularly polarized light propagating in the $\hat{\mathbf{u}}$ direction.

c. Calculate the polarization vector $\mathbf{E}$ for this wave

Solution

a. For a right-handed basis set, looking into $\mathbf{u}$, by the right hand rule, moving from $\mathbf{v}$ to $\mathbf{w}$ the vector rotates clockwise.

Starting the vector at $\hat{\mathbf{v}}$ and rotating toward $\hat{\mathbf{w}}$ in time yields $\mathbf{s}(t)$

To complete the orthonormal set,

$$\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{w}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

b.

$$\mathbf{s}(t) = \left\{ \frac{\cos[2\pi t\omega]}{\sqrt{6}} - \frac{\sin[2\pi t\omega]}{\sqrt{2}}, \frac{\cos[2\pi t\omega]}{\sqrt{6}} + \frac{\sin[2\pi t\omega]}{\sqrt{2}}, -\sqrt{\frac{2}{3}} \cos[2\pi t\omega] \right\}$$

c. The polarization vector $\mathbf{E}$ will have its real part as $\hat{\mathbf{v}}$ and imaginary part as $\hat{\mathbf{w}}$

$$\mathbf{E} = \begin{pmatrix} -\frac{i}{\sqrt{2}} + \frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Bonus problem sets

2.15 Polarization vectors and propagation vectors

Match each propagation vector $\mathbf{k}$ with the corresponding polarization vector $\mathbf{E}$.

$\mathbf{k}$	$\mathbf{E}$
a. $\left\{ \frac{1}{\sqrt{5}}, \sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}} \right\}$	(1) $\left\{ -\frac{2}{3}, \frac{11}{3}, \frac{1}{3} \right\}$
b. $\{0, 1, 0\}$	(2) $\left\{ \frac{2}{\sqrt{11}}, \frac{9}{\sqrt{11}}, -\frac{6}{\sqrt{11}} \right\}$
c. $\left\{ 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\}$	(3) $\left\{ \frac{1+i}{\sqrt{2}}, 0, \frac{1-i}{\sqrt{2}} \right\}$
d. $\left\{ \frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}} \right\}$	(4) $\left\{ 2\sqrt{\frac{10}{3}}, -2\sqrt{\frac{2}{15}}, 4\sqrt{\frac{2}{15}} \right\}$
e. $\left\{ \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right\}$	(5) $\left\{ 2\sqrt{2}, -1 + \frac{i}{\sqrt{2}}, -1 - \frac{i}{\sqrt{2}} \right\}$

Solution

k and **E** pairs

- a. and 5
- b. and 4
- c. and 2
- d. and 3
- e. and 1

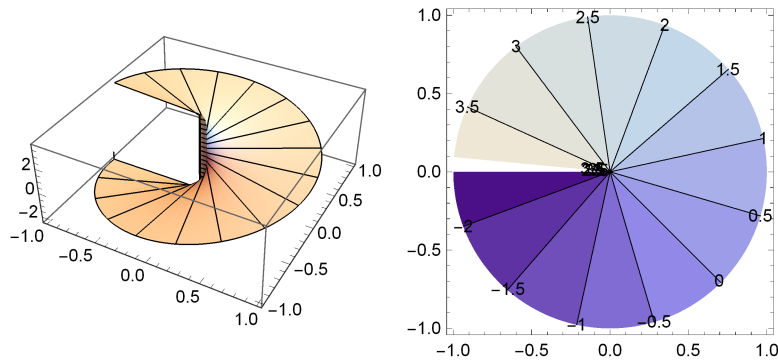
Fluxes:

- (1) 14
- (2) 11
- (3) 2
- (4) 16
- (5) 11

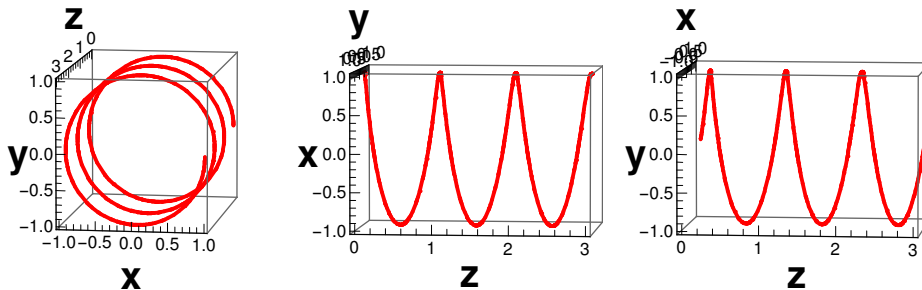
2.16 Identify the following functions

Write down an equation for the following functions.

a.



b.



Solution

a. $\tan^{-1}(y/x) + \frac{\pi}{4}$

b. $\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i2\pi z}$

2.17 Polarized Fluxes

a. Find the right and left circular polarized flux components of the following Jones vectors

$$\begin{aligned} \mathbf{e1} &= \{1, i\}; \\ \mathbf{e2} &= \{10i, -10\}; \\ \mathbf{e3} &= \{2, -2\}; \\ \mathbf{e4} &= \{1+i, 1-i\}; \end{aligned}$$

b. Find the matrix to transform from the usual cartesian basis for Jones vectors to the right $(1, -i)/\sqrt{2}$ and left circular $(1, i)/\sqrt{2}$ basis $\begin{pmatrix} E_R \\ E_L \end{pmatrix}$.

c. Transform the four Jones vectors into the circular basis.

Solution

a.

$$\mathbf{r} = \{1, -i\}/\sqrt{2}; \mathbf{l} = \{1, i\}/\sqrt{2};$$

$$\text{Conjugate}[\mathbf{e1}].\mathbf{r}=0$$

$$\{\text{Abs}[\text{Conjugate}[\mathbf{e1}].\mathbf{r}]^2, \text{Abs}[\text{Conjugate}[\mathbf{e1}].\mathbf{l}]^2\} = \{0, 2\}$$

$$\{\text{Abs}[\text{Conjugate}[\mathbf{e2}].\mathbf{r}]^2, \text{Abs}[\text{Conjugate}[\mathbf{e2}].\mathbf{l}]^2\} = \{0, 200\}$$

$$\{\text{Abs}[\text{Conjugate}[\mathbf{e3}].\mathbf{r}]^2, \text{Abs}[\text{Conjugate}[\mathbf{e3}].\mathbf{l}]^2\} = \{4, 4\}$$

$$\{\text{Abs}[\text{Conjugate}[\mathbf{e4}].\mathbf{r}]^2, \text{Abs}[\text{Conjugate}[\mathbf{e4}].\mathbf{l}]^2\} = \{4, 0\}$$

b.

$$\mathbf{R} = \{\mathbf{r}, \mathbf{l}\} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$\mathbf{R}.\mathbf{e1} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

$$\mathbf{R}.\mathbf{e2} = \begin{pmatrix} 10i\sqrt{2} \\ 0 \end{pmatrix}$$

$$\mathbf{R}.\mathbf{e3} = \begin{pmatrix} (1+i)\sqrt{2} \\ (1-i)\sqrt{2} \end{pmatrix}$$

$$\mathbf{R}.\mathbf{e4} = \begin{pmatrix} 0 \\ (1+i)\sqrt{2} \end{pmatrix}$$

2.18 Jones vectors

Consider the Jones vector

$$\mathbf{E}_a = \begin{pmatrix} 17 \\ 7 \end{pmatrix};$$

- Is this state linear?
- What is the orientation of the major axis?
- What is the component of the flux (intensity) in the following states: **H, V, 45, 135?**
- Convert **E** into the corresponding Stokes parameters?
- Rotate the stokes parameters 45° counterclockwise (x into y).

Solution

a. Yes, the phases of the x and y components are equal, both equal 0.

b. $\text{ArcTan}[17,7] = \text{ArcTan}[7/17] = 22.3801^\circ$.

c.

$$PH = \text{Abs}[\mathbf{E}_a \cdot \{1, 0\}]^2 = 289$$

$$PV = \text{Abs}[\mathbf{E}_a \cdot \{0, 1\}]^2 = 49$$

$$P45 = \text{Abs}[\mathbf{E}_a \cdot \{1, 1\} / \sqrt{2}]^2 = 288$$

$$P135 = \text{Abs}[\mathbf{E}_a \cdot \{1, -1\} / \sqrt{2}]^2 = 50$$

d.

$$S = \{PH + PV, PH - PV, P45 - P135, 0\} = \{338, 240, 238, 0\}$$

e.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 338 \\ 240 \\ 238 \\ 0 \end{pmatrix} = \begin{pmatrix} 338 \\ -238 \\ 240 \\ 0 \end{pmatrix}$$

Supplement materials

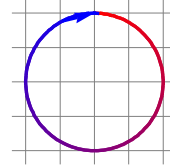
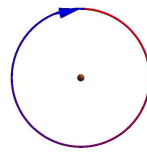
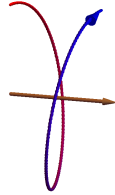
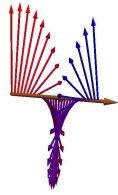
Illustrate: How **E** field evolving as light propagates in time?

The oscillation of electric field can be plotted as a function of time:

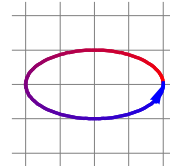
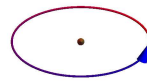
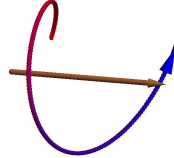
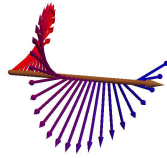
$\mathbf{E}(t) = \text{Re} \left[\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix} e^{-i \frac{2\pi}{T} t} \right]$. The following figures shows three examples of Jones vector **J**

which is the $\begin{pmatrix} E_x e^{-i\phi_x} \\ E_y e^{-i\phi_y} \end{pmatrix}$ part of $\mathbf{E}$ for a propagation along z axis. At any particular time t , $\mathbf{E}$ is a vector pointing orthogonal from its propagation axis (shown in the second column). We often connect the tip of these arrows to represent the $\mathbf{E}$ field (shown in the third column). Looking at the transverse plane (xy -plane) into the propagation along z axis (shown in the 3rd column), we see the electric field oscillating in its local transverse plane. We can estimate $\mathbf{J}$ by estimating where the arrow tip is at as t increases (shown in the 4th column). The arrows in the 3rd and 4th columns start at time $t=0$ (red) and end at one period $t=T$ (blue).

$$\mathbf{J} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$



$$\mathbf{J} = \begin{pmatrix} 1 \\ 0. + 0.5 i \\ 0 \end{pmatrix}$$



$$\mathbf{J} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

