Calculus Early Transcendental Functions 7th edition Larson Solutions Manual

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CHAPTER 2
Limits and Their Properties

Section 2.1 A Preview of Calculus

1. Calculus is the mathematics of change. Precalculus is more static. Answers will vary. *Sample answer:*

Precalculus: Area of a rectangle
Calculus: Area under a curve

Precalculus: Work done by a constant force
Calculus: Work done by a variable force

Precalculus: Center of a rectangle
Calculus: Centroid of a region

2. A secant line through a point P is a line joining P and another point Q on the graph.

The slope of the tangent line P is the limit of the slopes of the secant lines joining P and Q, as Q approaches P.

3. Precalculus: $(20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

4. Calculus required: Velocity is not constant.

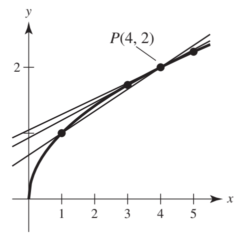
Distance $\approx (20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

5. Calculus required: Slope of the tangent line at $x = 2$ is the rate of change, and equals about 0.16.

6. Precalculus: rate of change $=$ slope $= 0.08$

7. $f(x) = \sqrt{x}$

(a)



(b) $$\text{slope} = m = \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{(\sqrt{x} + 2)(\sqrt{x} - 2)} = \frac{1}{\sqrt{x} + 2}, \; x \neq 4$$

$$x = 1: m = \frac{1}{\sqrt{1} + 2} = \frac{1}{3}$$

$$x = 3: m = \frac{1}{\sqrt{3} + 2} \approx 0.2679$$

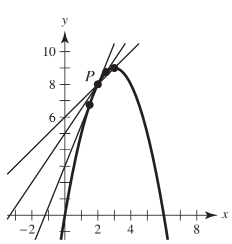
$$x = 5: m = \frac{1}{\sqrt{5} + 2} \approx 0.2361$$

(c) At $P(4, 2)$ the slope is $\dfrac{1}{\sqrt{4} + 2} = \dfrac{1}{4} = 0.25.$

You can improve your approximation of the slope at $x = 4$ by considering x-values very close to 4.

8. $f(x) = 6x - x^2$

(a)



(b) $$\text{slope} = m = \frac{(6x - x^2) - 8}{x - 2} = \frac{(x - 2)(4 - x)}{x - 2} = (4 - x), \; x \neq 2$$

For $x = 3, m = 4 - 3 = 1$

For $x = 2.5, m = 4 - 2.5 = 1.5 = \dfrac{3}{2}$

For $x = 1.5, m = 4 - 1.5 = 2.5 = \dfrac{5}{2}$

(c) At $P(2, 8)$, the slope is 2. You can improve your approximation by considering values of x close to 2.

9. (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

Area $\approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$

(b) You could improve the approximation by using more rectangles.

10. Answers will vary. *Sample answer*:

The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16 + 16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

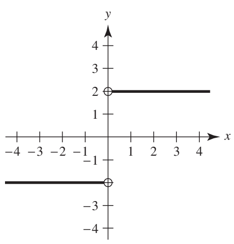
$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

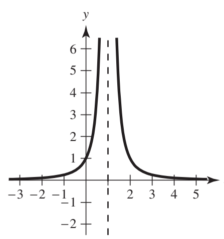
Section 2.2 Finding Limits Graphically and Numerically

1. As the graph of the function approaches 8 on the horizontal axis, the graph approaches 25 on the vertical axis.

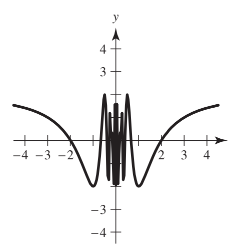
2. (i) The values of f approach different numbers as x approaches c from different sides of c:



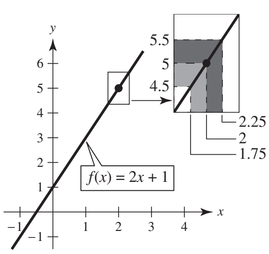
(ii) The values of f increase without bound as x approaches c:



(iii) The values of f oscillate between two fixed numbers as x approaches c:



3.



4. No. For example, consider Example 2 from this section.

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$\lim_{x \to 2} f(x) = 1$, but $f(2) = 0$

5.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	0.3448	0.3344	0.3334	?	0.3332	0.3322	0.3226

$\lim_{x \to 4} \frac{x - 4}{x^2 - 5x - 4} \approx 0.3333$ $\left(\text{Actual limit is } \frac{1}{3}.\right)$

6.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} \approx 0.5000$ $\left(\text{Actual limit is } \frac{1}{2}.\right)$

7.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$\lim_{x\to 0} \dfrac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.) (Make sure you use radian mode.)

8.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	–0.0005	–0.0050	–0.0500

$\lim_{x\to 0} \dfrac{\cos x - 1}{x} \approx 0.0000$ (Actual limit is 0.) (Make sure you use radian mode.)

9.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.9516	0.9950	0.9995	1.0005	1.0050	1.0517

$\lim_{x\to 0} \dfrac{e^x - 1}{x} \approx 1.0000$ (Actual limit is 1.)

10.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	1.0536	1.0050	1.0005	0.9995	0.9950	0.9531

$\lim_{x\to 0} \dfrac{\ln(x + 1)}{x} \approx 1.0000$ (Actual limit is 1.)

11.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$\lim_{x\to 1} \dfrac{x - 2}{x^2 + x - 6} \approx 0.2500$ $\left(\text{Actual limit is } \dfrac{1}{4}.\right)$

12.

x	–4.1	–4.01	–4.001	–4	–3.999	–3.99	–3.9
$f(x)$	1.1111	1.0101	1.0010	?	0.9990	0.9901	0.9091

$\lim_{x\to -4} \dfrac{x + 4}{x^2 + 9x + 20} \approx 1.0000$ (Actual limit is 1.)

13.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.7340	0.6733	0.6673	0.6660	0.6600	0.6015

$\lim_{x\to 1} \dfrac{x^4 - 1}{x^6 - 1} \approx 0.6666$ $\left(\text{Actual limit is } \dfrac{2}{3}.\right)$

14.

x	–3.1	–3.01	–3.001	–3	–2.999	–2.99	–2.9
$f(x)$	27.91	27.0901	27.0090	?	26.9910	26.9101	26.11

$\lim_{x\to -3} \dfrac{x^3 + 27}{x + 3} \approx 27.0000$ (Actual limit is 27.)

15.

x	–6.1	–6.01	–6.001	–6	–5.999	–5.99	–5.9
$f(x)$	–0.1248	–0.1250	–0.1250	?	–0.1250	–0.1250	–0.1252

$$\lim_{x\to -6}\frac{\sqrt{10-x}-4}{x+6}\approx -0.1250 \qquad \left(\text{Actual limit is } -\frac{1}{8}.\right)$$

16.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	0.1149	0.115	0.1111	?	0.1111	0.1107	0.1075

$$\lim_{x\to 2}\frac{x/(x+1)-2/3}{x-2}\approx 0.1111 \qquad \left(\text{Actual limit is } \frac{1}{9}.\right)$$

17.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	1.9867	1.9999	2.0000	2.0000	1.9999	1.9867

$$\lim_{x\to 0}\frac{\sin 2x}{x}\approx 2.0000 \qquad (\text{Actual limit is 2.})\ (\text{Make sure you use radian mode.})$$

18.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.4950	0.5000	0.5000	0.5000	0.5000	0.4950

$$\lim_{x\to 0}\frac{\tan x}{\tan 2x}\approx 0.5000 \qquad \left(\text{Actual limit is } \frac{1}{2}.\right)$$

19.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.5129	0.5013	0.5001	0.4999	0.4988	0.4879

$$\lim_{x\to 2}\frac{\ln x-\ln 2}{x-2}\approx 0.5000 \qquad \left(\text{Actual limit is } \frac{1}{2}.\right)$$

20.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.3866	0.3697	0.3681	0.3677	0.3660	0.3498

$$\lim_{x\to 1}\frac{1-x}{e-e^x}\approx 0.3679 \qquad \left(\text{Actual limit is } \frac{1}{e}.\right)$$

21.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	3.99982	4	4	0	0	0.00018

$$\lim_{x\to 0}\frac{4}{1+e^{1/x}}\ \text{does not exist.}$$

22. $f(x)=\dfrac{3|x|}{x^2}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	30	300	3000	?	3000	300	30

As x approaches 0 from either side, the function increases without bound.

23. $\lim_{x\to 3}(4 - x) = 1$

24. $\lim_{x\to 0}\sec x = 1$

25. $\lim_{x\to 2} f(x) = \lim_{x\to 2}(4 - x) = 2$

26. $\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(x^2 + 3\right) = 4$

27. $\lim_{x\to 2}\dfrac{|x - 2|}{x - 2}$ does not exist.

For values of x to the left of 2, $\dfrac{|x - 2|}{(x - 2)} = -1$, whereas

for values of x to the right of 2, $\dfrac{|x - 2|}{(x - 2)} = 1$.

28. $\lim_{x\to 0}\dfrac{4}{2 + e^{1/x}}$ does not exist. The function approaches 2 from the left side of 0 and it approaches 0 from the right side of 0.

29. $\lim_{x\to 0}\cos(1/x)$ does not exist because the function oscillates between -1 and 1 as x approaches 0.

30. $\lim_{x\to \pi/2}\tan x$ does not exist because the function increases without bound as x approaches $\dfrac{\pi}{2}$ from the left and decreases without bound as x approaches $\dfrac{\pi}{2}$ from the right.

31. (a) $f(1)$ exists. The black dot at $(1, 2)$ indicates that $f(1) = 2$.

(b) $\lim_{x\to 1} f(x)$ does not exist. As x approaches 1 from the left, $f(x)$ approaches 3.5, whereas as x approaches 1 from the right, $f(x)$ approaches 1.

(c) $f(4)$ does not exist. The hollow circle at $(4, 2)$ indicates that f is not defined at 4.

(d) $\lim_{x\to 4} f(x)$ exists. As x approaches 4, $f(x)$ approaches 2: $\lim_{x\to 4} f(x) = 2$.

32. (a) $f(-2)$ does not exist. The vertical dotted line indicates that f is not defined at -2.

(b) $\lim_{x\to -2} f(x)$ does not exist. As x approaches -2, the values of $f(x)$ do not approach a specific number.

(c) $f(0)$ exists. The black dot at $(0, 4)$ indicates that $f(0) = 4$.

(d) $\lim_{x\to 0} f(x)$ does not exist. As x approaches 0 from the left, $f(x)$ approaches $\frac{1}{2}$, whereas as x approaches 0 from the right, $f(x)$ approaches 4.

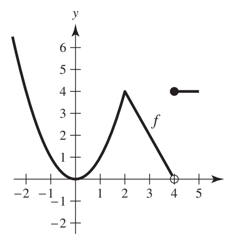
(e) $f(2)$ does not exist. The hollow circle at $\left(2, \frac{1}{2}\right)$ indicates that $f(2)$ is not defined.

(f) $\lim_{x\to 2} f(x)$ exists. As x approaches 2, $f(x)$ approaches $\frac{1}{2}$: $\lim_{x\to 2} f(x) = \frac{1}{2}$.

(g) $f(4)$ exists. The black dot at $(4, 2)$ indicates that $f(4) = 2$.

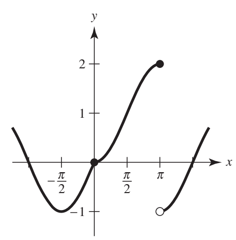
(h) $\lim_{x\to 4} f(x)$ does not exist. As x approaches 4, the values of $f(x)$ do not approach a specific number.

33.



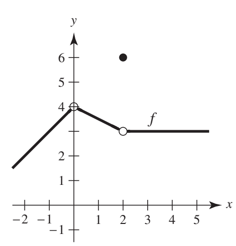
$\lim_{x\to c} f(x)$ exists for all values of $c \neq 4$.

34.

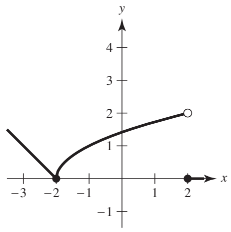


$\lim_{x\to c} f(x)$ exists for all values of $c \neq \pi$.

35. One possible answer is



36. One possible answer is



37. You need $|f(x) - 3| = |(x + 1) - 3| = |x - 2| < 0.4$. So, take $\delta = 0.4$. If $0 < |x - 2| < 0.4$, then $|x - 2| = |(x + 1) - 3| = |f(x) - 3| < 0.4$, as desired.

38. You need $|f(x) - 1| = \left|\frac{1}{x-1} - 1\right| = \left|\frac{2-x}{x-1}\right| < 0.01$. Let $\delta = \frac{1}{101}$. If $0 < |x - 2| < \frac{1}{101}$, then

$$-\frac{1}{101} < x - 2 < \frac{1}{101} \Rightarrow 1 - \frac{1}{101} < x - 1 < 1 + \frac{1}{101}$$

$$\Rightarrow \frac{100}{101} < x - 1 < \frac{102}{101}$$

$$\Rightarrow |x - 1| > \frac{100}{101}$$

and you have

$$|f(x) - 1| = \left|\frac{1}{x-1} - 1\right| = \left|\frac{2-x}{x-1}\right| < \frac{1/101}{100/101} = \frac{1}{100} = 0.01.$$

39. You need to find δ such that $0 < |x - 1| < \delta$ implies $|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1$. That is,

$$-0.1 < \frac{1}{x} - 1 < 0.1$$

$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$

$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$

$$\frac{10}{9} > x > \frac{10}{11}$$

$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$

$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < |x - 1| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$

$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1.$$

40. You need to find δ such that $0 < |x - 1| < \delta$ implies $|f(x) - 1| = \left|2 - \frac{1}{x} - 1\right| = \left|1 - \frac{1}{x}\right| < \varepsilon$.

$$-\varepsilon < \frac{1}{x} - 1 < \varepsilon$$

$$1 - \varepsilon < \frac{1}{x} < 1 + \varepsilon$$

$$\frac{1}{1-\varepsilon} > x > \frac{1}{1+\varepsilon}$$

$$\frac{1}{1-\varepsilon} - 1 > x - 1 > \frac{1}{1+\varepsilon} - 1$$

$$\frac{\varepsilon}{1-\varepsilon} > x - 1 > \frac{-\varepsilon}{1+\varepsilon}$$

For $\varepsilon = 0.05$, take $\delta = \frac{0.05}{1 - 0.05} \approx 0.05$.

For $\varepsilon = 0.01$, take $\delta = \frac{0.01}{1 - 0.01} \approx 0.01$.

For $\varepsilon = 0.005$, take $\delta = \frac{0.005}{1 - 0.005} \approx 0.005$.

As ε decreases, so does δ.

41. $\lim_{x\to 2}(3x+2) = 3(2)+2 = 8 = L$

(a) $\left|(3x+2)-8\right| < 0.01$

$\left|3x-6\right| < 0.01$

$3\left|x-2\right| < 0.01$

$0 < \left|x-2\right| < \frac{0.01}{3} \approx 0.0033 = \delta$

So, if $0 < \left|x-2\right| < \delta = \frac{0.01}{3}$, you have

$3\left|x-2\right| < 0.01$

$\left|3x-6\right| < 0.01$

$\left|(3x+2)-8\right| < 0.01$

$\left|f(x)-L\right| < 0.01.$

(b) $\left|(3x+2)-8\right| < 0.005$

$\left|3x-6\right| < 0.005$

$3\left|x-2\right| < 0.005$

$0 < \left|x-2\right| < \dfrac{0.005}{3} \approx 0.00167 = \delta$

Finally, as in part (a), if $0 < \left|x-2\right| < \dfrac{0.005}{3}$, you have $\left|(3x+2)-8\right| < 0.005.$

42. $\lim_{x\to 6}\left(6-\dfrac{x}{3}\right) = 6-\dfrac{6}{3} = 4 = L$

(a) $\left|\left(6-\dfrac{x}{3}\right)-4\right| < 0.01$

$\left|2-\dfrac{x}{3}\right| < 0.01$

$\left|-\frac{1}{3}(x-6)\right| < 0.01$

$\left|x-6\right| < 0.03$

$0 < \left|x-6\right| < 0.03 = \delta$

So, if $0 < \left|x-6\right| < \delta = 0.03$, you have

$\left|-\frac{1}{3}(x-6)\right| < 0.01$

$\left|2-\dfrac{x}{3}\right| < 0.01$

$\left|\left(6-\dfrac{x}{3}\right)-4\right| < 0.01$

$\left|f(x)-L\right| < 0.01.$

(b) $\left|\left(6-\dfrac{x}{3}\right)-4\right| < 0.005$

$\left|2-\dfrac{x}{3}\right| < 0.005$

$\left|-\dfrac{1}{3}(x-6)\right| < 0.005$

$\left|x-6\right| < 0.015$

$0 < \left|x-6\right| < 0.015 = \delta$

As in part (a), if $0 < \left|x-6\right| < 0.015$, you have

$\left|\left(6-\dfrac{x}{3}\right)-4\right| < 0.005.$

43. $\lim_{x\to 2}\left(x^2-3\right) = 2^2-3 = 1 = L$

(a) $\left|\left(x^2-3\right)-1\right| < 0.01$

$\left|x^2-4\right| < 0.01$

$\left|(x+2)(x-2)\right| < 0.01$

$\left|x+2\right|\left|x-2\right| < 0.01$

$\left|x-2\right| < \dfrac{0.01}{\left|x+2\right|}$

If you assume $1 < x < 3$, then $\delta \approx 0.01/5 = 0.002.$

So, if $0 < \left|x-2\right| < \delta \approx 0.002$, you have

$\left|x-2\right| < 0.002 = \dfrac{1}{5}(0.01) < \dfrac{1}{\left|x+2\right|}(0.01)$

$\left|x+2\right|\left|x-2\right| < 0.01$

$\left|x^2-4\right| < 0.01$

$\left|\left(x^2-3\right)-1\right| < 0.01$

$\left|f(x)-L\right| < 0.01.$

(b) $\left|\left(x^2-3\right)-1\right| < 0.005$

$\left|x^2-4\right| < 0.005$

$\left|(x+2)(x-2)\right| < 0.005$

$\left|x+2\right|\left|x-2\right| < 0.005$

$\left|x-2\right| < \dfrac{0.005}{\left|x+2\right|}$

If you assume $1 < x < 3$, then $\delta = \dfrac{0.005}{5} = 0.001.$

Finally, as in part (a), if $0 < \left|x-2\right| < 0.001$, you have $\left|\left(x^2-3\right)-1\right| < 0.005.$

44. $\lim_{x\to 4}\left(x^2+6\right) = 4^2+6 = 22 = L$

(a) $\left|\left(x^2+6\right)-22\right| < 0.01$

$\left|x^2-16\right| < 0.01$

$\left|(x+4)(x-4)\right| < 0.01$

$\left|x+4\right|\left|x-4\right| < 0.01$

$\left|x-4\right| < \dfrac{0.01}{\left|x+4\right|}$

If you assume $3 < x < 5$, then

$\delta = \dfrac{0.01}{9} \approx 0.00111.$

So, if $0 < \left|x-4\right| < \delta \approx \dfrac{0.01}{9}$, you have

$\left|x-4\right| < \dfrac{0.01}{9} < \dfrac{0.01}{\left|x+4\right|}$

$\left|(x+4)(x-4)\right| < 0.01$

$\left|x^2-16\right| < 0.01$

$\left|\left(x^2+6\right)-22\right| < 0.01$

$\left|f(x)-L\right| < 0.01.$

(b) $\left|\left(x^2+6\right)-22\right| < 0.005$

$\left|x^2-16\right| < 0.005$

$\left|(x-4)(x+4)\right| < 0.005$

$\left|x-4\right|\left|x+4\right| < 0.005$

$\left|x-4\right| < \dfrac{0.05}{\left|x+4\right|}$

If you assume $3 < x < 5$, then

$\delta = \dfrac{0.005}{9} \approx 0.00056.$

Finally, as in part (a), if $0 < \left|x-4\right| < \dfrac{0.005}{9}$,

you have $\left|\left(x^2+6\right)-22\right| < 0.005.$

45. $\lim_{x\to 4}\left(x^2-x\right) = 16-4 = 12 = L$

(a) $\left|\left(x^2-x\right)-12\right| < 0.01$

$\left|(x-4)(x+3)\right| < 0.01$

$\left|x-4\right|\left|x+3\right| < 0.01$

$\left|x-4\right| < \dfrac{0.01}{\left|x+3\right|}$

If you assume $3 < x < 5$, then

$\delta = \dfrac{0.01}{8} = 0.00125.$

So, if $0 < \left|x-4\right| < \dfrac{0.01}{8}$, you have

$\left|x-4\right| < \dfrac{0.01}{\left|x+3\right|}$

$\left|x-4\right|\left|x+3\right| < 0.01$

$\left|x^2-x-12\right| < 0.01$

$\left|\left(x^2-x\right)-12\right| < 0.01$

$\left|f(x)-L\right| < 0.01$

(b) $\left|\left(x^2-x\right)-12\right| < 0.005$

$\left|(x-4)(x+3)\right| < 0.005$

$\left|x-4\right|\left|x+3\right| < 0.005$

$\left|x-4\right| < \dfrac{0.005}{\left|x+3\right|}$

If you assume $3 < x < 5$, then

$\delta = \dfrac{0.005}{8} = 0.000625.$

Finally, as in part (a), if $0 < \left|x-4\right| < \dfrac{0.005}{8}$,

you have $\left|\left(x^2-x\right)-12\right| < 0.005.$

46. $\lim_{x \to 3} x^2 = 3^2 = 9 = L$

(a)
$$\begin{aligned} |x^2 - 9| &< 0.01 \\ |(x-3)(x+3)| &< 0.01 \\ |x-3||x+3| &< 0.01 \\ |x-3| &< \frac{0.01}{|x+3|} \end{aligned}$$

If you assume $2 < x < 4$, then

$\delta = \dfrac{0.01}{7} \approx 0.0014.$

So, if $0 < |x-3| < \dfrac{0.01}{7}$, you have

$$\begin{aligned} |x-3| &< \frac{0.01}{|x+3|} \\ |x-3||x+3| &< 0.01 \\ |x^2 - 9| &< 0.01 \\ |f(x) - L| &< 0.01 \end{aligned}$$

(b)
$$\begin{aligned} |x^2 - 9| &< 0.005 \\ |(x-3)(x+3)| &< 0.005 \\ |x-3||x+3| &< 0.005 \\ |x-3| &< \frac{0.005}{|x+3|} \end{aligned}$$

If you assume $2 < x < 4$, then

$\delta = \dfrac{0.005}{7} \approx 0.00071.$

Finally, as in part (a), if $0 < |x-3| < \dfrac{0.005}{7}$, you have $|x^2 - 9| < 0.005.$

47. $\lim_{x \to 4} (x + 2) = 4 + 2 = 6$

Given $\varepsilon > 0$:

$$\begin{aligned} |(x+2) - 6| &< \varepsilon \\ |x-4| &< \varepsilon = \delta \end{aligned}$$

So, let $\delta = \varepsilon$. So, if $0 < |x-4| < \delta = \varepsilon$, you have

$$\begin{aligned} |x-4| &< \varepsilon \\ |(x+2) - 6| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

48. $\lim_{x \to -2} (4x + 5) = 4(-2) + 5 = -3$

Given $\varepsilon > 0$:

$$\begin{aligned} |(4x+5) - (-3)| &< \varepsilon \\ |4x+8| &< \varepsilon \\ 4|x+2| &< \varepsilon \\ |x+2| &< \frac{\varepsilon}{4} = \delta \end{aligned}$$

So, let $\delta = \dfrac{\varepsilon}{4}.$

So, if $0 < |x+2| < \delta = \dfrac{\varepsilon}{4}$, you have

$$\begin{aligned} |x+2| &< \frac{\varepsilon}{4} \\ |4x+8| &< \varepsilon \\ |(4x+5) - (-3)| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

49. $\lim_{x \to -4} \left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

$$\begin{aligned} \left|\left(\tfrac{1}{2}x - 1\right) - (-3)\right| &< \varepsilon \\ \left|\tfrac{1}{2}x + 2\right| &< \varepsilon \\ \tfrac{1}{2}|x - (-4)| &< \varepsilon \\ |x - (-4)| &< 2\varepsilon \end{aligned}$$

So, let $\delta = 2\varepsilon.$

So, if $0 < |x - (-4)| < \delta = 2\varepsilon$, you have

$$\begin{aligned} |x - (-4)| &< 2\varepsilon \\ \left|\tfrac{1}{2}x + 2\right| &< \varepsilon \\ \left|\left(\tfrac{1}{2}x - 1\right) + 3\right| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

50. $\lim_{x \to 3}\left(\frac{3}{4}x + 1\right) = \frac{3}{4}(3) + 1 = \frac{13}{4}$

Given $\varepsilon > 0$:

$$\left|\left(\frac{3}{4}x + 1\right) - \frac{13}{4}\right| < \varepsilon$$
$$\left|\frac{3}{4}x - \frac{9}{4}\right| < \varepsilon$$
$$\frac{3}{4}|x - 3| < \varepsilon$$
$$|x - 3| < \frac{4}{3}\varepsilon$$

So, let $\delta = \frac{4}{3}\varepsilon$.

So, if $0 < |x - 3| < \delta = \frac{4}{3}\varepsilon$, you have

$$|x - 3| < \frac{4}{3}\varepsilon$$
$$\frac{3}{4}|x - 3| < \varepsilon$$
$$\left|\frac{3}{4}x - \frac{9}{4}\right| < \varepsilon$$
$$\left|\left(\frac{3}{4}x + 1\right) - \frac{13}{4}\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

51. $\lim_{x \to 6} 3 = 3$

Given $\varepsilon > 0$:

$$|3 - 3| < \varepsilon$$
$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$|3 - 3| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

52. $\lim_{x \to 2}(-1) = -1$

Given $\varepsilon > 0$: $|-1 - (-1)| < \varepsilon$

$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$|(-1) - (-1)| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

53. $\lim_{x \to 0} \sqrt[3]{x} = 0$

Given $\varepsilon > 0$: $\left|\sqrt[3]{x} - 0\right| < \varepsilon$

$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$|x| < \varepsilon^3 = \delta$$

So, let $\delta = \varepsilon^3$.

So, for $0|x - 0|\delta = \varepsilon^3$, you have

$$|x| < \varepsilon^3$$
$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$\left|\sqrt[3]{x} - 0\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

54. $\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$

Given $\varepsilon > 0$: $\left|\sqrt{x} - 2\right| < \varepsilon$

$$\left|\sqrt{x} - 2\right|\left|\sqrt{x} + 2\right| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$|x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\varepsilon$. Then,

$$0 < |x - 4| < \delta = 3\varepsilon \Rightarrow |x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$\Rightarrow \left|\sqrt{x} - 2\right| < \varepsilon.$$

55. $\lim_{x \to -5}|x - 5| = |(-5) - 5| = |-10| = 10$

Given $\varepsilon > 0$: $\big||x - 5| - 10\big| < \varepsilon$

$$|-(x - 5) - 10| < \varepsilon \qquad (x - 5 < 0)$$
$$|-x - 5| < \varepsilon$$
$$|x - (-5)| < \varepsilon$$

So, let $\delta = \varepsilon$.

So for $|x - (-5)| < \delta = \varepsilon$, you have

$$|-(x + 5)| < \varepsilon$$
$$|-(x - 5) - 10| < \varepsilon$$
$$\big||x - 5| - 10\big| < \varepsilon \qquad (\text{because } x - 5 < 0)$$
$$|f(x) - L| < \varepsilon.$$

56. $\lim_{x\to 3} |x - 3| = |3 - 3| = 0$

Given $\varepsilon > 0$: $\left||x - 3| - 0\right| < \varepsilon$

$|x - 3| < \varepsilon$

So, let $\delta = \varepsilon$.

So, for $0 < |x - 3| < \delta = \varepsilon$, you have

$$|x - 3| < \varepsilon$$
$$\left||x - 3| - 0\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

57. $\lim_{x\to 1}(x^2 + 1) = 1^2 + 1 = 2$

Given $\varepsilon > 0$: $\left|(x^2 + 1) - 2\right| < \varepsilon$

$$|x^2 - 1| < \varepsilon$$
$$|(x + 1)(x - 1)| < \varepsilon$$
$$|x - 1| < \frac{\varepsilon}{|x + 1|}$$

If you assume $0 < x < 2$, then $\delta = \varepsilon/3$.

So for $0 < |x - 1| < \delta = \frac{\varepsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\varepsilon < \frac{1}{|x + 1|}\varepsilon$$
$$|x^2 - 1| < \varepsilon$$
$$\left|(x^2 + 1) - 2\right| < \varepsilon$$
$$|f(x) - 2| < \varepsilon.$$

58. $\lim_{x\to -4}(x^2 + 4x) = (-4)^2 + 4(-4) = 0$

Given $\varepsilon > 0$: $\left|(x^2 + 4x) - 0\right| < \varepsilon$

$$|x(x + 4)| < \varepsilon$$
$$|x + 4| < \frac{\varepsilon}{|x|}$$

If you assume $-5 < x < -3$, then $\delta = \frac{\varepsilon}{5}$.

So for $0 < |x - (-4)| < \delta = \frac{\varepsilon}{5}$, you have

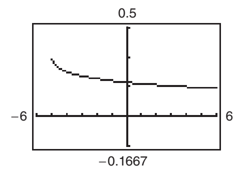
$$|x + 4| < \frac{\varepsilon}{5} < \frac{1}{|x|}\varepsilon$$
$$|x(x + 4)| < \varepsilon$$
$$\left|(x^2 + 4x) - 0\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

59. $\lim_{x\to\pi} f(x) = \lim_{x\to\pi} 4 = 4$

60. $\lim_{x\to\pi} f(x) = \lim_{x\to\pi} x = \pi$

61. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

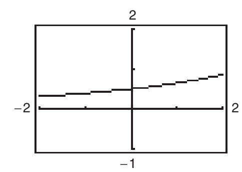
$\lim_{x\to 4} f(x) = \dfrac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \frac{1}{6}\right)$.

62. $f(x) = \dfrac{e^{x/2} - 1}{x}$

$\lim_{x\to 0} f(x) = \dfrac{1}{2}$



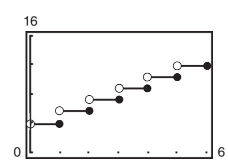
The domain is all $x \neq 0$. The graphing utility does not show the hole at $\left(0, \frac{1}{2}\right)$.

63. $C(t) = 9.99 - 0.79[\![1 - t]\!],\ t > 0$

(a) $C(10.75) = 9.99 - 0.79[\![1 - 10.75]\!]$
$= 9.99 - 0.79(-10)$
$= \$17.89$

$C(10.75)$ represents the cost of a 10-minute, 45-second call.

(b)

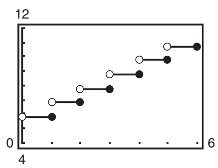


(c) The limit does not exist because the limits from the left and right are not equal.

64. $C(t) = 5.79 - 0.99[\![1 - t]\!], t > 0$

(a) $$\begin{aligned} C(10.75) &= 5.79 - 0.99[\![1 - 10.75]\!] \\ &= 5.79 - 0.99(-10) \\ &= \$15.69 \end{aligned}$$

$C(10.75)$ represents the cost of a 10-minute, 45-second call.

(b) 

(c) The limit does not exist because the limits from the left and right are not equal.

65. Choosing a smaller positive value of δ will still satisfy the inequality $|f(x) - L| < \varepsilon$.

66. In the definition of $\lim_{x\to c} f(x)$, f must be defined on both sides of c, but does not have to be defined at c itself. The value of f at c has no bearing on the limit as x approaches c.

67. No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

68. No. The fact that $\lim_{x\to 2} f(x) = 4$ has no bearing on the value of f at 2.

69. (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) When $C = 5.5$: $r = \dfrac{5.5}{2\pi} \approx 0.87535$ cm

When $C = 6.5$: $r = \dfrac{6.5}{2\pi} \approx 1.03451$ cm

So $0.87535 < r < 1.03451$.

(c) $\lim_{x\to 3/\pi} (2\pi r) = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

70. $V = \dfrac{4}{3}\pi r^3$, $V = 2.48$

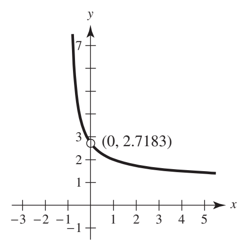
(a) $$\begin{aligned} 2.48 &= \frac{4}{3}\pi r^3 \\ r^3 &= \frac{1.86}{\pi} \\ r &\approx 0.8397 \text{ in.} \end{aligned}$$

(b) $$\begin{aligned} 2.45 \le\ & V \le 2.51 \\ 2.45 \le\ & \frac{4}{3}\pi r^3 \le 2.51 \\ 0.5849 \le\ & r^3 \le 0.5992 \\ 0.8363 \le\ & r \le 0.8431 \end{aligned}$$

(c) For $\varepsilon = 2.51 - 2.48 = 0.03$, $\delta \approx 0.003$

71. $f(x) = (1 + x)^{1/x}$

$\lim_{x\to 0} (1 + x)^{1/x} = e \approx 2.71828$



x	$f(x)$
–0.1	2.867972
–0.01	2.731999
–0.001	2.719642
–0.0001	2.718418
–0.00001	2.718295
–0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

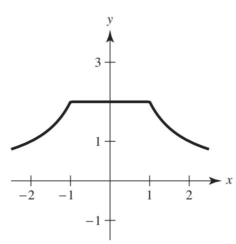
72. $f(x) = \dfrac{|x + 1| - |x - 1|}{x}$

x	–1	–0.5	–0.1	0	0.1	0.5	1.0
$f(x)$	2	2	2	Undef.	2	2	2

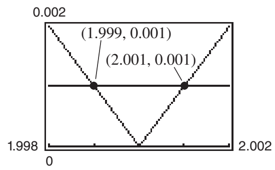
$\lim_{x\to 0} f(x) = 2$

Note that for

$-1 < x < 1, x \ne 0$, $f(x) = \dfrac{(x + 1) + (x - 1)}{x} = 2.$



73.



Using the zoom and trace feature, $\delta = 0.001$. So $(2 - \delta, 2 + \delta) = (1.999, 2.001)$.

Note: $\dfrac{x^2 - 4}{x - 2} = x + 2$ for $x \neq 2$.

74. (a) $\lim_{x \to c} f(x)$ exists for all $c \neq -3$.

(b) $\lim_{x \to c} f(x)$ exists for all $c \neq -2, 0$.

75. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

76. True

77. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$$f(2) = 0$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x - 4) = 2 \neq 0$$

78. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x - 4) = 2 \text{ and } f(2) = 0 \neq 2$$

79. $f(x) = \sqrt{x}$

$\lim_{x \to 0.25} \sqrt{x} = 0.5$ is true.

As x approaches $0.25 = \frac{1}{4}$ from either side, $f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

80. $f(x) = \sqrt{x}$

$\lim_{x \to 0} \sqrt{x} = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of f is $x \geq 0$.

81. Using a graphing utility, you see that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\sin 2x}{x} = 2, \text{ etc.}$$

So, $\lim_{x \to 0} \dfrac{\sin nx}{x} = n$.

82. Using a graphing utility, you see that

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \to 0} \frac{\tan 2x}{x} = 2, \quad \text{etc.}$$

So, $\lim_{x \to 0} \dfrac{\tan(nx)}{x} = n$.

83. If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, you have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$. Therefore, $|L_1 - L_2| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

84. $f(x) = mx + b$, $m \neq 0$. Let $\varepsilon > 0$ be given.

Take $\delta = \dfrac{\varepsilon}{|m|}$.

If $0 < |x - c| < \delta = \dfrac{\varepsilon}{|m|}$, then

$$|m||x - c| < \varepsilon$$

$$|mx - mc| < \varepsilon$$

$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that $\lim_{x \to c} (mx + b) = mc + b$.

85. $\lim_{x \to c} [f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$$|(f(x) - L) - 0| < \varepsilon.$$

This means the same as $|f(x) - L| < \varepsilon$ when $0 < |x - c| < \delta$.

So, $\lim_{x \to c} f(x) = L$.

86. (a) $(3x+1)(3x-1)x^2 + 0.01 = (9x^2-1)x^2 + \frac{1}{100}$

$$= 9x^4 - x^2 + \frac{1}{100}$$

$$= \frac{1}{100}(10x^2-1)(90x^2-1)$$

So, $(3x+1)(3x-1)x^2 + 0.01 > 0$ if

$10x^2 - 1 < 0$ and $90x^2 - 1 < 0$.

Let $(a, b) = \left(-\frac{1}{\sqrt{90}}, \frac{1}{\sqrt{90}}\right)$.

For all $x \neq 0$ in (a, b), the graph is positive.

You can verify this with a graphing utility.

(b) You are given $\lim_{x\to c} g(x) = L > 0$. Let $\varepsilon = \frac{1}{2}L$.

There exists $\delta > 0$ such that $0 < |x - c| < \delta$

implies that $|g(x) - L| < \varepsilon = \frac{L}{2}$. That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$

$$\frac{L}{2} < \quad g(x) \quad < \frac{3L}{2}$$

For x in the interval $(c-\delta, c+\delta)$, $x \neq c$, you

have $g(x) > \frac{L}{2} > 0$, as desired.

87. The radius OP has a length equal to the altitude z of the triangle plus $\frac{h}{2}$. So, $z = 1 - \frac{h}{2}$.

$$\text{Area triangle} = \frac{1}{2}b\left(1 - \frac{h}{2}\right)$$

$$\text{Area rectangle} = bh$$

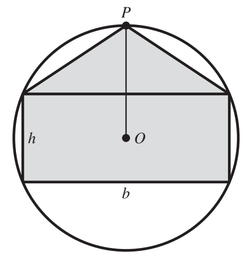
Because these are equal,

$$\frac{1}{2}b\left(1-\frac{h}{2}\right) = bh$$

$$1 - \frac{h}{2} = 2h$$

$$\frac{5}{2}h = 1$$

$$h = \frac{2}{5}.$$



88. Consider a cross section of the cone, where EF is a diagonal of the inscribed cube. $AD = 3$, $BC = 2$.

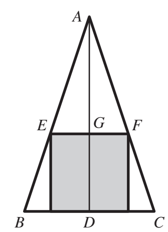
Let x be the length of a side of the cube.

Then $EF = x\sqrt{2}$.

By similar triangles,

$$\frac{EF}{BC} = \frac{AG}{AD}$$

$$\frac{x\sqrt{2}}{2} = \frac{3-x}{3}$$



Solving for x,

$$3\sqrt{2}x = 6 - 2x$$

$$(3\sqrt{2} + 2)x = 6$$

$$x = \frac{6}{3\sqrt{2}+2} = \frac{9\sqrt{2}-6}{7} \approx 0.96.$$

Section 2.3 Evaluating Limits Analytically

1. For polynomial functions $p(x)$, substitute c for x, and simplify.

2. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x\to c} \frac{f(x)}{g(x)}$$

for which $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$

3. If a function f is squeezed between two functions h and g, $h(x) \leq f(x) \leq g(x)$, and h and g have the same limit L as $x \to c$, then $\lim_{x\to c} f(x)$ exists and equals L

4. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

$$\lim_{x\to 0} (1+x)^{1/x} = e$$

5. $\lim_{x\to -3} (2x+5) = 2(-3) + 5 = -1$

6. $\lim_{x\to 9} (4x-1) = 4(9) - 1 = 36 - 1 = 35$

7. $\lim_{x\to -3} (x^2+3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

8. $\lim_{x\to 1} (2x^3 - 6x + 5) = 2(1)^3 - 6(1) + 5$

$$= 2 - 6 + 5 = 1$$

9. $\lim_{x\to 3} \sqrt{x+8} = \sqrt{3+8} = \sqrt{11}$

10. $\lim_{x\to 2} \sqrt[3]{12x+3} = \sqrt[3]{12(2)+3}$

$= \sqrt[3]{24+3} = \sqrt[3]{27} = 3$

11. $\lim_{x\to -4} (1-x)^3 = [1-(-4)]^3 = 5^3 = 125$

12. $\lim_{x\to 0} (3x-2)^4 = (3(0)-2)^4 = (-2)^4 = 16$

13. $\lim_{x\to 2} \dfrac{3}{2x+1} = \dfrac{3}{2(2)+1} = \dfrac{3}{5}$

14. $\lim_{x\to -5} \dfrac{5}{x+3} = \dfrac{5}{-5+3} = -\dfrac{5}{2}$

15. $\lim_{x\to 1} \dfrac{x}{x^2+4} = \dfrac{1}{1^2+4} = \dfrac{1}{5}$

16. $\lim_{x\to 1} \dfrac{3x+5}{x+1} = \dfrac{3(1)+5}{1+1} = \dfrac{3+5}{2} = \dfrac{8}{2} = 4$

17. $\lim_{x\to 7} \dfrac{3x}{\sqrt{x+2}} = \dfrac{3(7)}{\sqrt{7+2}} = \dfrac{21}{3} = 7$

18. $\lim_{x\to 3} \dfrac{\sqrt{x+6}}{x+2} = \dfrac{\sqrt{3+6}}{3+2} = \dfrac{\sqrt{9}}{5} = \dfrac{3}{5}$

19. (a) $\lim_{x\to 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x\to 4} g(x) = 4^3 = 64$

(c) $\lim_{x\to 1} g(f(x)) = g(f(1)) = g(4) = 64$

20. (a) $\lim_{x\to -3} f(x) = (-3) + 7 = 4$

(b) $\lim_{x\to 4} g(x) = 4^2 = 16$

(c) $\lim_{x\to -3} g(f(x)) = g(4) = 16$

21. (a) $\lim_{x\to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x\to 3} g(x) = \sqrt{3+1} = 2$

(c) $\lim_{x\to 1} g(f(x)) = g(3) = 2$

22. (a) $\lim_{x\to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x\to 21} g(x) = \sqrt[3]{21+6} = 3$

(c) $\lim_{x\to 4} g(f(x)) = g(21) = 3$

23. $\lim_{x\to \pi/2} \sin x = \sin \dfrac{\pi}{2} = 1$

24. $\lim_{x\to \pi} \tan x = \tan \pi = 0$

25. $\lim_{x\to 1} \cos \dfrac{\pi x}{3} = \cos \dfrac{\pi}{3} = \dfrac{1}{2}$

26. $\lim_{x\to 2} \sin \dfrac{\pi x}{12} = \sin \dfrac{\pi(2)}{12} = \sin \dfrac{\pi}{6} = \dfrac{1}{2}$

27. $\lim_{x\to 0} \sec 2x = \sec 0 = 1$

28. $\lim_{x\to \pi} \cos 3x = \cos 3\pi = -1$

29. $\lim_{x\to 5\pi/6} \sin x = \sin \dfrac{5\pi}{6} = \dfrac{1}{2}$

30. $\lim_{x\to 5\pi/3} \cos x = \cos \dfrac{5\pi}{3} = \dfrac{1}{2}$

31. $\lim_{x\to 3} \tan\left(\dfrac{\pi x}{4}\right) = \tan \dfrac{3\pi}{4} = -1$

32. $\lim_{x\to 7} \sec\left(\dfrac{\pi x}{6}\right) = \sec \dfrac{7\pi}{6} = \dfrac{-2\sqrt{3}}{3}$

33. $\lim_{x\to 0} e^x \cos 2x = e^0 \cos 0 = 1$

34. $\lim_{x\to 0} e^{-x} \sin \pi x = e^0 \sin 0 = 0$

35. $\lim_{x\to 1} \left(\ln 3x + e^x\right) = \ln 3 + e$

36. $\lim_{x\to 1} \ln\left(\dfrac{x}{e^x}\right) = \ln\left(\dfrac{1}{e}\right) = \ln e^{-1} = -1$

37. $\lim_{x\to c} f(x) = \dfrac{2}{5}, \lim_{x\to c} g(x) = 2$

(a) $\lim_{x\to c} [5g(x)] = 5 \lim_{x\to c} g(x) = 5(2) = 10$

(b) $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

$= \dfrac{2}{5} + 2$

$= \dfrac{12}{5}$

(c) $\lim_{x\to c} [f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right]$

$= \dfrac{2}{5}(2)$

$= \dfrac{4}{5}$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{2/5}{2} = \dfrac{1}{5}$

38. $\lim_{x \to c} f(x) = 2, \lim_{x \to c} g(x) = \frac{3}{4}$

(a) $\lim_{x \to c} [4f(x)] = 4 \lim_{x \to c} f(x) = 4(2) = 8$

(b) $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = 2 + \frac{3}{4} = \frac{11}{4}$

(c) $\lim_{x \to c} [f(x)g(x)] = \left[\lim_{x \to c} f(x)\right]\left[\lim_{x \to c} g(x)\right] = 2\left(\frac{3}{4}\right) = \frac{3}{2}$

(d) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{2}{(3/4)} = \frac{8}{3}$

39. $\lim_{x \to c} f(x) = 16$

(a) $\lim_{x \to c} [f(x)]^2 = \left[\lim_{x \to c} f(x)\right]^2 = (16)^2 = 256$

(b) $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)} = \sqrt{16} = 4$

(c) $\lim_{x \to c} [3f(x)] = 3\left[\lim_{x \to c} f(x)\right] = 3(16) = 48$

(d) $\lim_{x \to c} [f(x)]^{3/2} = \left[\lim_{x \to c} f(x)\right]^{3/2} = (16)^{3/2} = 64$

40. $\lim_{x \to c} f(x) = 27$

(a) $\lim_{x \to c} \sqrt[3]{f(x)} = \sqrt[3]{\lim_{x \to c} f(x)} = \sqrt[3]{27} = 3$

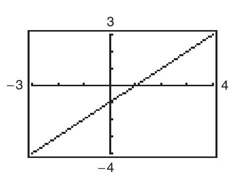
(b) $\lim_{x \to c} \frac{f(x)}{18} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} 18} = \frac{27}{18} = \frac{3}{2}$

(c) $\lim_{x \to c} [f(x)]^2 = \left[\lim_{x \to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x \to c} [f(x)]^{2/3} = \left[\lim_{x \to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

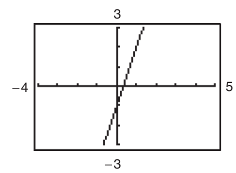
41. $f(x) = \frac{x^2 - 1}{x + 1} = \frac{(x + 1)(x - 1)}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$\lim_{x \to -1} f(x) = \lim_{x \to -1} g(x) = \lim_{x \to -1} (x - 1) = -1 - 1 = -2$



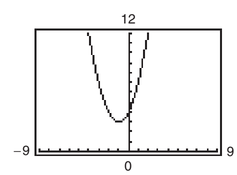
42. $f(x) = \frac{3x^2 + 5x - 2}{x + 2} = \frac{(x + 2)(3x - 1)}{x + 2}$ and $g(x) = 3x - 1$ agree except at $x = -2$.

$\lim_{x \to -2} f(x) = \lim_{x \to -2} g(x) = \lim_{x \to -2} (3x - 1)$
$= 3(-2) - 1 = -7$



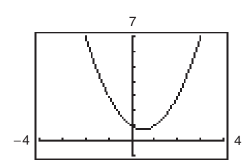
43. $f(x) = \frac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} (x^2 + 2x + 4)$
$= 2^2 + 2(2) + 4 = 12$



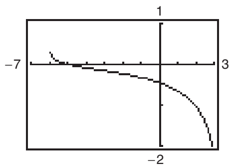
44. $f(x) = \frac{x^3 + 1}{x + 1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

$\lim_{x \to -1} f(x) = \lim_{x \to -1} g(x) = \lim_{x \to -1} (x^2 - x + 1)$
$= (-1)^2 - (-1) + 1 = 3$



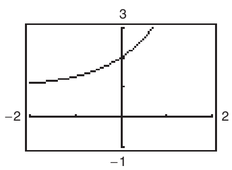
45. $f(x) = \dfrac{(x+4)\ln(x+6)}{x^2-16}$ and $g(x) = \dfrac{\ln(x+6)}{x-4}$ agree except at $x = -4$.

$$\lim_{x\to -4} f(x) = \lim_{x\to -4} g(x) = \frac{\ln 2}{-8} \approx -0.0866$$



46. $f(x) = \dfrac{e^{2x}-1}{e^x-1}$ and $g(x) = e^x + 1$ agree except at $x = 0$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = e^0 + 1 = 2$$



47. $\displaystyle\lim_{x\to 0}\frac{x}{x^2-x} = \lim_{x\to 0}\frac{x}{x(x-1)} = \lim_{x\to 0}\frac{1}{x-1} = \frac{1}{0-1} = -1$

48. $\displaystyle\lim_{x\to 0}\frac{7x^3-x^2}{x} = \lim_{x\to 0}\left(7x^2-x\right) = 0 - 0 = 0$

49. $\displaystyle\lim_{x\to 4}\frac{x-4}{x^2-16} = \lim_{x\to 4}\frac{x-4}{(x+4)(x-4)}$

$$= \lim_{x\to 4}\frac{1}{x+4} = \frac{1}{4+4} = \frac{1}{8}$$

50. $\displaystyle\lim_{x\to 5}\frac{5-x}{x^2-25} = \lim_{x\to 5}\frac{-(x-5)}{(x-5)(x+5)}$

$$= \lim_{x\to 5}\frac{-1}{x+5} = \frac{-1}{5+5} = -\frac{1}{10}$$

51. $\displaystyle\lim_{x\to -3}\frac{x^2+x-6}{x^2-9} = \lim_{x\to -3}\frac{(x+3)(x-2)}{(x+3)(x-3)}$

$$= \lim_{x\to -3}\frac{x-2}{x-3} = \frac{-3-2}{-3-3} = \frac{-5}{-6} = \frac{5}{6}$$

52. $\displaystyle\lim_{x\to 2}\frac{x^2+2x-8}{x^2-x-2} = \lim_{x\to 2}\frac{(x-2)(x+4)}{(x-2)(x+1)}$

$$= \lim_{x\to 2}\frac{x+4}{x+1} = \frac{2+4}{2+1} = \frac{6}{3} = 2$$

53. $\displaystyle\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}\cdot\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$$= \lim_{x\to 4}\frac{(x+5)-9}{(x-4)\left(\sqrt{x+5}+3\right)}$$

$$= \lim_{x\to 4}\frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

54. $\displaystyle\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$

$$= \lim_{x\to 3}\frac{x-3}{(x-3)\left[\sqrt{x+1}+2\right]}$$

$$= \lim_{x\to 3}\frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

55. $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x} = \lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}\cdot\frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}$

$$= \lim_{x\to 0}\frac{(x+5)-5}{x\left(\sqrt{x+5}+\sqrt{5}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x+5}+\sqrt{5}} = \frac{1}{\sqrt{5}+\sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$$

56. $\displaystyle\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}\cdot\frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}$

$$= \lim_{x\to 0}\frac{2+x-2}{\left(\sqrt{2+x}+\sqrt{2}\right)x} = \lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

57. $\displaystyle\lim_{x\to 0}\frac{\frac{1}{3+x}-\frac{1}{3}}{x} = \lim_{x\to 0}\frac{3-(3+x)}{(3+x)3(x)} = \lim_{x\to 0}\frac{-x}{(3+x)(3)(x)} = \lim_{x\to 0}\frac{-1}{(3+x)3} = \frac{-1}{(3)3} = -\frac{1}{9}$

58. $\lim_{x\to 0} \dfrac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x} = \lim_{x\to 0} \dfrac{\dfrac{4-(x+4)}{4(x+4)}}{x} = \lim_{x\to 0} \dfrac{-1}{4(x+4)} = \dfrac{-1}{4(4)} = -\dfrac{1}{16}$

59. $\lim_{\Delta x\to 0} \dfrac{2(x+\Delta x) - 2x}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{2\Delta x}{\Delta x} = \lim_{\Delta x\to 0} 2 = 2$

60. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x\to 0} (2x + \Delta x) = 2x$

61. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^2 - 2(x+\Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x}$

$= \lim_{\Delta x\to 0} (2x + \Delta x - 2) = 2x - 2$

62. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{\Delta x\left(3x^2 + 3x\Delta x + (\Delta x)^2\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2\right) = 3x^2$

63. $\lim_{x\to 0} \dfrac{\sin x}{5x} = \lim_{x\to 0} \left[\left(\dfrac{\sin x}{x}\right)\left(\dfrac{1}{5}\right)\right] = (1)\left(\dfrac{1}{5}\right) = \dfrac{1}{5}$

64. $\lim_{x\to 0} \dfrac{3(1-\cos x)}{x} = \lim_{x\to 0} \left[3\left(\dfrac{(1-\cos x)}{x}\right)\right] = (3)(0) = 0$

65. $\lim_{x\to 0} \dfrac{(\sin x)(1-\cos x)}{x^2} = \lim_{x\to 0} \left[\dfrac{\sin x}{x} \cdot \dfrac{1-\cos x}{x}\right]$

$= (1)(0) = 0$

66. $\lim_{\theta\to 0} \dfrac{\cos\theta \tan\theta}{\theta} = \lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

67. $\lim_{x\to 0} \dfrac{\sin^2 x}{x} = \lim_{x\to 0} \left[\dfrac{\sin x}{x} \sin x\right] = (1)\sin 0 = 0$

68. $\lim_{x\to 0} \dfrac{\tan^2 x}{x} = \lim_{x\to 0} \dfrac{\sin^2 x}{x\cos^2 x} = \lim_{x\to 0} \left[\dfrac{\sin x}{x} \cdot \dfrac{\sin x}{\cos^2 x}\right]$

$= (1)(0) = 0$

69. $\lim_{h\to 0} \dfrac{(1-\cos h)^2}{h} = \lim_{h\to 0} \left[\dfrac{1-\cos h}{h}(1-\cos h)\right]$

$= (0)(0) = 0$

70. $\lim_{\phi\to\pi} \phi\sec\phi = \pi(-1) = -\pi$

71. $\lim_{x\to 0} \dfrac{6 - 6\cos x}{3} = \dfrac{6 - 6\cos 0}{3} = \dfrac{6-6}{3} = 0$

72. $\lim_{x\to 0} \dfrac{\cos x - \sin x - 1}{2x} = \lim_{x\to 0} \dfrac{-\sin x}{2x} + \lim_{x\to 0} \dfrac{\cos x - 1}{2x}$

$= -\dfrac{1}{2}\lim_{x\to 0} \dfrac{\sin x}{x} - \dfrac{1}{2}\lim_{x\to 0} \dfrac{1-\cos x}{x}$

$= -\dfrac{1}{2}(1) - \dfrac{1}{2}(0) = -\dfrac{1}{2}$

73. $\lim_{x\to 0} \dfrac{1 - e^{-x}}{e^x - 1} = \lim_{x\to 0} \dfrac{1 - e^{-x}}{e^x - 1} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim_{x\to 0} \dfrac{(1 - e^{-x})e^{-x}}{1 - e^{-x}}$

$= \lim_{x\to 0} e^{-x} = 1$

74. $\lim_{x\to 0} \dfrac{4(e^{2x} - 1)}{e^x - 1} = \lim_{x\to 0} \dfrac{4(e^x - 1)(e^x + 1)}{e^x - 1}$

$= \lim_{x\to 0} 4(e^x + 1) = 4(2) = 8$

75. $\lim_{t\to 0} \dfrac{\sin 3t}{2t} = \lim_{t\to 0} \left(\dfrac{\sin 3t}{3t}\right)\left(\dfrac{3}{2}\right) = (1)\left(\dfrac{3}{2}\right) = \dfrac{3}{2}$

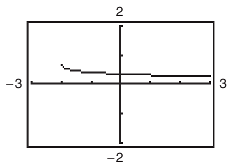
76. $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \left[2\left(\dfrac{\sin 2x}{2x}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{3x}{\sin 3x}\right)\right]$

$= 2(1)\left(\dfrac{1}{3}\right)(1) = \dfrac{2}{3}$

77. $f(x) = \dfrac{\sqrt{x+2} - \sqrt{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	?	0.354	0.353	0.349

It appears that the limit is 0.354.



The graph has a hole at $x = 0$.

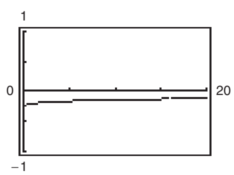
Analytically, $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} \cdot \frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$$= \lim_{x\to 0} \frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$$

78. $f(x) = \dfrac{4 - \sqrt{x}}{x - 16}$

x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	–0.1252	–0.125	–0.125	?	–0.125	–0.125	–0.1248

It appears that the limit is –0.125.



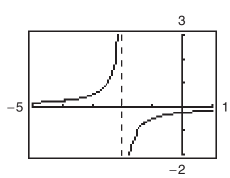
The graph has a hole at $x = 16$.

Analytically, $\displaystyle\lim_{x\to 16} \frac{4-\sqrt{x}}{x-16} = \lim_{x\to 16} \frac{\left(4-\sqrt{x}\right)}{\left(\sqrt{x}+4\right)\left(\sqrt{x}-4\right)} = \lim_{x\to 16} \frac{-1}{\sqrt{x}+4} = -\frac{1}{8}.$

79. $f(x) = \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.263	–0.251	–0.250	?	–0.250	–0.249	–0.238

It appears that the limit is –0.250.



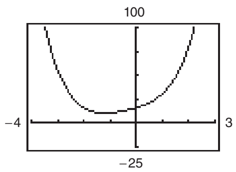
The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim_{x\to 0} \frac{2-(2+x)}{2(2+x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-x}{2(2+x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-1}{2(2+x)} = -\frac{1}{4}.$

80. $f(x) = \dfrac{x^5 - 32}{x - 2}$

x	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41

It appears that the limit is 80.



The graph has a hole at $x = 2$.

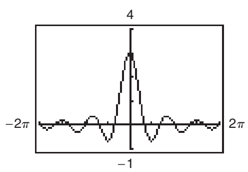
Analytically, $\lim_{x\to 2} \dfrac{x^5 - 32}{x - 2} = \lim_{x\to 2} \dfrac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} = \lim_{x\to 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$

(*Hint:* Use long division to factor $x^5 - 32$.)

81. $f(t) = \dfrac{\sin 3t}{t}$

t	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

It appears that the limit is 3.



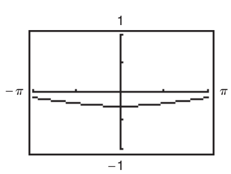
The graph has a hole at $t = 0$.

Analytically, $\lim_{t\to 0} \dfrac{\sin 3t}{t} = \lim_{t\to 0} 3\left(\dfrac{\sin 3t}{3t}\right) = 3(1) = 3.$

82. $f(x) = \dfrac{\cos x - 1}{2x^2}$

x	–1	–0.1	–0.01	0.01	0.1	1
$f(x)$	–0.2298	–0.2498	–0.25	–0.25	–0.2498	–0.2298

It appears that the limit is –0.25.



The graph has a hole at $x = 0$.

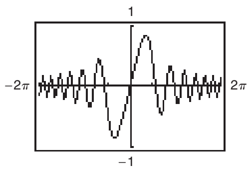
Analytically, $\dfrac{\cos x - 1}{2x^2} \cdot \dfrac{\cos x + 1}{\cos x + 1} = \dfrac{\cos^2 x - 1}{2x^2(\cos x + 1)} = \dfrac{-\sin^2 x}{2x^2(\cos x + 1)} = \dfrac{\sin^2 x}{x^2} \cdot \dfrac{-1}{2(\cos x + 1)}$

$$\lim_{x\to 0}\left[\frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}\right] = 1\left(\frac{-1}{4}\right) = -\frac{1}{4} = -0.25$$

83. $f(x) = \dfrac{\sin x^2}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.099998	–0.01	–0.001	?	0.001	0.01	0.099998

It appears that the limit is 0.



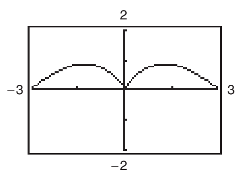
The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{x\to 0} x\left(\frac{\sin x^2}{x^2}\right) = 0(1) = 0.$

84. $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215

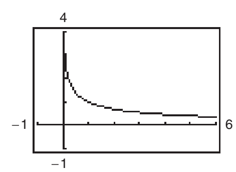
It appears that the limit is 0.



The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x\to 0} \frac{\sin x}{\sqrt[3]{x}} = \lim_{x\to 0} \sqrt[3]{x^2}\left(\frac{\sin x}{x}\right) = (0)(1) = 0.$

85. $f(x) = \dfrac{\ln x}{x - 1}$



x	0.5	0.9	0.99	1.01	1.1	1.5
$f(x)$	1.3863	1.0536	1.0050	0.9950	0.9531	0.8109

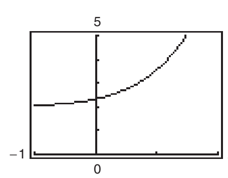
It appears that the limit is 1.

Analytically, $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1}\left(\frac{1}{x-1}\right)\ln x = \lim_{x\to 1} \ln x^{1/(x-1)}$

Let $y = x - 1$, then $x = y + 1$ and $y \to 0$ as $x \to 1$.

So, $\displaystyle\lim_{x\to 1} \ln x^{1/(x-1)} = \lim_{y\to 0} \ln(y+1)^{1/y} = \ln e = 1.$

86. $f(x) = \dfrac{e^{3x} - 8}{e^{2x} - 4}$



x	0.5	0.6	0.69	0.70	0.8	0.9
$f(x)$	2.7450	2.8687	2.9953	3.0103	3.1722	3.3565

It appears that the limit is 3.

Analytically, $\displaystyle\lim_{x\to \ln 2} \frac{e^{3x}-8}{e^{2x}-4} = \lim_{x\to \ln 2} \frac{(e^x-2)(e^{2x}+2e^x+4)}{(e^x-2)(e^x+2)} = \lim_{x\to \ln 2} \frac{e^{2x}+2e^x+4}{e^x+2} = \frac{4+4+4}{2+2} = 3.$

87. $f(x) = 3x - 2$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(x + \Delta x) - 2 - (3x - 2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x + 3\Delta x - 2 - 3x + 2}{\Delta x} = \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = 3$$

88. $f(x) = -6x + 3$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[-6(x + \Delta x) + 3] - [-6x + 3]}{\Delta x} = \lim_{\Delta x \to 0} \frac{-6x - 6\Delta x + 3 + 6x - 3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-6\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (-6) = -6$$

89. $f(x) = x^2 - 4x$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 4(x + \Delta x) - (x^2 - 4x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 4x - 4\Delta x - x^2 + 4x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 4)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 4) = 2x - 4$$

90. $f(x) = 3x^2 + 1$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[3(x + \Delta x)^2 + 1\right] - \left[3x^2 + 1\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(3x^2 + 6x\Delta x + 3(\Delta x)^2 + 1\right) - \left(3x^2 + 1\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\Delta x + 3(\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} 6x + 3\Delta x = 6x$$

91. $f(x) = 2\sqrt{x}$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\sqrt{x + \Delta x} - 2\sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\left(\sqrt{x + \Delta x} - \sqrt{x}\right)}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{2(x + \Delta x - x)}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{2}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}} = x^{-1/2}$$

92. $f(x) = \sqrt{x} - 5$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(\sqrt{x + \Delta x} - 5\right) - \left(\sqrt{x} - 5\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

93. $f(x) = \dfrac{1}{x + 3}$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x + 3} - \dfrac{1}{x + 3}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x + 3 - (x + \Delta x + 3)}{(x + \Delta x + 3)(x + 3)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + 3)(x + 3)} = \frac{-1}{(x + 3)^2}$$

94. $f(x) = \dfrac{1}{x^2}$

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\dfrac{1}{(x+\Delta x)^2}-\dfrac{1}{x^2}}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2-(x+\Delta x)^2}{x^2(x+\Delta x)^2\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{x^2-\left[x^2+2x\Delta x+(\Delta x)^2\right]}{x^2(x+\Delta x)^2\Delta x} = \lim_{\Delta x\to 0}\frac{-2x\Delta x-(\Delta x)^2}{x^2(x+\Delta x)^2\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{-2x-\Delta x}{x^2(x+\Delta x)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}$$

95. $\lim\limits_{x\to 0}\left(4-x^2\right) \le \lim\limits_{x\to 0} f(x) \le \lim\limits_{x\to 0}\left(3+x^2\right)$

$$4 \le \lim_{x\to 0} f(x) \le 4$$

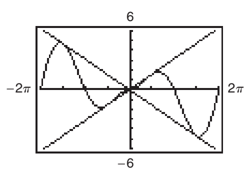
Therefore, $\lim\limits_{x\to 0} f(x) = 4.$

96. $\lim\limits_{x\to a}\left[b-|x-a|\right] \le \lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a}\left[b+|x-a|\right]$

$$b \le \lim_{x\to a} f(x) \le b$$

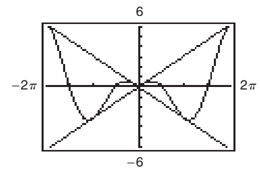
Therefore, $\lim\limits_{x\to a} f(x) = b.$

97. $f(x) = |x|\sin x$



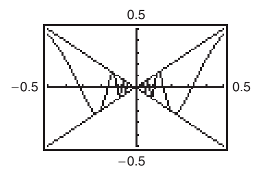
$\lim\limits_{x\to 0}|x|\sin x = 0$

98. $f(x) = |x|\cos x$



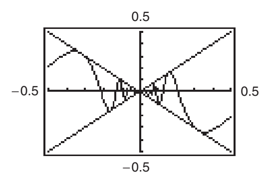
$\lim\limits_{x\to 0}|x|\cos x = 0$

99. $f(x) = x\sin\dfrac{1}{x}$



$\lim\limits_{x\to 0}\left(x\sin\dfrac{1}{x}\right) = 0$

100. $f(x) = x\cos\dfrac{1}{x}$



$\lim\limits_{x\to 0}\left(x\cos\dfrac{1}{x}\right) = 0$

101. (a) Two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

(b) $f(x) = \dfrac{x^2-1}{x-1} = \dfrac{(x+1)(x-1)}{x-1}$ and $g(x) = x+1$ agree at all points except $x = 1$.

(Other answers possible.)

102. Answers will vary. *Sample answers:*

(a) linear: $f(x) = \dfrac{1}{2}x$; $\lim\limits_{x\to 8}\dfrac{1}{2}x = \dfrac{1}{2}(8) = 4$

(b) polynomial of degree 2: $f(x) = x^2 - 60$; $\lim\limits_{x\to 8}\left(x^2-60\right) = 8^2 - 60 = 4$

(c) rational: $f(x) = \dfrac{x}{2x-14}$; $\lim\limits_{x\to 8}\dfrac{x}{2x-14} = \dfrac{8}{2(8)-14} = \dfrac{8}{2} = 4$

(d) radical: $f(x) = \sqrt{x+8}$; $\lim\limits_{x\to 8}\sqrt{x+8} = \sqrt{8+8} = \sqrt{16} = 4$

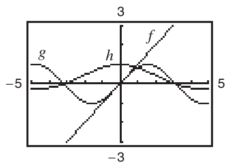
(e) cosine: $f(x) = 4\cos(\pi x)$; $\lim\limits_{x\to 8} 4\cos(\pi x) = 4\cos 8\pi = 4(1) = 4$

(f) sine: $f(x) = 4\sin\left(\dfrac{\pi}{16}x\right)$; $\lim\limits_{x\to 8} 4\sin\left(\dfrac{\pi}{16}x\right) = 4\sin\dfrac{\pi}{2} = 4(1) = 4$

(g) $f(x) = 4e^{x-8}$; $\lim\limits_{x\to 8} 4e^{x-8} = 4e^0 = 4$

(h) $f(x) = 4\ln[e(x-7)]$; $\lim\limits_{x\to 8} 4\ln[e(x-7)] = 4\ln e = 4$

103. $f(x) = x$, $g(x) = \sin x$, $h(x) = \dfrac{\sin x}{x}$



When the x-values are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. So, $|g|/|f| \approx 1$ when x is "close to" 0.

104. (a) Use the dividing out technique because the numerator and denominator have a common factor.

(b) Use the rationalizing technique because the numerator involves a radical expression.

105. $s(t) = -16t^2 + 500$

$$\lim_{t\to 2}\frac{s(2)-s(t)}{2-t} = \lim_{t\to 2}\frac{-16(2)^2+500-(-16t^2+500)}{2-t}$$
$$= \lim_{t\to 2}\frac{436+16t^2-500}{2-t}$$
$$= \lim_{t\to 2}\frac{16(t^2-4)}{2-t}$$
$$= \lim_{t\to 2}\frac{16(t-2)(t+2)}{2-t}$$
$$= \lim_{t\to 2} -16(t+2) = -64 \text{ ft/sec}$$

The paint can is falling at about 64 feet/second.

106. $s(t) = -16t^2 + 500 = 0$ when $t = \sqrt{\dfrac{500}{16}} = \dfrac{5\sqrt{5}}{2}$ sec. The velocity at time $a = \dfrac{5\sqrt{5}}{2}$ is

$$\lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{s\left(\frac{5\sqrt{5}}{2}\right)-s(t)}{\frac{5\sqrt{5}}{2}-t} = \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{0-(-16t^2+500)}{\frac{5\sqrt{5}}{2}-t}$$
$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{16\left(t^2-\frac{125}{4}\right)}{\frac{5\sqrt{5}}{2}-t}$$
$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)}\frac{16\left(t+\frac{5\sqrt{5}}{2}\right)\left(t-\frac{5\sqrt{5}}{2}\right)}{\frac{5\sqrt{5}}{2}-t}$$
$$= \lim_{t\to\frac{5\sqrt{5}}{2}}\left[-16\left(t+\frac{5\sqrt{5}}{2}\right)\right] = -80\sqrt{5}\text{ ft/sec} \approx -178.9 \text{ ft/sec.}$$

The velocity of the paint can when it hits the ground is about 178.9 ft/sec.

107. $s(t) = -4.9t^2 + 200$

$$\lim_{t\to 3}\frac{s(3)-s(t)}{3-t} = \lim_{t\to 3}\frac{-4.9(3)^2+200-(-4.9t^2+200)}{3-t}$$
$$= \lim_{t\to 3}\frac{4.9(t^2-9)}{3-t}$$
$$= \lim_{t\to 3}\frac{4.9(t-3)(t+3)}{3-t}$$
$$= \lim_{t\to 3}[-4.9(t+3)]$$
$$= -29.4 \text{ m/sec}$$

The object is falling about 29.4 m/sec.

108. $-4.9t^2 + 200 = 0$ when $t = \sqrt{\dfrac{200}{4.9}} = \dfrac{20\sqrt{5}}{7}$ sec. The velocity at time $a = \dfrac{20\sqrt{5}}{7}$ is

$$\begin{aligned}\lim_{t\to a}\frac{s(a)-s(t)}{a-t} &= \lim_{t\to a}\frac{0-\left[-4.9t^2+200\right]}{a-t}\\ &= \lim_{t\to a}\frac{4.9(t+a)(t-a)}{a-t}\\ &= \lim_{t\to\frac{20\sqrt{5}}{7}}\left[-4.9\left(t+\frac{20\sqrt{5}}{7}\right)\right] = -28\sqrt{5}\text{ m/sec}\\ &\approx -62.6\text{ m/sec.}\end{aligned}$$

The velocity of the object when it hits the ground is about 62.6 m/sec.

109. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist. However,

$\lim_{x\to 0}\left[f(x) + g(x)\right] = \lim_{x\to 0}\left[\dfrac{1}{x} + \left(-\dfrac{1}{x}\right)\right] = \lim_{x\to 0}[0] = 0$ and therefore does not exist.

110. Suppose, on the contrary, that $\lim_{x\to c} g(x)$ exists. Then, because $\lim_{x\to c} f(x)$ exists, so would $\lim_{x\to c}\left[f(x) + g(x)\right]$, which is a contradiction. So, $\lim_{x\to c} g(x)$ does not exist.

111. Given $f(x) = b$, show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \varepsilon$ whenever $|x - c| < \delta$. Because $|f(x) - b| = |b - b| = 0 < \varepsilon$ for every $\varepsilon > 0$, any value of $\delta > 0$ will work.

112. Given $f(x) = x^n$, n is a positive integer, then

$$\begin{aligned}\lim_{x\to c} x^n &= \lim_{x\to c}\left(xx^{n-1}\right)\\ &= \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right] = c\left[\lim_{x\to c}\left(xx^{n-2}\right)\right]\\ &= c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right] = c(c)\lim_{x\to c}\left(xx^{n-3}\right)\\ &= \cdots = c^n.\end{aligned}$$

113. If $b = 0$, the property is true because both sides are equal to 0. If $b \neq 0$, let $\varepsilon > 0$ be given. Because $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon/|b|$ whenever $0 < |x - c| < \delta$. So, whenever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \varepsilon \quad\text{or}\quad |bf(x) - bL| < \varepsilon$$

which implies that $\lim_{x\to c}\left[bf(x)\right] = bL$.

114. Given $\lim_{x\to c} f(x) = 0$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \varepsilon$ whenever $0 < |x - c| < \delta$.

Now $|f(x) - 0| = |f(x)| = \big||f(x)| - 0\big| < \varepsilon$ for $|x - c| < \delta$. Therefore, $\lim_{x\to c}|f(x)| = 0$.

115.

$$\begin{aligned}-M|f(x)| \le f(x)g(x) &\le M|f(x)|\\ \lim_{x\to c}\left(-M|f(x)|\right) \le \lim_{x\to c}\left[f(x)g(x)\right] &\le \lim_{x\to c}\left(M|f(x)|\right)\\ -M(0) \le \lim_{x\to c}\left[f(x)g(x)\right] &\le M(0)\\ 0 \le \lim_{x\to c}\left[f(x)g(x)\right] &\le 0\end{aligned}$$

Therefore, $\lim_{x\to c}\left[f(x)g(x)\right] = 0$.

116. (a) If $\lim_{x\to c}|f(x)| = 0$, then $\lim_{x\to c}\left[-|f(x)|\right] = 0$.

$$\begin{aligned}-|f(x)| \le f(x) &\le |f(x)|\\ \lim_{x\to c}\left[-|f(x)|\right] \le \lim_{x\to c} f(x) &\le \lim_{x\to c}|f(x)|\\ 0 \le \lim_{x\to c} f(x) &\le 0\end{aligned}$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$. Since $\big||f(x)| - |L|\big| \le |f(x) - L| < \varepsilon$ for $|x - c| < \delta$, then $\lim_{x\to c}|f(x)| = |L|$.

117. Let

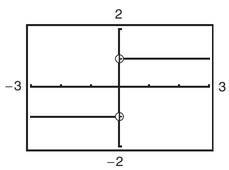
$$f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} 4 = 4.$

$\lim_{x\to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

118. The graphing utility was set in degree mode, instead of *radian* mode.

119. The limit does not exist because the function approaches 1 from the right side of 0 and approaches -1 from the left side of 0.



120. False. $\lim_{x\to\pi} \dfrac{\sin x}{x} = \dfrac{0}{\pi} = 0$

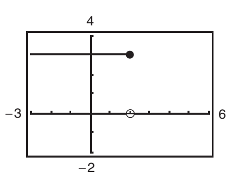
121. True.

122. False. Let

$$f(x) = \begin{cases} x & x \neq 1 \\ 3 & x = 1 \end{cases}, \quad c = 1.$$

Then $\lim_{x\to 1} f(x) = 1$ but $f(1) \neq 1$.

123. False. The limit does not exist because $f(x)$ approaches 3 from the left side of 2 and approaches 0 from the right side of 2.



124. False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$.

Then $f(x) < g(x)$ for all $x \neq 0$. But

$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$

125.
$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x} &= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} \\
&= \left[\lim_{x\to 0} \frac{\sin x}{x}\right]\left[\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right] \\
&= (1)(0) = 0
\end{aligned}$$

126. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

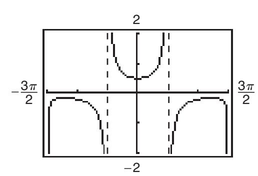
$\lim_{x\to 0} g(x) = 0$

when x is "close to" 0, both parts of the function are "close to" 0.

127. $f(x) = \dfrac{\sec x - 1}{x^2}$

(a) The domain of f is all $x \neq 0, \pi/2 + n\pi$.

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x\to 0} f(x) = \dfrac{1}{2}$

(d)
$$\begin{aligned}
\frac{\sec x - 1}{x^2} &= \frac{\sec x - 1}{x^2} \cdot \frac{\sec x + 1}{\sec x + 1} \\
&= \frac{\sec^2 x - 1}{x^2(\sec x + 1)} = \frac{\tan^2 x}{x^2(\sec x + 1)} \\
&= \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}
\end{aligned}$$

So,
$$\begin{aligned}
\lim_{x\to 0} \frac{\sec x - 1}{x^2} &= \lim_{x\to 0} \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1} \\
&= 1(1)\left(\frac{1}{2}\right) = \frac{1}{2}.
\end{aligned}$$

128. (a) $\lim_{x\to0}\frac{1-\cos x}{x^2}=\lim_{x\to0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}$

$=\lim_{x\to0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$

$=\lim_{x\to0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}$

$=(1)\left(\frac{1}{2}\right)=\frac{1}{2}$

(b) From part (a),

$\frac{1-\cos x}{x^2}\approx\frac{1}{2}\Rightarrow 1-\cos x$

$\approx\frac{1}{2}x^2\Rightarrow\cos x$

$\approx 1-\frac{1}{2}x^2$ for x

$\approx 0.$

(c) $\cos(0.1)\approx 1-\frac{1}{2}(0.1)^2=0.995$

(d) $\cos(0.1)\approx 0.9950$, which agrees with part (c).

Section 2.4 Continuity and One-Sided Limits

1. A function f is continuous at a point c if there is no interruption of the graph at c.

2. $c=-1$ because $\lim_{x\to-1^+}2\sqrt{x+1}=2\sqrt{-1+1}=0$

3. The limit exists because the limit from the left and the limit from the right and equivalent.

4. If f is continuous on a close interval $[a, b]$ and $f(a)\neq f(b)$, then f takes on all values between $f(a)$ and $f(b)$.

5. (a) $\lim_{x\to4^+}f(x)=3$

(b) $\lim_{x\to4^-}f(x)=3$

(c) $\lim_{x\to4}f(x)=3$

The function is continuous at $x=4$ and is continuous on $(-\infty,\infty)$.

6. (a) $\lim_{x\to-2^+}f(x)=-2$

(b) $\lim_{x\to-2^-}f(x)=-2$

(c) $\lim_{x\to-2}f(x)=-2$

The function is continuous at $x=-2$.

7. (a) $\lim_{x\to3^+}f(x)=0$

(b) $\lim_{x\to3^-}f(x)=0$

(c) $\lim_{x\to3}f(x)=0$

The function is NOT continuous at $x=3$.

8. (a) $\lim_{x\to-3^+}f(x)=3$

(b) $\lim_{x\to-3^-}f(x)=3$

(c) $\lim_{x\to-3}f(x)=3$

The function is NOT continuous at $x=-3$ because $f(-3)=4\neq\lim_{x\to-3}f(x)$.

9. (a) $\lim_{x\to2^+}f(x)=-3$

(b) $\lim_{x\to2^-}f(x)=3$

(c) $\lim_{x\to2}f(x)$ does not exist

The function is NOT continuous at $x=2$.

10. (a) $\lim_{x\to-1^+}f(x)=0$

(b) $\lim_{x\to-1^-}f(x)=2$

(c) $\lim_{x\to-1}f(x)$ does not exist.

The function is NOT continuous at $x=-1$.

11. $\lim_{x\to8^+}\frac{1}{x+8}=\frac{1}{8+8}=\frac{1}{16}$

12. $\lim_{x\to3^+}\frac{2}{x+3}=\frac{2}{3+3}=\frac{1}{3}$

13. $\lim_{x\to5^+}\frac{x-5}{x^2-25}=\lim_{x\to5^+}\frac{x-5}{(x+5)(x-5)}$

$=\lim_{x\to5^+}\frac{1}{x+5}=\frac{1}{10}$

14. $\lim_{x\to4^+}\frac{4-x}{x^2-16}=\lim_{x\to4^+}\frac{-(x-4)}{(x+4)(x-4)}=\lim_{x\to4^+}\frac{-1}{x+4}$

$=\frac{-1}{4+4}=-\frac{1}{8}$

15. $\lim_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$ does not exist because $\dfrac{x}{\sqrt{x^2-9}}$ decreases without bound as $x \to -3^-$.

16. $$\lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4^-} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

17. $$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

18. $$\lim_{x\to 10^+} \frac{|x-10|}{x-10} = \lim_{x\to 10^+} \frac{x-10}{x-10} = 1$$

19. $$\lim_{\Delta x\to 0^-} \frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{x-(x+\Delta x)}{x(x+\Delta x)}\cdot\frac{1}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{x(x+\Delta x)}\cdot\frac{1}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-1}{x(x+\Delta x)} = \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

20. $$\lim_{\Delta x\to 0^+} \frac{(x+\Delta x)^2+(x+\Delta x)-(x^2+x)}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{x^2+2x(\Delta x)+(\Delta x)^2+x+\Delta x-x^2-x}{\Delta x}$$
$$= \lim_{\Delta x\to 0^+} \frac{2x(\Delta x)+(\Delta x)^2+\Delta x}{\Delta x} = \lim_{\Delta x\to 0^+} (2x+\Delta x+1) = 2x+0+1 = 2x+1$$

21. $$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x+2}{2} = \frac{5}{2}$$

22. $$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-4x+6) = 9-12+6 = 3$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (-x^2+4x-2) = -9+12-2 = 1$$
Since these one-sided limits disagree, $\lim_{x\to 3} f(x)$ does not exist.

23. $\lim_{x\to\pi} \cot x$ does not exist because $\lim_{x\to\pi^+} \cot x$ and $\lim_{x\to\pi^-} \cot x$ do not exist.

24. $\lim_{x\to\pi/2} \sec x$ does not exist because $\lim_{x\to(\pi/2)^+} \sec x$ and $\lim_{x\to(\pi/2)^-} \sec x$ do not exist.

25. $$\lim_{x\to 4^-} (5[\![x]\!]-7) = 5(3)-7 = 8$$
$([\![x]\!] = 3 \text{ for } 3 \le x < 4)$

26. $$\lim_{x\to 2^+} (2x-[\![x]\!]) = 2(2)-2 = 2$$

27. $$\lim_{x\to -1} \left(\left[\!\left[\frac{x}{3}\right]\!\right]+3\right) = \left[\!\left[-\frac{1}{3}\right]\!\right]+3 = -1+3 = 2$$

28. $$\lim_{x\to 1} \left(1-\left[\!\left[-\frac{x}{2}\right]\!\right]\right) = 1-(-1) = 2$$

29. $\lim_{x\to 3^+} \ln(x-3) = \ln 0$ does not exist.

30. $\lim_{x\to 6^-} \ln(6-x) = \ln 0$ does not exist.

31. $$\lim_{x\to 2^-} \ln\left[x^2(3-x)\right] = \ln\left[4(1)\right] = \ln 4$$

32. $$\lim_{x\to 5^+} \ln\frac{x}{\sqrt{x-4}} = \ln\frac{5}{1} = \ln 5$$

33. $f(x) = \dfrac{1}{x^2-4}$ has discontinuities at $x = -2$ and $x = 2$ because $f(-2)$ and $f(2)$ are not defined.

34. $f(x) = \dfrac{x^2-1}{x+1}$ has a discontinuity at $x = -1$ because $f(-1)$ is not defined.

35. $f(x) = \dfrac{[\![x]\!]}{2}+x$ has discontinuities at each integer k because $\lim_{x\to k^-} f(x) \ne \lim_{x\to k^+} f(x)$.

36. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has a discontinuity at $x = 1$

because $f(1) = 2 \neq \lim_{x \to 1} f(x) = 1$.

37. $g(x) = \sqrt{49 - x^2}$ is continuous on $[-7, 7]$.

38. $f(t) = 3 - \sqrt{9 - t^2}$ is continuous on $[-3, 3]$.

39. $\lim_{x \to 0^-} f(x) = 3 = \lim_{x \to 0^+} f(x)$. f is continuous on $[-1, 4]$.

40. $g(2)$ is not defined. g is continuous on $[-1, 2)$.

41. $f(x) = \dfrac{4}{x - 6}$ has a nonremovable discontinuity at $x = 6$ because $\lim_{x \to 6} f(x)$ does not exist.

42. $f(x) = \dfrac{1}{x^2 + 1}$ is continuous for all real x.

43. $f(x) = 3x - \cos x$ is continuous for all real x.

44. $f(x) = \sin x - 8x$ is continuous for all real x.

45. $f(x) = \dfrac{x}{x^2 - x}$ is not continuous at $x = 0, 1$.

Because $\dfrac{x}{x^2 - x} = \dfrac{1}{x - 1}$ for $x \neq 0$, $x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

46. $f(x) = \dfrac{x}{x^2 - 4}$ has nonremovable discontinuities at $x = 2$ and $x = -2$ because $\lim_{x \to 2} f(x)$ and $\lim_{x \to -2} f(x)$ do not exist.

47. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10} = \dfrac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ because $\lim_{x \to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

48. $f(x) = \dfrac{x + 2}{x^2 - x - 6} = \dfrac{x + 2}{(x - 3)(x + 2)}$

has a nonremovable discontinuity at $x = 3$ because $\lim_{x \to 3} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 3} = -\frac{1}{5}.$$

49. $f(x) = \dfrac{|x + 7|}{x + 7}$

has a nonremovable discontinuity at $x = -7$ because $\lim_{x \to -7} f(x)$ does not exist.

50. $f(x) = \dfrac{2|x - 3|}{x - 3}$ has a nonremovable discontinuity at $x = 3$ because $\lim_{x \to 3} f(x)$ does not exist.

51. $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = \dfrac{2}{2} + 1 = 2$

2. $\left.\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} \left(\frac{x}{2} + 1\right) = 2 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (3 - x) = 1 \end{aligned}\right\} \lim_{x \to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

52. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} (-2x) = -4 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} \left(x^2 - 4x + 1\right) = -3 \end{aligned}\right\} \lim_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

53. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

$= \begin{cases} \tan \dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$

has **possible** discontinuities at $x = -1, x = 1$.

1. $f(-1) = -1$ $\quad f(1) = 1$

2. $\lim_{x\to -1} f(x) = -1$ $\quad \lim_{x\to 1} f(x) = 1$

3. $f(-1) = \lim_{x\to -1} f(x)$ $\quad f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

54. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

$= \begin{cases} \csc \dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$

has **possible** discontinuities at $x = 1, x = 5$.

1. $f(1) = \csc \dfrac{\pi}{6} = 2$ $\quad f(5) = \csc \dfrac{5\pi}{6} = 2$

2. $\lim_{x\to 1} f(x) = 2$ $\quad \lim_{x\to 5} f(x) = 2$

3. $f(1) = \lim_{x\to 1} f(x)$ $\quad f(5) = \lim_{x\to 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x.

55. $f(x) = \begin{cases} \ln(x + 1), & x \ge 0 \\ 1 - x^2, & x < 0 \end{cases}$

has a **possible** discontinuity at $x = 0$.

1. $f(0) = \ln(0 + 1) = \ln 1 = 0$

2. $\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= 1 - 0 = 1 \\ \lim_{x\to 0^+} f(x) &= 0 \end{aligned}\right\} \lim_{x\to 0} f(x)$ does not exist.

So, f has a nonremovable discontinuity at $x = 0$.

56. $f(x) = \begin{cases} 10 - 3e^{5-x}, & x > 5 \\ 10 - \frac{3}{5}x, & x \le 5 \end{cases}$

has a **possible** discontinuity at $x = 5$.

1. $f(5) = 7$

2. $\left.\begin{aligned} \lim_{x\to 5^+} f(x) &= 10 - 3e^{5-5} = 7 \\ \lim_{x\to 5^-} f(x) &= 10 - \tfrac{3}{5}(5) = 7 \end{aligned}\right\} \lim_{x\to 5} f(x) = 7$

3. $f(5) = \lim_{x\to 5} f(x)$

f is continuous at $x = 5$, so, f is continuous for all real x.

57. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

58. $f(x) = \tan \dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

59. $f(x) = [\![x - 8]\!]$ has nonremovable discontinuities at each integer k.

60. $f(x) = 5 - [\![x]\!]$ has nonremovable discontinuities at each integer k.

61. $f(2) = 8$

Find a so that $\lim_{x\to 2^+} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2$.

62. $f(1) = 3$

Find a so that $\lim_{x\to 1^-} (ax - 4) = 3$

$$a(1) - 4 = 3$$
$$a = 7.$$

63. $\lim_{x\to a} g(x) = \lim_{x\to a} \frac{x^2 - a^2}{x - a}$

$= \lim_{x\to a}(x + a) = 2a$

Find a such $2a = 8 \Rightarrow a = 4.$

64. $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{4 \sin x}{x} = 4$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}(a - 2x) = a$

Let $a = 4.$

65. $f(1) = \arctan(1 - 1) + 2 = 2$

Find a such that $\lim_{x\to 1^-}\left(ae^{x-1} + 3\right) = 2$

$ae^{1-1} + 3 = 2$

$a + 3 = 2$

$a = -1.$

66. $f(4) = 2e^{4a} - 2$

Find a such that $\lim_{x\to 4^+} \ln(x - 3) + x^2 = 2e^{4a} - 2$

$\ln(4 - 3) + 4^2 = 2e^{4a} - 2$

$16 = 2e^{4a} - 2$

$9 = e^{4a}$

$\ln 9 = 4a$

$a = \frac{\ln 9}{4} = \frac{\ln 3^2}{4} = \frac{\ln 3}{2}.$

67. $f(g(x)) = \frac{1}{(x^2 + 5) - 6} = \frac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

68. $f(g(x)) = \frac{1}{\sqrt{x - 1}}$

Nonremovable discontinuity at $x = 1$; continuous for all $x > 1$

69. $f(g(x)) = \tan\frac{x}{2}$

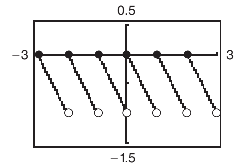
Not continuous at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ Continuous on the open intervals $\ldots, (-3\pi, -\pi), (-\pi, \pi), (\pi, 3\pi), \ldots$

70. $f(g(x)) = \sin x^2$

Continuous for all real x

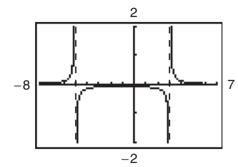
71. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



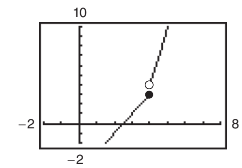
72. $h(x) = \frac{1}{x^2 + 2x - 15} = \frac{1}{(x + 5)(x - 3)}$

Nonremovable discontinuities at $x = -5$ and $x = 3$



73. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

Nonremovable discontinuity at $x = 4$



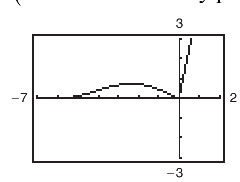
74. $f(x) = \begin{cases} \frac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

$f(0) = 5(0) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{(\cos x - 1)}{x} = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(5x) = 0$

Therefore, $\lim_{x\to 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line.
($x = 0$ was the only possible discontinuity.)



75. $f(x) = \frac{x}{x^2 + x + 2}$

Continuous on $(-\infty, \infty)$

76. $f(x) = \frac{x + 1}{\sqrt{x}}$

Continuous on $(0, \infty)$

77. $f(x) = 3 - \sqrt{x}$

Continuous on $[0, \infty)$

78. $f(x) = x\sqrt{x+3}$

Continuous on $[-3, \infty)$

79. $f(x) = \sec\dfrac{\pi x}{4}$

Continuous on:

$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

80. $f(x) = \cos\dfrac{1}{x}$

Continuous on $(-\infty, 0)$ and $(0, \infty)$

81. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

Since $\lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{x^2 - 1}{x - 1} = \lim_{x\to 1} \dfrac{(x-1)(x+1)}{x-1}$

$= \lim_{x\to 1} (x + 1) = 2,$

f is continuous on $(-\infty, \infty)$.

82. $f(x) = \begin{cases} 2x - 4, & x \neq 3 \\ 1, & x = 3 \end{cases}$

Since $\lim_{x\to 3} f(x) = \lim_{x\to 3} (2x - 4) = 2 \neq 1,$

f is continuous on $(-\infty, 3)$ and $(3, \infty)$.

83. $f(x) = \frac{1}{12}x^4 - x^3 + 4$ is continuous on the interval $[1, 2]$. $f(1) = \frac{37}{12}$ and $f(2) = -\frac{8}{3}$. By the Intermediate Value Theorem, there exists a number c in $[1, 2]$ such that $f(c) = 0$.

84. $f(x) = x^3 + 5x - 3$ is continuous on the interval $[0, 1]$. $f(0) = -3$ and $f(1) = 3$. By the Intermediate Value Theorem, there exists a number c in $[0, 1]$ such that $f(c) = 0$.

85. $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$. $f(0) = -3$ and $f(\pi) = \pi^2 - 1 \approx 8.87 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and π.

86. $f(x) = -\dfrac{5}{x} + \tan\left(\dfrac{\pi x}{10}\right)$ is continuous on the interval $[1, 4]$.

$f(1) = -5 + \tan\left(\dfrac{\pi}{10}\right) \approx -4.7$ and

$f(4) = -\dfrac{5}{4} + \tan\left(\dfrac{2\pi}{5}\right) \approx 1.8$. By the Intermediate Value Theorem, there exists a number c in $[1, 4]$ such that $f(c) = 0$.

87. $h(x) = -2e^{-x/2}\cos 2x$ is continuous on the interval $\left[0, \dfrac{\pi}{2}\right]$. $h(0) = -2 < 0$ and $h\left(\dfrac{\pi}{2}\right) \approx 0.91 > 0$.

By the Intermediate Value Theorem, there exists a number c in $\left[0, \dfrac{\pi}{2}\right]$ such that $h(c) = 0$.

88. $g(t) = (t^3 + 2t - 2)\ln(t^2 + 4)$ is continuous on the interval $[0, 1]$. $g(0) \approx -2.77 < 0$ and $g(1) \approx 1.61 > 0$.

By the Intermediate Value Theorem, there exists a number c in $[0, 1]$ such that $g(c) = 0$.

89. Consider the intervals $[1, 3]$ and $[3, 5]$ for $f(x) = (x - 3)^2 = 2$.

$f(1) = 2 > 0$ and $f(3) = -2 < 0$, so f has at least one zero in $[1, 3]$.

$f(3) = -2 < 0$ and $f(5) = 2 > 0$, so f has at least one zero in $[3, 5]$.

So, f has at least two zeros in $[1, 5]$.

90. Consider the intervals $[1, 3]$ and $[3, 5]$ for $f(x) = 2\cos x$.

$f(1) = 2\cos 1 \approx 1.08 > 0$ and $f(3) = 2\cos 3 \approx -1.98 < 0$, so f has at least one zero in $[1, 3]$.

$f(3) = 2\cos 3 \approx -1.98 < 0$ and $f(5) = 2\cos 5 \approx 0.57 > 0$, so f has at least one zero in $[3, 5]$.

So, f has at least two zeros in $[1, 5]$.

91. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.68$. Using the *root* feature, you find that $x \approx 0.6823$.

92. $f(x) = x^4 - x^2 + 3x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.37$. Using the *root* feature, you find that $x \approx 0.3733$.

93. $f(x) = \sqrt{x^2 + 17x + 19} - 6$

f is continuous on $[0, 1]$.

$f(0) = \sqrt{19} - 6 \approx -1.64 < 0$

$f(1) = \sqrt{37} - 6 \approx 0.08 > 0$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.95$. Using the *root* feature, you find that $x \approx 0.9472$.

94. $f(x) = \sqrt{x^4 + 39x + 13} - 4$

f is continuous on $[0, 1]$.

$f(0) = \sqrt{13} - 4 \approx -0.39 < 0$

$f(1) = \sqrt{53} - 4 \approx 3.28 > 0$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.08$. Using the *root* feature, you find that $x \approx 0.0769$.

95. $g(t) = 2\cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $g(t)$, you find that $t \approx 0.56$. Using the *root* feature, you find that $t \approx 0.5636$.

96. $h(\theta) = \tan\theta + 3\theta - 4$ is continuous on $[0, 1]$.

$h(0) = -4$ and $h(1) = \tan(1) - 1 \approx 0.557$.

By the Intermediate Value Theorem, $h(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $h(\theta)$, you find that $\theta \approx 0.91$. Using the *root* feature, you obtain $\theta \approx 0.9071$.

97. $f(x) = x + e^x - 3$

f is continuous on $[0, 1]$.

$f(0) = e^0 - 3 = -2 < 0$ and

$f(1) = 1 + e - 3 = e - 2 > 0$.

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.79$. Using the *root* feature, you find that $x \approx 0.7921$.

98. $g(x) = 5\ln(x + 1) - 2$

g is continuous on $[0, 1]$.

$g(0) = 5\ln(0 + 1) - 2 = -2$ and

$g(1) = 5\ln(2) - 2 > 0$.

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $g(x)$, you find that $x \approx 0.49$. Using the *root* feature, you find that $x \approx 0.4918$.

99. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$\begin{aligned} x^2 + x - 1 &= 11 \\ x^2 + x - 12 &= 0 \\ (x + 4)(x - 3) &= 0 \\ x = -4 \text{ or } x &= 3 \end{aligned}$$

$c = 3$ ($x = -4$ is not in the interval.)

So, $f(3) = 11$.

100. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$-1 < 0 < 8$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$

$c = 2$ ($x = 4$ is not in the interval.)

So, $f(2) = 0$.

101. $f(x) = \sqrt{x + 7} - 2$

f is continuous on $[0, 5]$.

$f(0) = \sqrt{7} - 2 \approx 0.6458 < 1$

$f(5) = \sqrt{12} - 2 \approx 1.4641 > 1$

The Intermediate Value Theorem applies.

$$\sqrt{x + 7} - 2 = 1$$
$$\sqrt{x + 7} = 3$$
$$x + 7 = 9$$
$$x = 2$$
$$c = 2$$

So, $f(2) = 1$.

102. $f(x) = \sqrt[3]{x} + 8$

f is continuous on $[-9, -6]$.

$f(-9) = (-9)^{1/3} + 8 \approx 5.9199 < 6$

$f(-6) = (-6)^{1/3} + 8 \approx 6.1829 > 6$

The Intermediate Value Theorem applies.

$$\sqrt[3]{x} + 8 = 6$$
$$\sqrt[3]{x} = -2$$
$$x = (-2)^3 = -8$$
$$c = -8$$

So, $f(-8) = 6$.

103. $f(x) = \dfrac{x - x^3}{x - 4}$

f is continuous on $[1, 3]$. The nonremovable discontinuity, $x = 4$, lies outside the interval.

$f(1) = \dfrac{1 - 1}{1 - 4} = 0 < 3$

$f(3) = 24 > 3$

The Intermediate Value Theorem applies.

$$\frac{x - x^3}{x - 4} = 3$$
$$x - x^3 = 3x - 12$$
$$x^3 + 2x - 12 = 0$$
$$(x - 2)(x^2 + 2x + 6) = 0$$
$$x = 2$$

($x^2 + 2x + 6$ has no real solution.)

$$c = 2$$

So, $f(2) = 3$.

104. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\dfrac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$f\left(\dfrac{5}{2}\right) = \dfrac{35}{6}$ and $f(4) = \dfrac{20}{3}$

$\dfrac{35}{6} < 6 < \dfrac{20}{3}$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

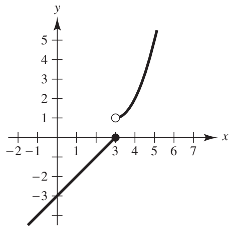
$c = 3$ ($x = 2$ is not in the interval.)

So, $f(3) = 6$.

105. Answers will vary. *Sample answer:*

$$f(x) = \frac{1}{(x - a)(x - b)}$$

106. Answers will vary. *Sample answer*:



The function is not continuous at $x = 3$ because $\lim_{x\to 3^+} f(x) = 1 \neq 0 = \lim_{x\to 3^-} f(x)$.

107. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

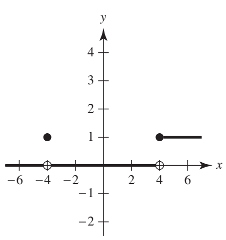
108. A discontinuity at c is removable if the function f can be made continuous at c by appropriately defining (or redefining) $f(c)$. Otherwise, the discontinuity is nonremovable.

(a) $f(x) = \dfrac{|x - 4|}{x - 4}$

(b) $f(x) = \dfrac{\sin(x + 4)}{x + 4}$

(c) $f(x) = \begin{cases} 1, & x \geq 4 \\ 0, & -4 < x < 4 \\ 1, & x = -4 \\ 0, & x < -4 \end{cases}$

$x = 4$ is nonremovable, $x = -4$ is removable



109. True

1. $f(c) = L$ is defined.

2. $\lim_{x\to c} f(x) = L$ exists.

3. $f(c) = \lim_{x\to c} f(x)$

All of the conditions for continuity are met.

110. True. If $f(x) = g(x)$, $x \neq c$, then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$ (if they exist) and at least one of these limits then does not equal the corresponding function value at $x = c$.

111. False. $f(x) = \cos x$ has two zeros in $[0, 2\pi]$. However, $f(0)$ and $f(2\pi)$ have the same sign.

112. True. For $x \in (-1, 0)$, $[\![x]\!] = -1$, which implies that $\lim_{x\to 0^-} [\![x]\!] = -1$.

113. False. A rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

114. False. $f(1)$ is not defined and $\lim_{x\to 1} f(x)$ does not exist.

115. The functions agree for integer values of x:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{for } x \text{ an integer}$$

However, for non-integer values of x, the functions differ by 1.

$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!]$.

For example,

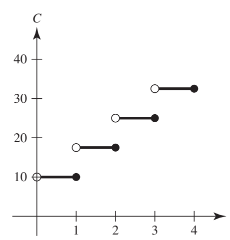
$f\left(\frac{1}{2}\right) = 3 + 0 = 3$, $g\left(\frac{1}{2}\right) = 3 - (-1) = 4$.

116. $\lim_{t\to 4^-} f(t) \approx 28$

$\lim_{t\to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 ounces. At the beginning of day 4, more chlorine was added, and the amount is now about 56 ounces.

117. $C(t) = 10 - 7.5[\![1 - t]\!]$, $t > 0$

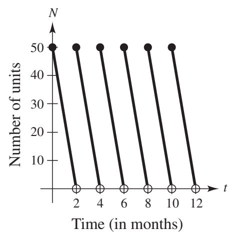


There is a nonremovable discontinuity at every integer value of t, or gigabyte.

118. $N(t) = 25\left(2\left[\!\left[\dfrac{t+2}{2}\right]\!\right] - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

There is a nonremovable discontinuity at every positive even integer. The company replenishes its inventory every two months.



119. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.),
$f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:00 A.M.), $f(10) = s(10) - r(10) > 0$.

Because $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \le t \le 10$, the position functions for the run up and the run down are equal.

120. Let $V = \dfrac{4}{3}\pi r^3$ be the volume of a sphere with radius r.

V is continuous on $[5, 8]$. $V(5) = \dfrac{500\pi}{3} \approx 523.6$ and

$V(8) = \dfrac{2048\pi}{3} \approx 2144.7$. Because

$523.6 < 1500 < 2144.7$, the Intermediate Value Theorem guarantees that there is at least one value r between 5 and 8 such that $V(r) = 1500$. (In fact, $r \approx 7.1012$.)

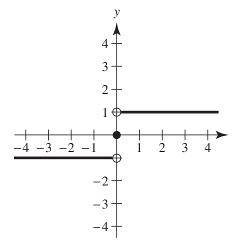
121. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). So, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

122. Let c be any real number. Then $\lim\limits_{x\to c} f(x)$ does not exist because there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

123. If $x = 0$, then $f(0) = 0$ and $\lim\limits_{x\to 0} f(x) = 0$. So, f is continuous at $x = 0$.

If $x \ne 0$, then $\lim\limits_{t\to x} f(t) = 0$ for x rational, whereas $\lim\limits_{t\to x} f(t) = \lim\limits_{t\to x} kt = kx \ne 0$ for x irrational. So, f is not continuous for all $x \ne 0$.

124. $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

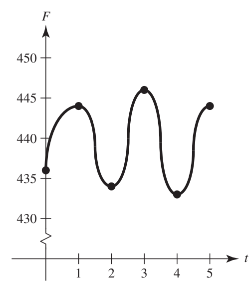


(a) $\lim\limits_{x\to 0^-} \operatorname{sgn}(x) = -1$

(b) $\lim\limits_{x\to 0^+} \operatorname{sgn}(x) = 1$

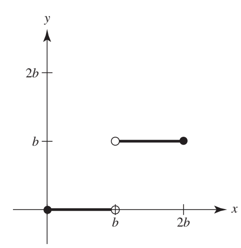
(c) $\lim\limits_{x\to 0} \operatorname{sgn}(x)$ does not exist.

125. (a)



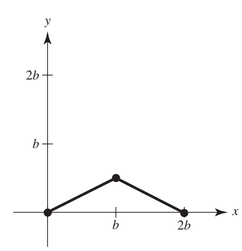
(b) No. The frequency is oscillating.

126. (a) $f(x) = \begin{cases} 0, & 0 \le x < b \\ b, & b < x \le 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \dfrac{x}{2}, & 0 \le x \le b \\ b - \dfrac{x}{2}, & b < x \le 2b \end{cases}$



Continuous on $[0, 2b]$.

127. $f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$

f is continuous for $x < c$ and for $x > c$. At $x = c$, you need $1 - c^2 = c$. Solving $c^2 + c - 1$, you obtain

$$c = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

128. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

129. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}, c > 0$

Domain: $x + c^2 \ge 0 \Rightarrow x \ge -c^2$ and $x \ne 0, \left[-c^2, 0\right) \cup (0, \infty)$

$$\lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c} = \lim_{x \to 0} \frac{\left(x + c^2\right) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x \to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

130. **1.** $f(c)$ is defined.

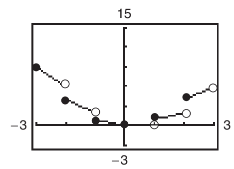
2. $\lim_{x \to c} f(x) = \lim_{\Delta x \to 0} f(c + \Delta x) = f(c)$ exists.

[Let $x = c + \Delta x$. As $x \to c, \Delta x \to 0$]

3. $\lim_{x \to c} f(x) = f(c)$.

Therefore, f is continuous at $x = c$.

131. $h(x) = x[\![x]\!]$



h has nonremovable discontinuities at $x = \pm 1, \pm 2, \pm 3, \ldots$.

132. (a) Define $f(x) = f_2(x) - f_1(x)$. Because f_1 and f_2 are continuous on $[a, b]$, so is f.

$f(a) = f_2(a) - f_1(a) > 0$ and $f(b) = f_2(b) - f_1(b) < 0$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$

(b) Let $f_1(x) = x$ and $f_2(x) = \cos x$, continuous on $[0, \pi/2]$, $f_1(0) < f_2(0)$ and $f_1(\pi/2) > f_2(\pi/2)$.

So by part (a), there exists c in $[0, \pi/2]$ such that $c = \cos(c)$.

Using a graphing utility, $c \approx 0.739$.

133. The statement is true.

If $y \ge 0$ and $y \le 1$, then $y(y - 1) \le 0 \le x^2$, as desired. So assume $y > 1$. There are now two cases.

Case 1:

If $x \le y - \frac{1}{2}$, then $2x + 1 \le 2y$ and

$$\begin{aligned} y(y - 1) &= y(y + 1) - 2y \\ &\le (x + 1)^2 - 2y \\ &= x^2 + 2x + 1 - 2y \\ &\le x^2 + 2y - 2y \\ &= x^2 \end{aligned}$$

Case 2:

If $x \ge y - \frac{1}{2}$

$$\begin{aligned} x^2 &\ge \left(y - \tfrac{1}{2}\right)^2 \\ &= y^2 - y + \tfrac{1}{4} \\ &> y^2 - y \\ &= y(y - 1) \end{aligned}$$

In both cases, $y(y - 1) \le x^2$.

134. $P(1) = P(0^2 + 1) = P(0)^2 + 1 = 1$

$P(2) = P(1^2 + 1) = P(1)^2 + 1 = 2$

$P(5) = P(2^2 + 1) = P(2)^2 + 1 = 5$

Continuing this pattern, you see that $P(x) = x$ for infinitely many values of x.

So, the finite degree polynomial must be constant: $P(x) = x$ for all x.

Section 2.5 Infinite Limits

1. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$$\lim_{x\to c} f(x) = \infty$$

Says how the limit fails to exist.

2. The line $x = c$ is a vertical asymptote if the graph of f approaches $\pm\infty$ as x approaches c.

3. $$\lim_{x\to -2^+} 2\left|\frac{x}{x^2 - 4}\right| = \infty$$

$$\lim_{x\to -2^-} 2\left|\frac{x}{x^2 - 4}\right| = \infty$$

4. $$\lim_{x\to -2^+} \frac{1}{x + 2} = \infty$$

$$\lim_{x\to -2^-} \frac{1}{x + 2} = -\infty$$

5. $$\lim_{x\to -2^+} \tan\frac{\pi x}{4} = -\infty$$

$$\lim_{x\to -2^-} \tan\frac{\pi x}{4} = \infty$$

6. $$\lim_{x\to -2^+} \sec\frac{\pi x}{4} = \infty$$

$$\lim_{x\to -2^-} \sec\frac{\pi x}{4} = -\infty$$

7. $f(x) = \dfrac{1}{x - 4}$

As x approaches 4 from the left, $x - 4$ is a small negative number. So,

$$\lim_{x\to 4^-} f(x) = -\infty$$

As x approaches 4 from the right, $x - 4$ is a small positive number. So,

$$\lim_{x\to 4^+} f(x) = \infty$$

8. $f(x) = \dfrac{-1}{x - 4}$

As x approaches 4 from the left, $x - 4$ is a small negative number. So,

$$\lim_{x\to 4^-} f(x) = \infty.$$

As x approaches 4 from the right, $x - 4$ is a small positive number. So,

$$\lim_{x\to 4^+} f(x) = -\infty.$$

9. $f(x) = \dfrac{1}{(x - 4)^2}$

As x approaches 4 from the left or right, $(x - 4)^2$ is a small positive number. So,

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x) = \infty.$$

10. $f(x) = \dfrac{-1}{(x - 4)^2}$

As x approaches 4 from the left or right, $(x - 4)^2$ is a small positive number. So,

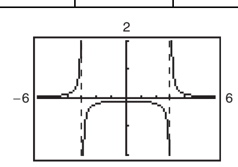
$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = -\infty.$$

11. $f(x) = \dfrac{1}{x^2 - 9}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	0.308	1.639	16.64	166.6	–166.7	–16.69	–1.695	–0.364

$$\lim_{x\to -3^-} f(x) = \infty$$

$$\lim_{x\to -3^+} f(x) = -\infty$$

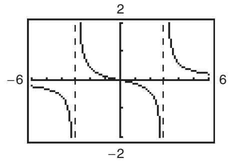


12. $f(x) = \dfrac{x}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1.077	-5.082	-50.08	-500.1	499.9	49.92	4.915	0.9091

$\lim_{x \to -3^-} f(x) = -\infty$

$\lim_{x \to -3^+} f(x) = \infty$

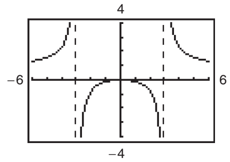


13. $f(x) = \dfrac{x^2}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	-1499	-149.3	-14.25	-2.273

$\lim_{x \to -3^-} f(x) = \infty$

$\lim_{x \to -3^+} f(x) = -\infty$

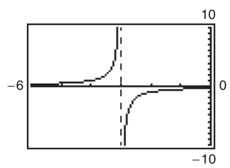


14. $f(x) = -\dfrac{1}{3 + x}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	2	10	100	1000	-1000	-100	-10	-2

$\lim_{x \to -3^-} f(x) = \infty$

$\lim_{x \to -3^+} f(x) = -\infty$

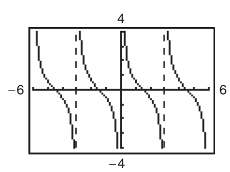


15. $f(x) = \cot \dfrac{\pi x}{3}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1.7321	-9.514	-95.49	-954.9	954.9	95.49	9.514	1.7321

$\lim_{x \to -3^-} f(x) = -\infty$

$\lim_{x \to -3^+} f(x) = \infty$

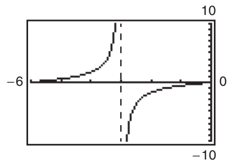


16. $f(x) = \tan \dfrac{\pi x}{6}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	3.73	19.08	190.98	1909.9	–11909.9	–190.98	–19.08	–3.73

$\lim_{x\to -3^-} f(x) = \infty$

$\lim_{x\to -3^+} f(x) = -\infty$



17. $f(x) = \dfrac{x^2}{x^2 - 4} = \dfrac{x^2}{(x+2)(x-2)}$

$\lim_{x\to -2^-} \dfrac{x^2}{x^2-4} = \infty$ and $\lim_{x\to -2^+} \dfrac{x^2}{x^2-4} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x\to 2^-} \dfrac{x^2}{x^2-4} = -\infty$ and $\lim_{x\to 2^+} \dfrac{x^2}{x^2-4} = \infty$

Therefore, $x = 2$ is a vertical asymptote.

18. $g(t) = \dfrac{t-1}{t^2+1}$

No vertical asymptotes because the denominator is never zero.

19. $f(x) = \dfrac{3}{x^2+x-2} = \dfrac{3}{(x+2)(x-1)}$

$\lim_{x\to -2^-} \dfrac{3}{x^2+x-2} = \infty$ and $\lim_{x\to -2^+} \dfrac{3}{x^2+x-2} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x\to 1^-} \dfrac{3}{x^2+x-2} = -\infty$ and $\lim_{x\to 1^+} \dfrac{3}{x^2+x-2} = \infty$

Therefore, $x = 1$ is a vertical asymptote.

20. $g(x) = \dfrac{x^2-5x+25}{x^3+125}$

$= \dfrac{x^2-5x+25}{(x+5)(x^2-5x+25)}$

$= \dfrac{1}{x+5}$

$\lim_{x\to -5^-} \dfrac{1}{x+5} = -\infty$ and $\lim_{x\to -5^+} \dfrac{1}{x+5} = \infty$

Therefore, $x = -5$ is a vertical asymptote.

21. $f(x) = \dfrac{4(x^2+x-6)}{x(x^3-2x^2-9x+18)}$

$= \dfrac{4(x+3)(x-2)}{x(x-2)(x^2-9)}$

$= \dfrac{4}{x(x-3)}, x \neq -3, 2$

$\lim_{x\to 0^-} f(x) = \infty$ and $\lim_{x\to 0^+} f(x) = -\infty$

Therefore, $x = 0$ is a vertical asymptote.

$\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = \infty$

Therefore, $x = 3$ is a vertical asymptote.

$\lim_{x\to 2} f(x) = \dfrac{4}{2(2-3)} = -2$

and

$\lim_{x\to -3} f(x) = \dfrac{4}{-3(-3-3)} = \dfrac{2}{9}$

Therefore, the graph has holes at $x = 2$ and $x = -3$.

22. $h(x) = \dfrac{x^2-9}{x^3+3x^2-x-3}$

$= \dfrac{(x-3)(x+3)}{(x-1)(x+1)(x+3)}$

$= \dfrac{x-3}{(x+1)(x-1)}, x \neq -3$

$\lim_{x\to -1^-} h(x) = -\infty$ and $\lim_{x\to -1^+} h(x) = \infty$

Therefore, $x = -1$ is a vertical asymptote.

$\lim_{x\to 1^-} h(x) = \infty$ and $\lim_{x\to 1^+} h(x) = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

$\lim_{x\to -3} h(x) = \dfrac{-3-3}{(-3+1)(-3-1)} = -\dfrac{3}{4}$

Therefore, the graph has a hole at $x = -3$.

23. $f(x) = \dfrac{e^{-2x}}{x - 1}$

$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} = \infty$

Therefore, $x = 1$ is a vertical asymptote.

24. $g(x) = xe^{-2x}$

The function is continuous for all x. Therefore, there are no vertical asymptotes.

25. $h(t) = \dfrac{\ln(t^2 + 1)}{t + 2}$

$\lim\limits_{t\to -2^-} h(t) = -\infty$ and $\lim\limits_{t\to -2^+} = \infty$

Therefore, $t = -2$ is a vertical asymptote.

26. $f(z) = \ln(z^2 - 4) = \ln[(z + 2)(z - 2)]$

$= \ln(z + 2) + \ln(z - 2)$

The function is undefined for $-2 < z < 2$.

$\lim\limits_{z\to -2^-} f(z) = -\infty$ and $\lim\limits_{z\to -2^+} f(z) = -\infty$

Therefore, the graph vertical asymptotes at $z = \pm 2$.

27. $f(x) = \dfrac{1}{e^x - 1}$

$\lim\limits_{x\to 0^-} f(x) = -\infty$ and $\lim\limits_{x\to 0^+} f(x) = \infty$

Therefore, $x = 0$ is a vertical asymptote.

28. $f(x) = \ln(x + 3)$

$\lim\limits_{x\to -3} f(x) = -\infty$

Therefore, $x = -3$ is a vertical asymptote.

29. $f(x) = \csc \pi x = \dfrac{1}{\sin \pi x}$

Let n be any integer.

$\lim\limits_{x\to n} f(x) = -\infty$ or ∞

Therefore, the graph has vertical asymptotes at $x = n$.

30. $f(x) = \tan \pi x = \dfrac{\sin \pi x}{\cos \pi x}$

$\cos \pi x = 0$ for $x = \dfrac{2n + 1}{2}$, where n is an integer.

$\lim\limits_{x\to \frac{2n+1}{2}} f(x) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at

$x = \dfrac{2n + 1}{2}$.

31. $s(t) = \dfrac{t}{\sin t}$

$\sin t = 0$ for $t = n\pi$, where n is an integer.

$\lim\limits_{t\to n\pi} s(t) = \infty$ or $-\infty$ (for $n \neq 0$)

Therefore, the graph has vertical asymptotes at $t = n\pi$, for $n \neq 0$.

$\lim\limits_{t\to 0} s(t) = 1$

Therefore, the graph has a hole at $t = 0$.

32. $g(\theta) = \dfrac{\tan \theta}{\theta} = \dfrac{\sin \theta}{\theta \cos \theta}$

$\cos \theta = 0$ for $\theta = \dfrac{\pi}{2} + n\pi$, where n is an integer.

$\lim\limits_{\theta\to \frac{\pi}{2} + n\pi} g(\theta) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at

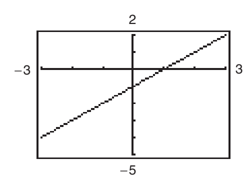
$\theta = \dfrac{\pi}{2} + n\pi$.

$\lim\limits_{\theta\to 0} g(\theta) = 1$

Therefore, the graph has a hole at $\theta = 0$.

33. $\lim\limits_{x\to -1} \dfrac{x^2 - 1}{x + 1} = \lim\limits_{x\to -1} (x - 1) = -2$

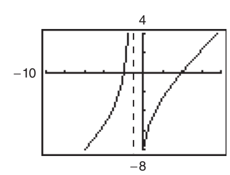
Removable discontinuity at $x = -1$



34. $\lim\limits_{x\to -1^-} \dfrac{x^2 - 2x - 8}{x + 1} = \infty$

$\lim\limits_{x\to -1^+} \dfrac{x^2 - 2x - 8}{x + 1} = -\infty$

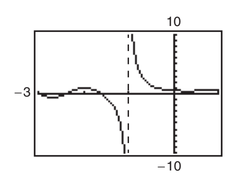
Vertical asymptote at $x = -1$



35. $\lim\limits_{x\to -1^-} \dfrac{\cos(x^2 - 1)}{x + 1} = -\infty$

$\lim\limits_{x\to -1^+} \dfrac{\cos(x^2 - 1)}{x + 1} = \infty$

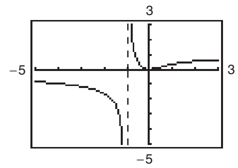
Vertical asymptote at $x = 1$



36. $\lim_{x\to -1^+} \dfrac{\ln(x^2+1)}{x+1} = \infty$

$\lim_{x\to -1^-} \dfrac{\ln(x^2+1)}{x+1} = -\infty$

Vertical asymptote at $x = -1$



37. $\lim_{x\to 2^+} \dfrac{x}{x-2} = \infty$

38. $\lim_{x\to 2^-} \dfrac{x^2}{x^2+4} = \dfrac{4}{4+4} = \dfrac{1}{2}$

39. $\lim_{x\to -3^-} \dfrac{x+3}{(x^2+x-6)} = \lim_{x\to -3^-} \dfrac{x+3}{(x+3)(x-2)}$

$= \lim_{x\to -3^-} \dfrac{1}{x-2} = -\dfrac{1}{5}$

40. $\lim_{x\to -(1/2)^+} \dfrac{6x^2+x-1}{4x^2-4x-3} = \lim_{x\to -(1/2)^+} \dfrac{(3x-1)(2x+1)}{(2x-3)(2x+1)}$

$= \lim_{x\to -(1/2)^+} \dfrac{3x-1}{2x-3} = \dfrac{5}{8}$

41. $\lim_{x\to 0^-} \left(1 + \dfrac{1}{x}\right) = -\infty$

42. $\lim_{x\to 0^+} \left(6 - \dfrac{1}{x^3}\right) = -\infty$

43. $\lim_{x\to -4^-} \left(x^2 + \dfrac{2}{x+4}\right) = -\infty$

44. $\lim_{x\to 0^+} \left(x - \dfrac{1}{x} + 3\right) = -\infty$

45. $\lim_{x\to 0^+} \left(\sin x + \dfrac{1}{x}\right) = \infty$

46. $\lim_{x\to (\pi/2)^+} \dfrac{-2}{\cos x} = \infty$

47. $\lim_{x\to 8^-} \dfrac{e^x}{(x-8)^3} = -\infty$

48. $\lim_{x\to 4^+} \ln(x^2-16) = -\infty$

49. $\lim_{x\to (\pi/2)^-} \ln|\cos x| = \ln\left|\cos \dfrac{\pi}{2}\right| = \ln 0 = -\infty$

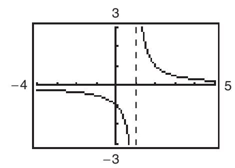
50. $\lim_{x\to 0^+} e^{-0.5x} \sin x = 1(0) = 0$

51. $\lim_{x\to (1/2)^-} x \sec \pi x = \lim_{x\to (1/2)^-} \dfrac{x}{\cos \pi x} = \infty$

52. $\lim_{x\to (1/2)^+} x^2 \tan \pi x = -\infty$

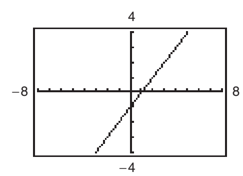
53. $f(x) = \dfrac{x^2+x+1}{x^3-1} = \dfrac{x^2+x+1}{(x-1)(x^2+x+1)}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{1}{x-1} = \infty$



54. $f(x) = \dfrac{x^3-1}{x^2+x+1} = \dfrac{(x-1)(x^2+x+1)}{x^2+x+1}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x-1) = 0$



55. $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = -2$

(a) $\lim_{x\to c} [f(x) + g(x)] = \infty - 2 = \infty$

(b) $\lim_{x\to c} [f(x)g(x)] = \infty(-2) = -\infty$

(c) $\lim_{x\to c} \dfrac{g(x)}{f(x)} = \dfrac{-2}{\infty} = 0$

56. $\lim_{x\to c} f(x) = -\infty$ and $\lim_{x\to c} g(x) = 3$

(a) $\lim_{x\to c} [f(x) + g(x)] = -\infty + 3 = -\infty$

(b) $\lim_{x\to c} [f(x)g(x)] = (-\infty)(3) = -\infty$

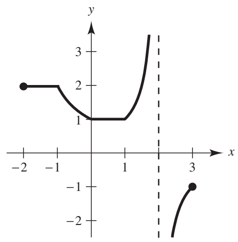
(c) $\lim_{x\to c} \dfrac{g(x)}{f(x)} = \dfrac{3}{-\infty} = 0$

57. One answer is

$f(x) = \dfrac{x-3}{(x-6)(x+2)} = \dfrac{x-3}{x^2-4x-12}.$

58. No. For example, $f(x) = \dfrac{1}{x^2+1}$ has no vertical asymptote.

59.

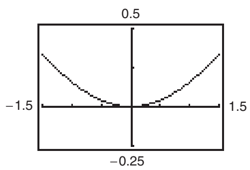


60. $m = \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}}$

$$\lim_{v \to c^-} m = \lim_{v \to c^-} \frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$$

61. (a)

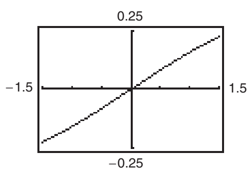
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$$\lim_{x \to 0^+} \frac{x - \sin x}{x} = 0$$

(b)

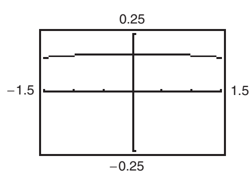
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$$\lim_{x \to 0^+} \frac{x - \sin x}{x^2} = 0$$

(c)

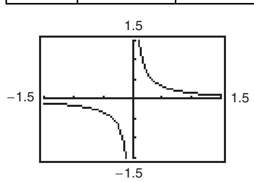
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$$\lim_{x \to 0^+} \frac{x - \sin x}{x^3} = 0.1667 = 1/6$$

(d)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$$\lim_{x \to 0^+} \frac{x - \sin x}{x^4} = \infty \text{ or } n > 3, \lim_{x \to 0^+} \frac{x - \sin x}{x^n} = \infty.$$

62. $\displaystyle\lim_{V \to 0^+} P = \infty$

As the volume of the gas decreases, the pressure increases.

63. (a) $r = \dfrac{2(7)}{\sqrt{625 - 49}} = \dfrac{7}{12}$ ft/sec

(b) $r = \dfrac{2(15)}{\sqrt{625 - 225}} = \dfrac{3}{2}$ ft/sec

(c) $\displaystyle\lim_{x \to 25^-} \frac{2x}{\sqrt{625 - x^2}} = \infty$

64. (a) $\text{Average speed} = \dfrac{\text{Total distance}}{\text{Total time}}$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

$$\frac{25x}{x - 25} = y$$

Domain: $x > 25$

(b)

x	30	40	50	60
y	150	66.667	50	42.857

(c) $\displaystyle\lim_{x\to 25^+} \frac{25x}{\sqrt{x - 25}} = \infty$

As x gets close to 25 miles per hour, y becomes larger and larger.

65. (a) $A = \dfrac{1}{2}bh - \dfrac{1}{2}r^2\theta$

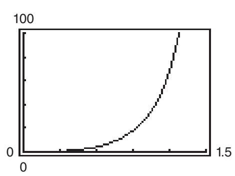
$$= \frac{1}{2}(10)(10 \tan \theta) - \frac{1}{2}(10)^2 \theta$$

$$= 50 \tan \theta - 50\,\theta$$

Domain: $\left(0, \dfrac{\pi}{2}\right)$

(b)

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1



(c) $\displaystyle\lim_{\theta\to \pi/2^-} A = \infty$

66. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.

(b) The direction of rotation is reversed.

(c) $2(20 \cot \phi) + 2(10 \cot \phi)$: straight sections. The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

So, the length of the belt around the pulleys is $20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi)$.

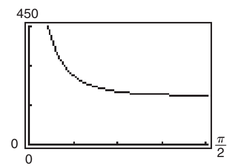
Total length $= 60 \cot \phi + 30(\pi + 2\phi)$

Domain: $\left(0, \dfrac{\pi}{2}\right)$

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5

(e)



(f) $\displaystyle\lim_{\phi\to (\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) When the angle θ approaches $\left(\dfrac{\pi}{2}\right)^-$, the total length of the belt approaches the sum of the circumferences of the two pulleys, which is $20\pi + 40\pi = 60\pi$ cm.

(h) $\displaystyle\lim_{\phi\to 0^+} L = \infty$

67. True. The function is undefined at a vertical asymptote.

68. True

69. False. The graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$ and $y = \csc x$ have vertical asymptotes.

70. False. Let

$$f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

71. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and $c = 0$.

$$\lim_{x\to 0} \frac{1}{x^2} = \infty \text{ and } \lim_{x\to 0} \frac{1}{x^4} = \infty, \text{ but } \lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{x^4}\right) = \lim_{x\to 0}\left(\frac{x^2 - 1}{x^4}\right) = -\infty \neq 0.$$

72. Given $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = L$:

(1) Difference:

Let $h(x) = -g(x)$. Then $\lim_{x\to c} h(x) = -L$, and $\lim_{x\to c}\left[f(x) - g(x)\right] = \lim_{x\to c}\left[f(x) + h(x)\right] = \infty$, by the Sum Property.

(2) Product:

If $L > 0$, then for $\varepsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $\left|g(x) - L\right| < L/2$ whenever $0 < \left|x - c\right| < \delta_1$. So, $L/2 < g(x) < 3L/2$. Because $\lim_{x\to c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $\left|x - c\right| < \delta_2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < \left|x - c\right| < \delta$, you have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x\to c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\varepsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\varepsilon$ whenever $0 < \left|x - c\right| < \delta_1$ and there exists $\delta_2 > 0$ such that $\left|g(x) - L\right| < L/2$ whenever $0 < \left|x - c\right| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < \left|x - c\right| < \delta$, you have

$$\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\varepsilon} = \varepsilon.$$

Therefore, $\displaystyle\lim_{x\to c} \frac{g(x)}{f(x)} = 0.$

73. Given $\lim_{x\to c} f(x) = \infty$, let $g(x) = 1$. Then $\displaystyle\lim_{x\to c}\frac{g(x)}{f(x)} = 0$ by Theorem 1.15.

74. Given $\displaystyle\lim_{x\to c}\frac{1}{f(x)} = 0$. Suppose $\lim_{x\to c} f(x)$ exists and equals L.

Then, $\displaystyle\lim_{x\to c}\frac{1}{f(x)} = \frac{\lim_{x\to c} 1}{\lim_{x\to c} f(x)} = \frac{1}{L} = 0.$

This is not possible. So, $\lim_{x\to c} f(x)$ does not exist.

75. $f(x) = \dfrac{1}{x - 3}$ is defined for all $x > 3$. Let $M > 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{1}{x - 3} > M$ whenever $3 < x < 3 + \delta$. Equivalently, $x - 3 < \dfrac{1}{M}$ whenever $\left|x - 3\right| < \delta$, $x > 3$. So take $\delta = \dfrac{1}{M}$. Then for $x > 3$ and $\left|x - 3\right| < \delta$, $\dfrac{1}{x - 3} > \dfrac{1}{8} = M$ and so $f(x) > M$. Thus, $\displaystyle\lim_{x\to 3^+}\frac{1}{x - 3} = \infty$.

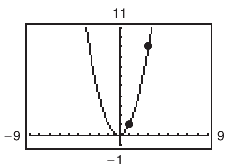
76. $f(x) = \dfrac{1}{x - 5}$ is defined for all $x < 5$. Let $N < 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{1}{x - 5} < N$ whenever $5 - \delta < x < 5$. Equivalently, $x - 5 > \dfrac{1}{N}$ whenever $\left|x - 5\right| < \delta$, $x < 5$. Equivalently, $\dfrac{1}{\left|x - 5\right|} < -\dfrac{1}{N}$ whenever $\left|x - 5\right| < \delta$, $x < 5$. So take $\delta = -\dfrac{1}{N}$. Note that $\delta > 0$ because $N < 0$. For $\left|x - 5\right| < \delta$ and $x < 5$, $\dfrac{1}{\left|x - 5\right|} > \dfrac{1}{\delta} = -N$, and $\dfrac{1}{x - 5} = -\dfrac{1}{\left|x - 5\right|} < N$. Thus, $\displaystyle\lim_{x\to 5^-}\frac{1}{x - 5} = -\infty$.

77. $f(x) = \dfrac{3}{8-x}$ is defined for all $x > 8$. Let $N < 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{3}{8-x} < N$ whenever $8 < x < 8 + \delta$. Equivalently, $\dfrac{8-x}{3} > \dfrac{1}{N}$ whenever $|x - 8| < \delta, x > 8$. Equivalently, $|8 - x| < \dfrac{-3}{N}$ whenever $|x - 8| < \delta, x > 8$. So, let $\delta = \dfrac{-3}{N}$. Note that $\delta > 0$ because $N < 0$. Finally, for $|x - 8| < \delta$ and $x > 8$, $\dfrac{1}{|x-8|} > \dfrac{1}{\delta} = \dfrac{N}{-3}, \dfrac{-3}{|x-8|} < N$, and $\dfrac{3}{8-x} < N$. Thus, $\lim\limits_{x \to 8^+} f(x) = -\infty$.

78. $f(x) = \dfrac{6}{9-x}$ is defined for all $x < 9$. Let $M > 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{6}{9-x} > M$ whenever $9 - \delta < x < 9$. Equivalently, $9 - x < \dfrac{6}{M}$ whenever $|x - 9| < \delta, x < 9$. So, let $\delta = \dfrac{6}{M}$. Finally, for $|x - 9| < \delta$ and $x < 9$, $|x - 9| < \dfrac{6}{M}, \dfrac{1}{|x-9|} > \dfrac{M}{6}$, and $\dfrac{6}{9-x} > M$. Thus, $\lim\limits_{x \to 9^-} f(x) = \delta$.

Review Exercises for Chapter 2

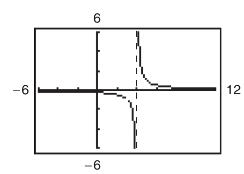
1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3. Or, the length is slightly longer than the distance between the two points, approximately 8.25.



2. Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

3. $f(x) = \dfrac{x-3}{x^2 - 7x + 12}$

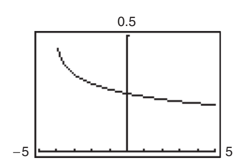
x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	–0.9091	–0.9901	–0.9990	?	–1.0010	–1.0101	–1.1111



$\lim\limits_{x \to 3} f(x) \approx -1.0000$ (Actual limit is -1.)

4. $f(x) = \dfrac{\sqrt{x+4} - 2}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.2516	0.2502	0.2500	?	0.2500	0.2498	0.2485



$\lim\limits_{x \to 0} f(x) \approx 0.2500$ $\left(\text{Actual limit is } \tfrac{1}{4}.\right)$

5. $h(x) = \left[\!\left[-\dfrac{x}{2} \right]\!\right] + x^2$

(a) The limit does not exist at $x = 2$. The function approaches 3 from the left side of 2, but it approaches 2 from the right side of 2.

(b) $\lim\limits_{x \to 1} h(x) = \left[\!\left[-\dfrac{1}{2} \right]\!\right] + x^2 = -1 + 1 = 0$

6. $f(t) = \dfrac{\ln(t + 2)}{t}$

(a) $\lim_{t \to 0} f(t)$ does not exist because $\lim_{t \to 0^-} f(t) = -\infty$ and $\lim_{t \to 0^+} f(t) = \infty$.

(b) $\lim_{t \to -1} f(t) = \dfrac{\ln 1}{-1} = 0$

7. $\lim_{x \to 1} (x + 4) = 1 + 4 = 5$

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then for $0 < |x - 1| < \delta = \varepsilon$, you have

$$\begin{aligned} |x - 1| &< \varepsilon \\ |(x + 4) - 5| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

8. $\lim_{x \to 9} \sqrt{x} = \sqrt{9} = 3$

Let $\varepsilon > 0$ be given. You need

$$\left|\sqrt{x} - 3\right| < \varepsilon \Rightarrow \left|\sqrt{x} + 3\right|\left|\sqrt{x} - 3\right| < \varepsilon\left|\sqrt{x} + 3\right| \Rightarrow |x - 9| < \varepsilon\left|\sqrt{x} + 3\right|.$$

Assuming $4 < x < 16$, you can choose $\delta = 5\varepsilon$.

So, for $0 < |x - 9| < \delta = 5\varepsilon$, you have

$$\begin{aligned} |x - 9| < 5\varepsilon &< \left|\sqrt{x} + 3\right|\varepsilon \\ \left|\sqrt{x} - 3\right| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

9. $\lim_{x \to 2} \left(1 - x^2\right) = 1 - 2^2 = -3$

Let $\varepsilon > 0$ be given. You need

$$\left|1 - x^2 - (-3)\right| < \varepsilon \Rightarrow \left|x^2 - 4\right| = |x - 2||x + 2| < \varepsilon \Rightarrow |x - 2| < \frac{1}{|x + 2|}\varepsilon$$

Assuming $1 < x < 3$, you can choose $\delta = \dfrac{\varepsilon}{5}$.

So, for $0 < |x - 2| < \delta = \dfrac{\varepsilon}{5}$, you have

$$\begin{aligned} |x - 2| < \frac{\varepsilon}{5} &< \frac{\varepsilon}{|x + 2|} \\ |x - 2||x + 2| &< \varepsilon \\ \left|x^2 - 4\right| &< \varepsilon \\ \left|4 - x^2\right| &< \varepsilon \\ \left|\left(1 - x^2\right) - (-3)\right| &< \varepsilon \\ |f(x) - L| &< \varepsilon. \end{aligned}$$

10. $\lim_{x\to 5} 9 = 9$. Let $\varepsilon > 0$ be given. δ can be any positive number. So, for $0 < |x - 5| < \delta$, you have

$$|9 - 9| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

11. $\lim_{x\to -6} x^2 = (-6)^2 = 36$

12. $\lim_{x\to 0}(5x - 3) = 5(0) - 3 = -3$

13. $\lim_{x\to 27}\left(\sqrt[3]{x} - 1\right)^4 = \left(\sqrt[3]{27} - 1\right)^4 = (3 - 1)^4 = 2^4 = 16$

14. $\lim_{x\to 2}\sqrt{x^3 + 1} = \sqrt{2^3 + 1} = \sqrt{8 + 1} = \sqrt{9} = 3$

15. $\lim_{x\to 4}\dfrac{4}{x - 1} = \dfrac{4}{4 - 1} = \dfrac{4}{3}$

16. $\lim_{x\to 2}\dfrac{x}{x^2 + 1} = \dfrac{2}{2^2 + 1} = \dfrac{2}{4 + 1} = \dfrac{2}{5}$

17.
$$\begin{aligned}\lim_{x\to -3}\frac{2x^2 + 11x + 15}{x + 3} &= \lim_{x\to -3}\frac{(2x + 5)(x + 3)}{x + 3}\\ &= \lim_{x\to -3}(2x + 5)\\ &= 2(-3) + 5\\ &= -1\end{aligned}$$

18.
$$\begin{aligned}\lim_{t\to 4}\frac{t^2 - 16}{t - 4} &= \lim_{t\to 4}\frac{(t - 4)(t + 4)}{t - 4}\\ &= \lim_{t\to 4}(t + 4) = 4 + 4 = 8\end{aligned}$$

19.
$$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{x - 3} - 1}{x - 4} &= \lim_{x\to 4}\frac{\sqrt{x - 3} - 1}{x - 4}\cdot\frac{\sqrt{x - 3} + 1}{\sqrt{x - 3} + 1}\\ &= \lim_{x\to 4}\frac{(x - 3) - 1}{(x - 4)\left(\sqrt{x - 3} + 1\right)}\\ &= \lim_{x\to 4}\frac{1}{\sqrt{x - 3} + 1} = \frac{1}{2}\end{aligned}$$

20.
$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{9 + x} - 3}{x} &= \lim_{x\to 0}\frac{\sqrt{9 + x} - 3}{x}\cdot\frac{\sqrt{9 + x} + 3}{\sqrt{9 + x} + 3}\\ &= \lim_{x\to 0}\frac{(9 + x) - 9}{x\left(\sqrt{9 + x} + 3\right)}\\ &= \lim_{x\to 0}\frac{x}{x\left(\sqrt{9 + x} + 3\right)}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{9 + x} + 3}\\ &= \frac{1}{\sqrt{9} + 3}\\ &= \frac{1}{6}\end{aligned}$$

21. $\lim_{x\to 0}\dfrac{[1/(x + 1)] - 1}{x} = \lim_{x\to 0}\dfrac{1 - (x + 1)}{x(x + 1)} = \lim_{x\to 0}\dfrac{-1}{x + 1} = -1$

22.
$$\begin{aligned}\lim_{s\to 0}\frac{\left(1/\sqrt{1 + s}\right) - 1}{s} &= \lim_{s\to 0}\left[\frac{\left(1/\sqrt{1 + s}\right) - 1}{s}\cdot\frac{\left(1/\sqrt{1 + s}\right) + 1}{\left(1/\sqrt{1 + s}\right) + 1}\right]\\ &= \lim_{s\to 0}\frac{[1/(1 + s)] - 1}{s\left[\left(1/\sqrt{1 + s}\right) + 1\right]} = \lim_{s\to 0}\frac{-1}{(1 + s)\left[\left(1/\sqrt{1 + s}\right) + 1\right]} = -\frac{1}{2}\end{aligned}$$

23. $\lim_{x\to 0}\dfrac{1 - \cos x}{\sin x} = \lim_{x\to 0}\left(\dfrac{x}{\sin x}\right)\left(\dfrac{1 - \cos x}{x}\right) = (1)(0) = 0$

24. $\lim_{x\to(\pi/4)}\dfrac{4x}{\tan x} = \dfrac{4(\pi/4)}{1} = \pi$

25. $\lim_{x\to 1} e^{x-1}\sin\dfrac{\pi x}{2} = e^0\sin\dfrac{\pi}{2} = 1$

26. $\lim_{x\to 2}\dfrac{\ln(x - 1)^2}{\ln(x - 1)} = \lim_{x\to 2}\dfrac{2\ln(x - 1)}{\ln(x - 1)} = \lim_{x\to 2} 2 = 2$

27.
$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{\sin(\pi/6)\cos\Delta x + \cos(\pi/6)\sin\Delta x - (1/2)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{1}{2}\cdot\frac{(\cos\Delta x - 1)}{\Delta x} + \lim_{\Delta x\to 0}\frac{\sqrt{3}}{2}\cdot\frac{\sin\Delta x}{\Delta x} = 0 + \frac{\sqrt{3}}{2}(1) = \frac{\sqrt{3}}{2}\end{aligned}$$

28. $$\lim_{\Delta x\to 0}\frac{\cos(\pi+\Delta x)+1}{\Delta x} = \lim_{\Delta x\to 0}\frac{\cos\pi\cos\Delta x - \sin\pi\sin\Delta x + 1}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\left[-\frac{(\cos\Delta x - 1)}{\Delta x}\right] - \lim_{\Delta x\to 0}\left[\sin\pi\frac{\sin\Delta x}{\Delta x}\right]$$
$$= -0 - (0)(1) = 0$$

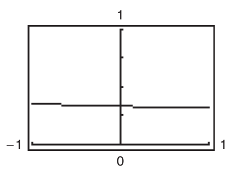
29. $$\lim_{x\to c}\left[f(x)g(x)\right] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right]$$
$$= (-6)\left(\tfrac{1}{2}\right) = -3$$

30. $$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{-6}{\left(\frac{1}{2}\right)} = -12$$

31. $$\lim_{x\to c}\left[f(x) + 2g(x)\right] = \lim_{x\to c} f(x) + 2\lim_{x\to c} g(x)$$
$$= -6 + 2\left(\tfrac{1}{2}\right) = -5$$

32. $$\lim_{x\to c}\left[f(x)\right]^2 = \left[\lim_{x\to c} f(x)\right]^2$$
$$= (-6)^2 = 36$$

33. $$f(x) = \frac{\sqrt{2x+9} - 3}{x}$$



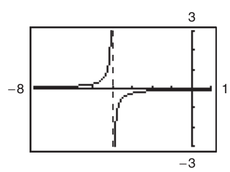
The limit appears to be $\frac{1}{3}$.

x	–0.01	–0.001	0	0.001	0.01
$f(x)$	0.3335	0.3333	?	0.3333	0.331

$$\lim_{x\to 0} f(x) \approx 0.3333$$

$$\lim_{x\to 0}\frac{\sqrt{2x+9} - 3}{x}\cdot\frac{\sqrt{2x+9} + 3}{\sqrt{2x+9} + 3} = \lim_{x\to 0}\frac{(2x+9) - 9}{x\left[\sqrt{2x+9} + 3\right]} = \lim_{x\to 0}\frac{2}{\sqrt{2x+9} + 3} = \frac{2}{\sqrt{9} + 3} = \frac{1}{3}$$

34. $$f(x) = \frac{\left[1/(x+4)\right] - (1/4)}{x}$$



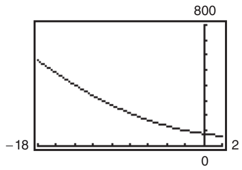
The limit appears to be $-\frac{1}{16}$

x	–0.01	–0.001	0	0.001	0.01
$f(x)$	–0.0627	–0.0625	?	–0.0625	–0.0623

$$\lim_{x\to 0} f(x) \approx -0.0625 = -\frac{1}{16}$$

$$\lim_{x\to 0}\frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \lim_{x\to 0}\frac{4 - (x+4)}{(x+4)4(x)} = \lim_{x\to 0}\frac{-1}{(x+4)4} = -\frac{1}{16}$$

35. $f(x) = \dfrac{x^3 + 729}{x + 9}$



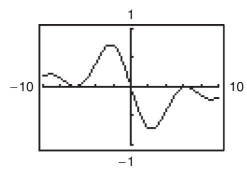
The limit appears to be 243.

x	−9.1	−9.01	−9.001	−9	−8.999	−8.99	−8.9
$f(x)$	245.7100	243.2701	243.0270	?	242.9730	242.7301	240.310

$$\lim_{x\to-9} \frac{x^3 + 729}{x + 9} \approx 243.00$$

$$\lim_{x\to-9} \frac{x^3 + 729}{x + 9} = \lim_{x\to-9} \frac{(x + 9)(x^2 - 9x + 81)}{x + 9} = \lim_{x\to-9} (x^2 - 9x + 81) = 81 + 81 + 81 = 243$$

36. $f(x) = \dfrac{\cos x - 1}{x}$



The limit appears to be 0.

x	−0.01	−0.001	0	0.001	0.01
$f(x)$	0.005	0.0005	0	−0.0005	−0.005

$$\lim_{x\to0} f(x) \approx 0.000$$

$$\begin{aligned}
\lim_{x\to0} \frac{\cos x - 1}{x} &= \lim_{x\to0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} \\
&= \lim_{x\to0} \frac{\cos^2 x - 1}{x(\cos x + 1)} \\
&= \lim_{x\to0} \frac{-\sin^2 x}{x(\cos x + 1)} \\
&= \lim_{x\to0} \left(\frac{\sin x}{x}\right)\left(\frac{-\sin x}{\cos x + 1}\right) \\
&= (1)\left(\frac{0}{2}\right) \\
&= 0
\end{aligned}$$

37.
$$\begin{aligned}
v &= \lim_{t\to4} \frac{s(4) - s(t)}{4 - t} \\
&= \lim_{t\to4} \frac{[-4.9(16) + 250] - [-4.9t^2 + 250]}{4 - t} \\
&= \lim_{t\to4} \frac{4.9(t^2 - 16)}{4 - t} \\
&= \lim_{t\to4} \frac{4.9(t - 4)(t + 4)}{4 - t} \\
&= \lim_{t\to4} [-4.9(t + 4)] = -39.2 \text{ m/sec}
\end{aligned}$$

The object is falling at about 39.2 m/sec.

38. $-4.9t^2 + 250 = 0 \Rightarrow t = \dfrac{50}{7} \approx 7.143$

The object will hit the ground after about 7.1 seconds.

When $a = \dfrac{50}{7}$, the velocity is

$$\begin{aligned}
\lim_{t\to a} \frac{s(a) - s(t)}{a - t} &= \lim_{t\to a} \frac{[-4.9a^2 + 250] - [-4.9t^2 + 250]}{a - t} \\
&= \lim_{t\to a} \frac{4.9(t^2 - a^2)}{a - t} \\
&= \lim_{t\to a} \frac{4.9(t - a)(t + a)}{a - t} \\
&= \lim_{t\to a} [-4.9(t + a)] \\
&= -4.9(2a) \qquad \left(a = \frac{50}{7}\right) \\
&= -70 \text{ m/sec.}
\end{aligned}$$

39. $\displaystyle \lim_{x\to3^+} \frac{1}{x + 3} = \frac{1}{3 + 3} = \frac{1}{6}$

40.
$$\begin{aligned}
\lim_{x\to6^-} \frac{x - 6}{x^2 - 36} &= \lim_{x\to6^-} \frac{x - 6}{(x - 6)(x + 6)} \\
&= \lim_{x\to6^-} \frac{1}{x + 6} = \frac{1}{12}
\end{aligned}$$

41.
$$\begin{aligned}
\lim_{x\to25^+} \frac{\sqrt{x} - 5}{x - 25} &= \lim_{x\to25^+} \frac{\sqrt{x} - 5}{(\sqrt{x} + 5)(\sqrt{x} - 5)} \\
&= \lim_{x\to25^+} \frac{1}{\sqrt{x} + 5} \\
&= \frac{1}{\sqrt{25} + 5} = \frac{1}{5 + 5} = \frac{1}{10}
\end{aligned}$$

42. $\displaystyle \lim_{x\to3^-} \frac{|x - 3|}{x - 3} = \lim_{x\to3^-} \frac{-(x - 3)}{x - 3} = -1$

43. $\lim_{x\to 2} f(x) = 0$

44. $\lim_{x\to 1^+} g(x) = 1 + 1 = 2$

45. $\lim_{t\to 1} h(t)$ does not exist because $\lim_{t\to 1^-} h(t) = 1 + 1 = 2$ and $\lim_{t\to 1^+} h(t) = \frac{1}{2}(1 + 1) = 1$.

46. $\lim_{s\to -2} f(s) = 2$

47. $\lim_{x\to 2^-} \left(2[\![x]\!] + 1\right) = 2(1) + 1 = 3$

48. $\lim_{x\to 4} [\![x - 1]\!]$ does not exist. There is a break in the graph at $x = 4$.

49. $\lim_{x\to 2^-} \frac{x^2 - 4}{|x - 2|} = \lim_{x\to 2^-} \frac{(x - 2)(x + 2)}{2 - x}$

$= \lim_{x\to 2^-} -(x + 2) = -(2 + 2) = -4$

50. $\lim_{x\to 1^+} \sqrt{x(x - 1)} = \sqrt{1(1 - 1)} = 0$

51. The function $g(x) = \sqrt{8 - x^3}$ is continuous on $[-2, 2]$ because $8 - x^3 \ge 0$ on $[-2, 2]$.

52. The function $h(x) = \frac{3}{5 - x}$ is not continuous on $[0, 5]$ because $h(5)$ is not defined.

53. $f(x) = x^4 - 81x$ is continuous for all real x.

54. $f(x) = x^2 - x + 20$ is continuous for all real x.

55. $f(x) = \frac{4}{x - 5}$ has a nonremovable discontinuity at $x = 5$ because $\lim_{x\to 5} f(x)$ does not exist.

56. $f(x) = \frac{1}{x^2 - 9} = \frac{1}{(x - 3)(x + 3)}$

has nonremovable discontinuities at $x = \pm 3$ because $\lim_{x\to 3} f(x)$ and $\lim_{x\to -3} f(x)$ do not exist.

57. $f(x) = \frac{x}{x^3 - x} = \frac{x}{x(x^2 - 1)} = \frac{1}{(x - 1)(x + 1)},\ x \ne 0$

has nonremovable discontinuities at $x = \pm 1$ because $\lim_{x\to -1} f(x)$ and $\lim_{x\to 1} f(x)$ do not exist, and has a removable discontinuity at $x = 0$ because

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{(x - 1)(x + 1)} = -1.$

58. $f(x) = \frac{x + 3}{x^2 - 3x - 18} = \frac{x + 3}{(x + 3)(x - 6)} = \frac{1}{x - 6},$
$x \ne -3$

has a nonremovable discontinuity at $x = 6$ because $\lim_{x\to 6} f(x)$ does not exist, and has a removable discontinuity at $x = -3$ because

$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{1}{x - 6} = -\frac{1}{9}.$

59. $f(2) = 5$

Find c so that $\lim_{x\to 2^+} (cx + 6) = 5$.

$c(2) + 6 = 5$
$2c = -1$
$c = -\frac{1}{2}$

60. $f(-1) = e^{-1+1} + 3 = 1 + 3 = 4$

So, find c such that $\lim_{x\to -1^-} \left(cx^2 - 2x\right) = 4$.

$c(-1)^2 - 2(-1) = 4$
$c + 2 = 4$
$c = 2$

61. $f(x) = -3x^2 + 7$

Continuous on $(-\infty, \infty)$

62. $f(x) = \frac{4x^2 + 7x - 2}{x + 2} = \frac{(4x - 1)(x + 2)}{x + 2}$

Continuous on $(-\infty, -2) \cup (-2, \infty)$. There is a removable discontinuity at $x = -2$.

63. $f(x) = \sqrt{x} + \cos x$ is continuous on $[0, \infty)$.

64. $f(x) = [\![x + 3]\!]$

$\lim_{x\to k^+} [\![x + 3]\!] = k + 3$ where k is an integer.

$\lim_{x\to k^-} [\![x + 3]\!] = k + 2$ where k is an integer.

Nonremovable discontinuity at each integer k

Continuous on $(k, k + 1)$ for all integers k

65. $g(x) = 2e^{[\![x]\!]/4}$ is continuous on all intervals $(n, n + 1)$, where n is an integer. g has nonremovable discontinuities at each n.

66. $h(x) = -2 \ln |5 - x|$

Because $|5 - x| > 0$ except for $x = 5$, h is continuous on $(-\infty, 5) \cup (5, \infty)$.

67. $f(x) = \dfrac{3x^2 - x - 2}{x - 1} = \dfrac{(3x + 2)(x - 1)}{x - 1}$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} (3x + 2) = 5$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

68. $f(x) = \begin{cases} 5 - x, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$

$\lim_{x\to 2^-} (5 - x) = 3$

$\lim_{x\to 2^+} (2x - 3) = 1$

Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

69. $f(x) = 2x^3 - 3$

f is continuous on $[1, 2]$. $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value c in $(1, 2)$ such that $2c^3 - 3 = 0$.

70. $f(x) = x^2 + x - 2$

Consider the intervals $[-3, 0]$ and $[0, 3]$.

$f(-3) = (-3)^2 - 3 - 2 = 4 > 0$

$f(0) = -2 < 0$

By the Intermediate Value Theorem, there is at least one zero in $[-3, 0]$.

$f(0) = -2 < 0$

$f(3) = (3)^2 + 3 - 2 = 10 > 0$

Again, there is at least one zero in $[0, 3]$.

So, there are at least two zeros in $[-3, 3]$.

71. $f(x) = x^2 + 5x - 4$

f is continuous on $[-1, 2]$.

$f(-1) = (-1)^2 + 5(-1) - 4 = -8 < 2$

$f(2) = 2^2 + 5(2) - 4 = 10 > 2$

The Intermediate Value Theorem applies.

$$\begin{aligned} x^2 + 5x - 4 &= 2 \\ x^2 + 5x - 6 &= 0 \\ (x + 6)(x - 1) &= 0 \\ x &= 1 \ (x = -6 \text{ lies outside the interval.}) \\ c &= 1 \end{aligned}$$

So, $f(1) = 2$.

72. $f(x) = (x - 6)^3 + 4$

f is continuous on $[4, 7]$.

$f(4) = (4 - 6)^3 + 4 = -8 + 4 = -4 < 3$

$f(7) = (7 - 6)^3 + 4 = 1 + 4 = 5 > 3$

The Intermediate Value Theorem applies.

$$\begin{aligned} (x - 6)^3 + 4 &= 3 \\ (x - 6)^3 &= -1 \\ x - 6 &= -1 \\ x &= 5 \\ c &= 5 \end{aligned}$$

So, $f(5) = 3$.

73. $\lim_{x\to 6^-} \dfrac{1}{x - 6} = -\infty$

$\lim_{x\to 6^+} \dfrac{1}{x - 6} = \infty$

74. $\lim_{x\to 6^-} \dfrac{-1}{(x - 6)^2} = -\infty$

$\lim_{x\to 6^+} \dfrac{-1}{(x - 6)^2} = -\infty$

75. $f(x) = \dfrac{3}{x}$

$\lim_{x\to 0^-} \dfrac{3}{x} = -\infty$

$\lim_{x\to 0^+} \dfrac{3}{x} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

76. $f(x) = \dfrac{5}{(x - 2)^4}$

$\lim_{x\to 2^-} \dfrac{5}{(x - 2)^4} = \infty = \lim_{x\to 2^+} \dfrac{5}{(x - 2)^4}$

Therefore, $x = 2$ is a vertical asymptote.

77. $f(x) = \dfrac{x^3}{x^2 - 9} = \dfrac{x^3}{(x + 3)(x - 3)}$

$\lim_{x\to -3^-} \dfrac{x^3}{x^2 - 9} = -\infty$ and $\lim_{x\to -3^+} \dfrac{x^3}{x^2 - 9} = \infty$

Therefore, $x = -3$ is a vertical asymptote.

$\lim_{x\to 3^-} \dfrac{x^3}{x^2 - 9} = -\infty$ and $\lim_{x\to 3^+} \dfrac{x^3}{x^2 - 9} = \infty$

Therefore, $x = 3$ is a vertical asymptote.

78. $f(x) = \dfrac{6x}{36 - x^2} = -\dfrac{6x}{(x+6)(x-6)}$

$\lim\limits_{x\to -6^-} \dfrac{6x}{36 - x^2} = \infty$ and $\lim\limits_{x\to -6^+} \dfrac{6x}{36 - x^2} = -\infty$

Therefore, $x = -6$ is a vertical asymptote.

$\lim\limits_{x\to 6^-} \dfrac{6x}{36 - x^2} = \infty$ and $\lim\limits_{x\to 6^+} \dfrac{6x}{36 - x^2} = -\infty$

Therefore, $x = 6$ is a vertical asymptote.

79. $f(x) = \sec \dfrac{\pi x}{2} = \dfrac{1}{\cos \dfrac{\pi x}{2}}$

$\cos \dfrac{\pi x}{2} = 0$ when $x = \pm 1, \pm 3, \ldots$

Therefore, the graph has vertical asymptotes at $x = 2n + 1$, where n is an integer.

80. $f(x) = \csc \pi x = \dfrac{1}{\sin \pi x}$

$\sin \pi x = 0$ for $x = n$, where n is an integer.

$\lim\limits_{x\to n} f(x) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at $x = n$.

81. $g(x) = \ln(25 - x^2) = \ln[(5 + x)(5 - x)]$

$\lim\limits_{x\to 5} \ln(25 - x^2) = 0$

$\lim\limits_{x\to -5} \ln(25 - x^2) = 0$

Therefore, the graph has holes at $x = \pm 5$. The graph does not have any vertical asymptotes.

82. $f(x) = 7e^{-3/x}$

$\lim\limits_{x\to 0^-} 7e^{-3/x} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

83. $\lim\limits_{x\to 1^-} \dfrac{x^2 + 2x + 1}{x - 1} = -\infty$

84. $\lim\limits_{x\to (1/2)^+} \dfrac{x}{2x - 1} = \infty$

85. $\lim\limits_{x\to -1^+} \dfrac{x + 1}{x^3 + 1} = \lim\limits_{x\to -1^+} \dfrac{1}{x^2 - x + 1} = \dfrac{1}{3}$

86. $\lim\limits_{x\to -1^-} \dfrac{x + 1}{x^4 - 1} = \lim\limits_{x\to -1^-} \dfrac{1}{(x^2 + 1)(x - 1)} = -\dfrac{1}{4}$

87. $\lim\limits_{x\to 0^+} \left(x - \dfrac{1}{x^3}\right) = -\infty$

88. $\lim\limits_{x\to 2^-} \dfrac{1}{\sqrt[3]{x^2 - 4}} = -\infty$

89. $\lim\limits_{x\to 0^+} \dfrac{\sin 4x}{5x} = \lim\limits_{x\to 0^+} \left[\dfrac{4}{5}\left(\dfrac{\sin 4x}{4x}\right)\right] = \dfrac{4}{5}$

90. $\lim\limits_{x\to 0^-} \dfrac{\sec x^3}{2x} = -\infty$

(**Note:** $\sec x^3 \approx 1$ for x near 0.)

91. $\lim\limits_{x\to 0^+} \dfrac{\csc 2x}{x} = \lim\limits_{x\to 0^+} \dfrac{1}{x \sin 2x} = \infty$

92. $\lim\limits_{x\to 0^-} \dfrac{\cos^2 x}{x} = -\infty$

93. $\lim\limits_{x\to 0^+} \ln(\sin x) = -\infty$

94. $\lim\limits_{x\to 0^-} 12e^{-2/x} = \infty$

95. $C = \dfrac{80{,}000p}{100 - p}, \; 0 \le p < 0$

(a) $C(50) = \dfrac{80{,}000(50)}{100 - 50} = \$80{,}000$

(b) $C(90) = \dfrac{80{,}000(90)}{100 - 90} = \$720{,}000$

(c) $\lim\limits_{p\to 100^-} C(p) = \infty$

It would be financially impossible to remove 100% of the pollutants.

Problem Solving for Chapter 2

1. (a) Perimeter $\Delta PAO = \sqrt{x^2 + (y - 1)^2} + \sqrt{x^2 + y^2} + 1$

$= \sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1$

Perimeter $\Delta PBO = \sqrt{(x - 1)^2 + y^2} + \sqrt{x^2 + y^2} + 1$

$= \sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$

(b) $r(x) = \dfrac{\sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$

x	4	2	1	0.1	0.01
Perimeter ΔPAO	33.02	9.08	3.41	2.10	2.01
Perimeter ΔPBO	33.77	9.60	3.41	2.00	2.00
$r(x)$	0.98	0.95	1	1.05	1.005

(c) $\displaystyle\lim_{x\to 0^+} r(x) = \frac{1 + 0 + 1}{1 + 0 + 1} = \frac{2}{2} = 1$

2. (a) Area $\Delta PAO = \dfrac{1}{2}bh = \dfrac{1}{2}(1)(x) = \dfrac{x}{2}$

Area $\Delta PBO = \dfrac{1}{2}bh = \dfrac{1}{2}(1)(y) = \dfrac{y}{2} = \dfrac{x^2}{2}$

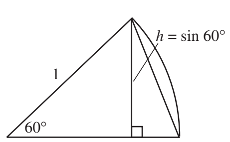
(b) $a(x) = \dfrac{\text{Area } \Delta PBO}{\text{Area } \Delta PAO} = \dfrac{x^2/2}{x/2} = x$

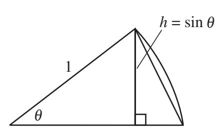
x	4	2	1	0.1	0.01
Area ΔPAO	2	1	1/2	1/20	1/200
Area ΔPBO	8	2	1/2	1/200	1/20,000
$a(x)$	4	2	1	1/10	1/100

(c) $\displaystyle\lim_{x\to 0^+} a(x) = \lim_{x\to 0^+} x = 0$

3. (a) There are 6 triangles, each with a central angle of $60° = \pi/3$. So,

Area hexagon $= 6\left[\dfrac{1}{2}bh\right] = 6\left[\dfrac{1}{2}(1)\sin\dfrac{\pi}{3}\right] = \dfrac{3\sqrt{3}}{2} \approx 2.598.$





Error $=$ Area (Circle) $-$ Area (Hexagon) $= \pi - \dfrac{3\sqrt{3}}{2} \approx 0.5435$

(b) There are n triangles, each with central angle of $\theta = 2\pi/n$. So,

$A_n = n\left[\dfrac{1}{2}bh\right] = n\left[\dfrac{1}{2}(1)\sin\dfrac{2\pi}{n}\right] = \dfrac{n\sin(2\pi/n)}{2}.$

(c)

n	6	12	24	48	96
A_n	2.598	3	3.106	3.133	3.139

As n gets larger and larger, $2\pi/n$ approaches 0. Letting $x = 2\pi/n$, $A_n = \dfrac{\sin(2\pi/n)}{2/n} = \dfrac{\sin(2\pi/n)}{(2\pi/n)}\pi = \dfrac{\sin x}{x}\pi$

which approaches $(1)\pi = \pi$, which is the area of the circle.

4. (a) Slope $= \dfrac{4-0}{3-0} = \dfrac{4}{3}$

(b) Slope $= -\dfrac{3}{4}$

Tangent line: $y - 4 = -\dfrac{3}{4}(x-3)$

$y = -\dfrac{3}{4}x + \dfrac{25}{4}$

(c) Let $Q = (x, y) = \left(x, \sqrt{25 - x^2}\right)$

$$m_x = \frac{\sqrt{25-x^2} - 4}{x - 3}$$

(d)
$$\lim_{x\to 3} m_x = \lim_{x\to 3} \frac{\sqrt{25-x^2}-4}{x-3}\cdot\frac{\sqrt{25-x^2}+4}{\sqrt{25-x^2}+4}$$
$$= \lim_{x\to 3}\frac{25-x^2-16}{(x-3)\left(\sqrt{25-x^2}+4\right)}$$
$$= \lim_{x\to 3}\frac{(3-x)(3+x)}{(x-3)\left(\sqrt{25-x^2}+4\right)}$$
$$= \lim_{x\to 3}\frac{-(3+x)}{\sqrt{25-x^2}+4} = \frac{-6}{4+4} = -\frac{3}{4}$$

This is the slope of the tangent line at P.

5. (a) Slope $= -\dfrac{12}{5}$

(b) Slope of tangent line is $\dfrac{5}{12}$.

$y + 12 = \dfrac{5}{12}(x-5)$

$y = \dfrac{5}{12}x - \dfrac{169}{12}$ Tangent line

(c) $Q = (x, y) = \left(x, -\sqrt{169-x^2}\right)$

$$m_x = \frac{-\sqrt{169-x^2}+12}{x-5}$$

(d)
$$\lim_{x\to 5} m_x = \lim_{x\to 5}\frac{12-\sqrt{169-x^2}}{x-5}\cdot\frac{12+\sqrt{169-x^2}}{12+\sqrt{169-x^2}}$$
$$= \lim_{x\to 5}\frac{144-\left(169-x^2\right)}{(x-5)\left(12+\sqrt{169-x^2}\right)}$$
$$= \lim_{x\to 5}\frac{x^2-25}{(x-5)\left(12+\sqrt{169-x^2}\right)}$$
$$= \lim_{x\to 5}\frac{(x+5)}{12+\sqrt{169-x^2}} = \frac{10}{12+12} = \frac{5}{12}$$

This is the same slope as part (b).

6.
$$\frac{\sqrt{a+bx}-\sqrt{3}}{x} = \frac{\sqrt{a+bx}-\sqrt{3}}{x}\cdot\frac{\sqrt{a+bx}+\sqrt{3}}{\sqrt{a+bx}+\sqrt{3}} = \frac{(a+bx)-3}{x\left(\sqrt{a+bx}+\sqrt{3}\right)}$$

Letting $a = 3$ simplifies the numerator.

So, $\displaystyle\lim_{x\to 0}\frac{\sqrt{3+bx}-\sqrt{3}}{x} = \lim_{x\to 0}\frac{bx}{x\left(\sqrt{3+bx}+\sqrt{3}\right)} = \lim_{x\to 0}\frac{b}{\sqrt{3+bx}+\sqrt{3}}$

Setting $\dfrac{b}{\sqrt{3}+\sqrt{3}} = \sqrt{3}$, you obtain $b = 6$. So, $a = 3$ and $b = 6$.

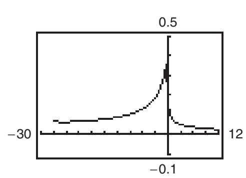
7. (a) $3 + x^{1/3} \ge 0$

$x^{1/3} \ge -3$

$x \ge -27$

Domain: $x \ge -27, x \ne 1$ or $[-27, 1) \cup (1, \infty)$

(b)



(c)
$$\lim_{x\to -27^+} f(x) = \frac{\sqrt{3+(-27)^{1/3}}-2}{-27-1} = \frac{-2}{-28} = \frac{1}{14} \approx 0.0714$$

(d) $$\lim_{x\to1} f(x) = \lim_{x\to1} \frac{\sqrt{3+x^{1/3}}-2}{x-1}\cdot\frac{\sqrt{3+x^{1/3}}+2}{\sqrt{3+x^{1/3}}+2} = \lim_{x\to1}\frac{3+x^{1/3}-4}{(x-1)\left(\sqrt{3+x^{1/3}}+2\right)}$$

$$= \lim_{x\to1}\frac{x^{1/3}-1}{\left(x^{1/3}-1\right)\left(x^{2/3}+x^{1/3}+1\right)\left(\sqrt{3+x^{1/3}}+2\right)} = \lim_{x\to1}\frac{1}{\left(x^{2/3}+x^{1/3}+1\right)\left(\sqrt{3+x^{1/3}}+2\right)}$$

$$= \frac{1}{(1+1+1)(2+2)} = \frac{1}{12}$$

8. $\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\left(a^2-2\right) = a^2-2$

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+}\frac{ax}{\tan x} = a\left(\text{because } \lim_{x\to0}\frac{\tan x}{x} = 1\right)$

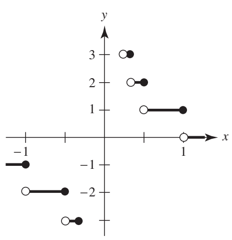
Thus,
$$\begin{aligned} a^2 - 2 &= a \\ a^2 - a - 2 &= 0 \\ (a-2)(a+1) &= 0 \\ a &= -1, 2 \end{aligned}$$

9. (a) $\lim_{x\to2} f(x) = 3$: g_1, g_4

(b) f continuous at 2: g_1

(c) $\lim_{x\to2^-} f(x) = 3$: g_1, g_3, g_4

10.



(a) $f\left(\frac{1}{4}\right) = [\![4]\!] = 4$

$f(3) = \left[\!\left[\frac{1}{3}\right]\!\right] = 0$

$f(1) = [\![1]\!] = 1$

(b) $\lim_{x\to1^-} f(x) = 1$

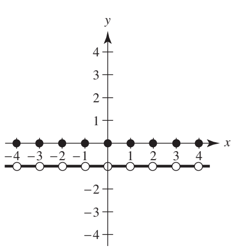
$\lim_{x\to1^+} f(x) = 0$

$\lim_{x\to0^-} f(x) = -\infty$

$\lim_{x\to0^+} f(x) = \infty$

(c) f is continuous for all real numbers except $x = 0, \pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

11.



(a) $f(1) = [\![1]\!] + [\![-1]\!] = 1 + (-1) = 0$

$f(0) = 0$

$f\left(\frac{1}{2}\right) = 0 + (-1) = -1$

$f(-2.7) = -3 + 2 = -1$

(b) $\lim_{x\to1^-} f(x) = -1$

$\lim_{x\to1^+} f(x) = -1$

$\lim_{x\to1/2} f(x) = -1$

(c) f is continuous for all real numbers except $x = 0, \pm1, \pm2, \pm3, \ldots$

12. (a)
$$\begin{aligned} v^2 &= \frac{192{,}000}{r} + v_0{}^2 - 48 \\ \frac{192{,}000}{r} &= v^2 - v_0{}^2 + 48 \\ r &= \frac{192{,}000}{v^2 - v_0{}^2 + 48} \\ \lim_{v\to0} r &= \frac{192{,}000}{48 - v_0{}^2} \end{aligned}$$

Let $v_0 = \sqrt{48} = 4\sqrt{3}$ mi/sec.

(b)
$$\begin{aligned} v^2 &= \frac{1920}{r} + v_0{}^2 - 2.17 \\ \frac{1920}{r} &= v^2 - v_0{}^2 + 2.17 \\ r &= \frac{1920}{v^2 - v_0{}^2 + 2.17} \\ \lim_{v\to0} r &= \frac{1920}{2.17 - v_0{}^2} \end{aligned}$$

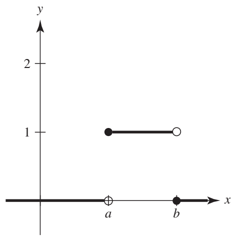
Let $v_0 = \sqrt{2.17}$ mi/sec $\quad$ ($\approx$ 1.47 mi/sec).

(c)
$$\begin{aligned} r &= \frac{10{,}600}{v^2 - v_0{}^2 + 6.99} \\ \lim_{v\to0} r &= \frac{10{,}600}{6.99 - v_0{}^2} \end{aligned}$$

Let $v_0 = \sqrt{6.99} \approx 2.64$ mi/sec.

Because this is smaller than the escape velocity for Earth, the mass is less.

13. (a)



(b) (i) $\lim_{x \to a^+} P_{a,b}(x) = 1$

(ii) $\lim_{x \to a^-} P_{a,b}(x) = 0$

(iii) $\lim_{x \to b^+} P_{a,b}(x) = 0$

(iv) $\lim_{x \to b^-} P_{a,b}(x) = 1$

(c) $P_{a,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of U, and above the x-axis, is 1.

14. Let $a \neq 0$ and let $\varepsilon > 0$ be given. There exists $\delta_1 > 0$ such that if $0 < |x - 0| < \delta_1$ then $|f(x) - L| < \varepsilon$. Let $\delta = \delta_1 / |a|$. Then for $0 < |x - 0| < \delta = \delta_1 / |a|$, you have

$$|x| < \frac{\delta_1}{|a|}$$

$$|ax| < \delta_1$$

$$|f(ax) - L| < \varepsilon.$$

As a counterexample, let $a = 0$ and

$$f(x) = \begin{cases} 1, & x \neq 0 \\ 2, & x = 0 \end{cases}.$$

Then $\lim_{x \to 0} f(x) = 1 = L$, but

$$\lim_{x \to 0} f(ax) = \lim_{x \to 0} f(0) = \lim_{x \to 0} 2 = 2.$$

Chapter 2 Limits and Their Properties

Chapter Comments

Section 2.1 gives a preview of calculus. On pages 67 and 68 of the textbook are examples of some of the concepts from precalculus extended to ideas that require the use of calculus. Review these ideas with your students to give them a feel for where the course is heading.

The idea of a limit is central to calculus. So, you should take the time to discuss the tangent line problem and/or the area problem in this section. Exercise 11 of Section 2.1 is yet another example of how limits will be used in calculus. A review of the formula for the distance between two points can be found in Appendix C.

The discussion of limits is difficult for most students the first time that they see it. For this reason, you should carefully go over the examples and the informal definition of a limit presented in Section 2.2. Stress to your students that a limit exists only if $f(x)$ moves arbitrarily close to a single real number as x approaches c from either side (the function does not have to exist at $x = c$). Otherwise, the limit fails to exist, as shown in Examples 3, 4, and 5 of this section. You might want to work Exercise 71 of Section 2.2 with your students in preparation for the definition of the number e coming up in Section 2.3. You may choose to omit the formal definition of a limit.

Carefully go over the properties of limits found in Section 2.3 to ensure that your students are comfortable with the idea of a limit and also with the notation used for limits. Suggest that students write these properties of limits on a sheet of paper for reference. By the time you get to Theorem 2.6, it should be obvious to your students that all of these properties amount to direct substitution.

When direct substitution for the limit of a quotient yields the indeterminate form $\frac{0}{0}$, remind your students that they must rewrite the fraction using legitimate algebra. Then do at least one problem using dividing out techniques and another using rationalizing techniques. Exercises 58 and 60 of Section 2.3 are examples of other algebraic techniques needed for the limit problems. It is important to go over the Squeeze Theorem, Theorem 2.8, with your students so that you can use it to prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$. The proof of Theorem 2.8 and many other theorems can be found in Appendix A. Your students need to memorize all three of the special limits in Theorem 2.9 as they will need these facts to do problems throughout the textbook. Most of your students will need help with Exercises 63–85 in this section.

Continuity, which is discussed in Section 2.4, is another idea that often puzzles students. However, if you describe a continuous function as one in which you can draw the entire graph without lifting your pencil, the idea seems to stay with them. Distinguishing between removable and nonremovable discontinuities will help students determine vertical asymptotes.

To discuss infinite limits, Section 2.5, remind your students of the graph of the function $f(x) = 1/x$ studied in Section 1.3. Be sure to make your students write a vertical asymptote as an equation, not just a number. For example, for the function $y = 1/x$, the vertical asymptote is $x = 0$.

Section 2.1 A Preview of Calculus

Section Comments

2.1 **A Preview of Calculus**—Understand what calculus is and how it compares with precalculus. Understand that the tangent line problem is basic to calculus. Understand that the area problem is also basic to calculus.

Teaching Tips

In this first section, students see how limits arise when attempting to find a tangent line to a curve. Be sure to say to students that they can think about the word *tangent* as meaning "touching" a curve at one particular point. Drawing a circle with a line that touches the circle at a specific point can illustrate tangency. Drawing a curve where a line intersects a curve twice illustrates a line that is not tangent to a curve as it crosses the curve more than once.

How Do You See It? Exercise

Page 71, Exercise 10 How would you describe the instantaneous rate of change of an automobile's position on a highway?

Solution

Answers will vary. *Sample answer:* The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

Suggested Homework Assignment

Page 71: 1–9 odd.

Section 2.2 Finding Limits Graphically and Numerically

Section Comments

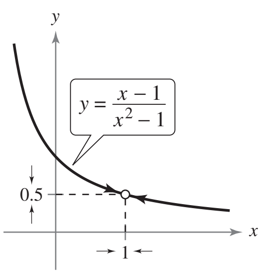
2.2 **Finding Limits Graphically and Numerically**—Estimate a limit using a numerical or graphical approach. Learn different ways that a limit can fail to exist. Study and use a formal definition of limit.

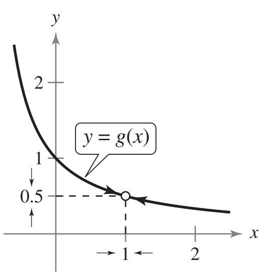
Teaching Tips

In this section, we turn our focus to limits in general, and numerical and graphical ways we can find them. Ask students to consider $f(x) = \dfrac{x^2 - 4}{x - 2}$ and what is happening to $f(x)$ as x approaches 2. This will lead into the definition of a limit.

Make sure that students find limits analytically instead of using their graping utilities, as they can be misleading. For example, present to the class, $\lim_{x \to 0} \sin \dfrac{\pi}{x}$. Constructing a table of values of $x = 1$, $\dfrac{1}{3}$, 0.1, $\dfrac{1}{2}$, $\dfrac{1}{4}$, and 0.01 leads to the assumption that $\lim_{x \to 0} \sin \dfrac{\pi}{x} = 0$. However, this limit does not exist.

When introducing the definition of a limit, using graphs will help students understand the meaning of the definition. Suggested graphs are below:

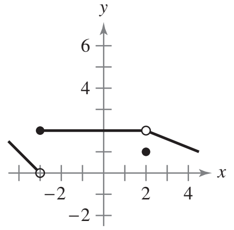




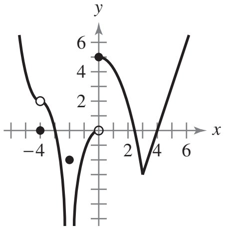
How Do You See It? Exercise

Page 82, Exercise 74 Use the graph of f to identify the values of c for which $\lim_{x\to c} f(x)$ exists.

(a)



(b)



Solution

(a) $\lim_{x\to c} f(x)$ exists for all $c \neq -3$.

(b) $\lim_{x\to c} f(x)$ exists for all $c \neq -2, 0$.

Suggested Homework Assignment

Pages 79–82: 1, 5, 7, 9, 15, and 19–35 odd.

Section 2.3 Evaluating Limits Analytically

Section Comments

2.3 **Evaluating Limits Analytically**—Evaluate a limit using properties of limits. Develop and use a strategy for finding limits. Evaluate a limit using the dividing out technique. Evaluate a limit using the rationalizing technique. Evaluate a limit using the Squeeze Theorem.

Teaching Tips

When starting this section, review how to factor using differences of two squares and two cubes, sum of two cubes, and rationalizing numerators. If you do not have time to review these concepts, encourage students to study the following material in *Precalculus*, 10th edition, by Larson.

- Factoring the difference of two squares: *Precalculus* Appendix A.3
- Factoring the difference of two cubes: *Precalculus* Appendix A.3
- Factoring the sum of two cubes: *Precalculus* Appendix A.3
- Rationalizing numerators: *Precalculus* Appendix A.2

State all limit properties as presented in this section and discuss the proper ways to write solutions when finding limits analytically. Give examples of direct substitution; some suggested examples include $\lim_{x\to 2} \dfrac{x^2 - 3x + 4}{x + 2}$ and $\lim_{x\to -3} 4x^3 - 5x + 1$.

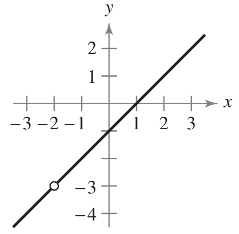
When rationalizing the numerator, start with an example without limits. A suggestion is $\dfrac{\sqrt{x+2} - \sqrt{2}}{x}$. After rationalizing the numerator, find the limit as x approaches 0.

When evaluating trigonometric limits, show students examples directly related to Theorem 2.9 in the text. Some examples are $\lim_{x\to 0} \dfrac{\sin 7x}{4x}$ and $\lim_{\theta\to 0} \dfrac{\cos\theta - 1}{\sin\theta}$.

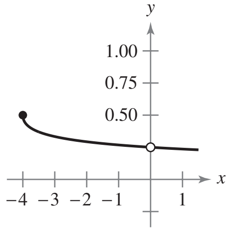
How Do You See It? Exercise

Page 92, Exercise 104 Would you use the dividing out technique or the rationalizing technique to find the limit of the function? Explain your reasoning.

(a) $\lim\limits_{x \to -2} \dfrac{x^2 + x - 2}{x + 2}$



(b) $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}$



Solution

(a) Use the dividing out technique because the numerator and denominator have a common factor.

(b) Use the rationalizing technique because the numerator involves a radical expression.

Suggested Homework Assignment

Pages 91–93: 1–75 odd, 89, 99, 105, 107, and 119–123 odd.

Section 2.4 Continuity and One-Sided Limits

Section Comments

2.4 **Continuity and One-Sided Limits**—Determine continuity at a point and continuity on an open interval. Determine one-sided limits and continuity on a closed interval. Use properties of continuity. Understand and use the Intermediate Value Theorem.

Teaching Tips

Students have trouble with limits involving trigonometric functions and rationalizing denominators and numerators. Students also have trouble with limits involving rational functions. Encourage students to sharpen their algebra skills by studying the following material in the text or in *Precalculus*, 10th edition, by Larson.

- Trigonometric functions: Section 1.4
- Rationalizing numerators: *Precalculus* Appendix A.2
- Simplifying rational expressions: *Precalculus* Appendix A.4

Address the following for this section:

- Describe the difference between a removable discontinuity and a nonremovable discontinuity.
- Write a function that has a removable discontinuity.
- Write a function that has a nonremovable discontinuity.
- Write a function that has a removable discontinuity and a nonremovable discontinuity.

Students will need to know these concepts when they study vertical asymptotes (Section 2.5), improper integrals (Section 8.8), and functions of several variables (Section 13.2).

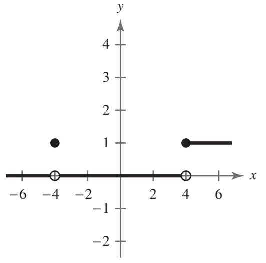
Discuss the difference between a removable discontinuity and a nonremovable discontinuity. Use the function $f(x) = \dfrac{|x - 4|}{x - 4}$ to illustrate a function with a nonremovable discontinuity.

Note that it is not possible to remove the discontinuity by redefining the function. When you illustrate a removable discontinuity, as in the function $f(x) = \dfrac{\sin(x + 4)}{x + 4}$, note that the function can be redefined to remove the discontinuity. Ask students how they would do this. (Let $f(-4) = 1$.) Finally, illustrate the example below. Show students the function and the graph, and then ask them if there are any removable or nonremovable discontinuities. Once they have properly identified $x = -4$ as removable and $x = 4$ as nonremovable, ask them to redefine f to remove the discontinuity at $x = -4$. (Let $f(-4) = 0$.)

Ask students to consider the following function with its graph below:

$$f(x) = \begin{cases} 0, & x < -4 \\ 1, & x = -4 \\ 0, & -4 < x < 4 \\ 1, & x \geq 4 \end{cases}$$

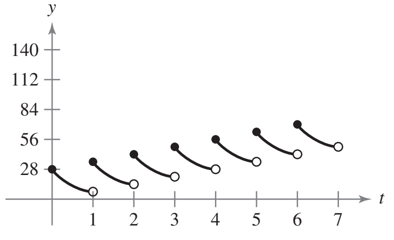
$x = 4$ is nonremovable. $x = -4$ is removable.



Use this example to show which are removable and nonremovable discontinuities. In addition, use a rational function with both a vertical asymptote and a hole to show the differences between removable and nonremovable discontinuities. A suggested example is $f(x) = \dfrac{8x^2 + 26x + 15}{2x^2 - x - 15}$.

How Do You See It? Exercise

Page 105, Exercise 116 Every day you dissolve 28 ounces of chlorine in a swimming pool. The graph shows the amount of chlorine $f(t)$ in the pool after t days. Estimate and interpret $\lim\limits_{t \to 4^-} f(t)$ and $\lim\limits_{t \to 4^+} f(t)$.



Solution

$\lim\limits_{t \to 4^-} f(t) \approx 28$

$\lim\limits_{t \to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 ounces. At the beginning of day 4, more chlorine was added, and the amount is now about 56 ounces.

Suggested Homework Assignment

Pages 103–105: 1, 3, 7–19 odd, 21, 23, 25, 29, 31, 33, 37, 39, 47, 49, 51, 55, 59, 61, 65, 67, 75–81 odd, 83, 99, 101, 105, and 109–113 odd.

Section 2.5 Infinite Limits

Section Comments

2.5 **Infinite Limits**—Determine infinite limits from the left and from the right. Find and sketch the vertical asymptotes of the graph of a function.

Teaching Tips

It is vital for students to know asymptotes of a rational function for this section. Encourage students to review material on asymptotes by studying Section 2.6 in *Precalculus*, 10th edition, by Larson.

Students need to be able to decipher when vertical asymptotes versus holes will occur in a rational function. A suggested problem for students to consider is to determine if $x = 1$ is a vertical asymptote of $f(x) = \dfrac{p(x)}{x - 1}$.

In class, you can ask students who believe the graph of f has a vertical asymptote at $x = 1$ to prove it. For those who think this is not always true, ask them to provide a counterexample. A sample answer is given in the solution. If desired, you could also ask students the following:

(a) At which x-values (if any) is f not continuous?

(b) Which of the discontinuities are removable?

Solution

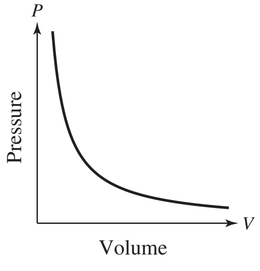
No, it is not always true. Consider $p(x) = x^2 - 1$. The function

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{p(x)}{x - 1}$$

has a hole at $(1, 2)$, not a vertical asymptote.

How Do You See It? Exercise

Page 113, Exercise 62 For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. What is the limit of P as V approaches 0 from the right? Explain what this means in the context of the problem.



Solution

$$\lim_{V \to 0^+} P = \infty$$

As the volume of the gas decreases, the pressure increases.

Suggested Homework Assignment

Pages 112–114: 1, 3, 7, 13, 17–31 odd, 37–51 odd, 55, 57, and 67–71 odd.

Chapter 2 Project

Medicine in the Bloodstream

A patient's kidneys purify 25% of the blood in her body in 4 hours.

Exercises

In Exercises 1–3, a patient takes one 16-milliliter dose of a medication.

1. Determine the amount of medication left in the patient's body after 4, 8, 12, and 16 hours.

2. Notice that after the first 4-hour period, $\frac{3}{4}$ of the 16 milliliters of medication is left in the body, after the second 4-hour period, $\frac{9}{16}$ of the 16 milliliters of medication is left in the body, and so on. Use this information to write an equation that represents the amount a of medication left in the patient's body after n 4-hour periods.

3. Can you find a value of n for which a equals 0? Explain.

In Exercises 4–9, the patient takes an additional 16-milliliter dose every 4 hours.

4. Determine the amount of medication in the patient's body immediately after taking the second dose.

5. Determine the amount of medication in the patient's body immediately after taking the third and fourth doses. What is happening to the amount of medication in the patient's body over time?

6. The medication is eliminated from the patient's body at a constant rate. Sketch a graph that shows the amount of medication in the patient's body during the first 16 hours. Let x represent the number of hours and y represent the amount of medication in the patient's body in milliliters.

7. Use the graph in Exercise 6 to find each limit.

(a) $\lim_{x\to 4^-} f(x)$

(b) $\lim_{x\to 4^+} f(x)$

(c) $\lim_{x\to 12^-} f(x)$

(d) $\lim_{x\to 12^+} f(x)$

8. Discuss the continuity of the function represented by the graph in Exercise 6. Interpret any discontinuities in the context of the problem.

9. The amount of medication in the patient's body remains constant when the amount eliminated in 4 hours is equal to the additional dose taken at the end of the 4-hour period. Write and solve an equation to find this amount.

Chapter 3 Differentiation

Chapter Comments

The material presented in Chapter 3 forms the basis for the remainder of calculus. Much of it needs to be memorized, beginning with the definition of a derivative of a function found on page 123. Students need to have a thorough understanding of the tangent line problem and they need to be able to find an equation of a tangent line. Frequently, students will use the function $f'(x)$ as the slope of the tangent line. They need to understand that $f'(x)$ is the formula for the slope and the actual value of the slope can be found by substituting into $f'(x)$ the appropriate value for x. On pages 125–126 of Section 3.1, you will find a discussion of situations where the derivative fails to exist. These examples (or similar ones) should be discussed in class.

As you teach this chapter, vary your notations for the derivative. One time write y'; another time write dy/dx or $f'(x)$. Terminology is also important. Instead of saying "find the derivative," sometimes say, "differentiate." This would be an appropriate time, also, to talk a little about Leibnitz and Newton and the discovery of calculus.

Sections 3.2, 3.3, and 3.4 present a number of rules for differentiation. Spending time on the notation of the higher derivatives can be beneficial. Have your students memorize the Product Rule and the Quotient Rule (Theorems 3.8 and 3.9) in words rather than symbols. Students tend to be lazy when it comes to trigonometry and therefore, you need to impress upon them that the formulas for the derivatives of the six trigonometric functions need to be memorized also. You will probably not have enough time in class to prove every one of these differentiation rules, so choose several to do in class and perhaps assign a few of the other proofs as homework.

The Chain Rule, in Section 3.4, will require two days of your class time. Students need a lot of practice with this and the algebra involved in these problems. Many students can find the derivative of $f(x) = x^2\sqrt{1 - x^2}$ without much trouble, but simplifying the answer is often difficult for them. Insist that they learn to factor and write the answer without negative exponents. Strive to get the answer in the form given in the back of the book. This will help them later on when the derivative is set equal to zero.

Implicit differentiation is often difficult for students. Have students think of y as a function of x and therefore y^3 is $[f(x)]^3$. This way they can relate implicit differentiation to the Chain Rule studied in the previous section.

Try to get your students to see that related rates, discussed in Section 3.7, are another use of the Chain Rule.

Section 3.1 The Derivative and the Tangent Line Problem

Section Comments

3.1 **The Derivative and the Tangent Line Problem**—Find the slope of the tangent line to a curve at a point. Use the limit definition to find the derivative of a function. Understand the relationship between differentiability and continuity.

Teaching Tips

Ask students what they think "the line tangent to a curve" means. Draw a curve with tangent lines to show a visual picture of tangent lines. For example:

CHAPTER 12
Vector-Valued Functions

Section 12.1 Vector-Valued Functions

1. You can use a vector-valued function to trace the graph of a curve. Recall that the terminal point of the position vector $\mathbf{r}(t)$ coincides with a point on the curve.

2. A vector-valued function $\mathbf{r}$ is continuous at the point $t = a$ if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

3. $\mathbf{r}(t) = \dfrac{1}{t+1}\mathbf{i} + \dfrac{t}{2}\mathbf{j} - 3t\mathbf{k}$

Component functions: $f(t) = \dfrac{1}{t+1}$

$g(t) = \dfrac{t}{2}$

$h(t) = -3t$

Domain: $(-\infty, -1) \cup (-1, \infty)$

4. $\mathbf{r}(t) = \sqrt{4 - t^2}\,\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$

Component functions: $f(t) = \sqrt{4 - t^2}$

$g(t) = t^2$

$h(t) = -6t$

Domain: $[-2, 2]$

5. $\mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$

Component functions: $f(t) = \ln t$

$g(t) = -e^t$

$h(t) = -t$

Domain: $(0, \infty)$

6. $\mathbf{r}(t) = \sin t\mathbf{i} + 4\cos t\mathbf{j} + t\mathbf{k}$

Component functions: $f(t) = \sin t$

$g(t) = 4\cos t$

$h(t) = t$

Domain: $(-\infty, \infty)$

7. $\mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t) = \left(\cos t\mathbf{i} - \sin t\mathbf{j} + \sqrt{t}\mathbf{k}\right) + \left(\cos t\mathbf{i} + \sin t\mathbf{j}\right) = 2\cos t\mathbf{i} + \sqrt{t}\mathbf{k}$

Domain: $[0, \infty)$

8. $\mathbf{r}(t) = \mathbf{F}(t) - G(t) = \left(\ln t\mathbf{i} + 5t\mathbf{j} - 3t^2\mathbf{k}\right) - \left(\mathbf{i} + 4t\mathbf{j} - 3t^2\mathbf{k}\right)$

$$= (\ln t - 1)\mathbf{i} + (5t - 4t)\mathbf{j} + \left(-3t^2 + 3t^2\right)\mathbf{k} = (\ln t - 1)\mathbf{i} + t\mathbf{j}$$

Domain: $(0, \infty)$

9. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & 0 \\ 0 & \sin t & \cos t \end{vmatrix} = \cos^2 t\mathbf{i} - \sin t\cos t\mathbf{j} + \sin^2 t\mathbf{k}$

Domain: $(-\infty, \infty)$

10. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^3 & -t & t \\ \sqrt[3]{t} & \dfrac{1}{t+1} & t+2 \end{vmatrix} = \left(-t(t+2) - \dfrac{t}{t+1}\right)\mathbf{i} - \left(t^3(t+2) - t\sqrt[3]{t}\right)\mathbf{j} + \left(\dfrac{t^3}{t+1} + t\sqrt[3]{t}\right)\mathbf{k}$

Domain: $(-\infty, -1), (-1, \infty)$

11. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t-1)\mathbf{j}$

(a) $\mathbf{r}(1) = \frac{1}{2}\mathbf{i}$

(b) $\mathbf{r}(0) = \mathbf{j}$

(c) $\mathbf{r}(s+1) = \frac{1}{2}(s+1)^2\mathbf{i} - (s+1-1)\mathbf{j} = \frac{1}{2}(s+1)^2\mathbf{i} - s\mathbf{j}$

(d) $\mathbf{r}(2+\Delta t) - \mathbf{r}(2) = \frac{1}{2}(2+\Delta t)^2\mathbf{i} - (2+\Delta t-1)\mathbf{j} - (2\mathbf{i}-\mathbf{j}) = \left(2 + 2\Delta t + \frac{1}{2}(\Delta t)^2\right)\mathbf{i} - (1+\Delta t)\mathbf{j} - 2\mathbf{i} + \mathbf{j}$

$= \left(2\Delta t + \frac{1}{2}(\Delta t)^2\right)\mathbf{i} - (\Delta t)\mathbf{j} = \frac{1}{2}\Delta t(\Delta t + 4)\mathbf{i} - \Delta t\mathbf{j}$

12. $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j}$

(a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

(c) $\mathbf{r}(\theta - \pi) = \cos(\theta-\pi)\mathbf{i} + 2\sin(\theta-\pi)\mathbf{j} = -\cos\theta\mathbf{i} - 2\sin\theta\mathbf{j}$

(d) $\mathbf{r}\left(\frac{\pi}{6} + \Delta t\right) - \mathbf{r}\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6} + \Delta t\right)\mathbf{i} + 2\sin\left(\frac{\pi}{6} + \Delta t\right)\mathbf{j} - \left(\cos\left(\frac{\pi}{6}\right)\mathbf{i} + 2\sin\frac{\pi}{6}\mathbf{j}\right)$

13. $P(0, 0, 0), Q(5, 2, 2)$

$\mathbf{v} = \overrightarrow{PQ} = \langle 5, 2, 2\rangle$

$\mathbf{r}(t) = 5t\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k}, 0 \le t \le 1$

$x = 5t, y = 2t, z = 2t, 0 \le t \le 1$, Parametric equation

(Answers may vary.)

14. $P(0, 2, -1), Q(4, 7, 2)$

$\mathbf{v} = \overrightarrow{PQ} = \langle 4, 5, 3\rangle$

$\mathbf{r}(t) = 4t\mathbf{i} + (2+5t)\mathbf{j} + (-1+3t)\mathbf{k},\ 0 \le t \le 1$

$x = 4t, y = 2 + 5t, z = -1 + 3t,$

$0 \le t \le 1$, Parametric equation

(Answers may vary.)

15. $P(-3, -6, -1), Q(-1, -9, -8)$

$\mathbf{v} = \overrightarrow{PQ} = \langle -1+3, -9+6, -8+1\rangle = \langle 2, -3, -7\rangle$

$\mathbf{r}(t) = (-3+2t)\mathbf{i} + (-6-3t)\mathbf{j} + (-1-7t)\mathbf{k}, 0 \le t \le 1$

$x = -3 + 2t, y = -6 - 3t, z = -1 - 7t, 0 \le t \le 1$, Parametric equation

(Answers may vary.)

16. $P(1, -6, 8), Q(-3, -2, 5)$

$\mathbf{v} = \overrightarrow{PQ} = \langle -4, 4, -3\rangle$

$\mathbf{r}(t) = (1-4t)\mathbf{i} + (-6+4t)\mathbf{j} + (8-3t)\mathbf{k},\ 0 \le t \le 1$

$x = 1 - 4t,\ y = -6 + 4t,\ z = 8 - 3t,\ \ 0 \le t \le 1$, Parametric equation

(Answers may vary.)

17. $\mathbf{r}(t)\cdot\mathbf{u}(t) = (3t-1)(t^2) + \left(\frac{1}{4}t^3\right)(-8) + 4(t^3)$

$= 3t^3 - t^2 - 2t^3 + 4t^3 = 5t^3 - t^2$, a scalar.

No, the dot product is a scalar-valued function.

18. $\mathbf{r}(t)\cdot\mathbf{u}(t) = (3\cos t)(4\sin t) + (2\sin t)(-6\cos t) + (t-2)(t^2) = t^3 - 2t^2$, a scalar.

No, the dot product is a scalar-valued function.

19. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k},\ -2 \le t \le 2$

$x = t, y = 2t, z = t^2$

So, $z = x^2$. Matches (b)

20. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k},\ -1 \le t \le 1$

$x = \cos(\pi t), y = \sin(\pi t), z = t^2$

So, $x^2 + y^2 = 1$. Matches (c)

21. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, -2 \le t \le 2$

$x = t, y = t^2, z = e^{0.75t}$

So, $y = x^2$. Matches (d)

22. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \dfrac{2t}{3}\mathbf{k}, 0.1 \le t \le 5$

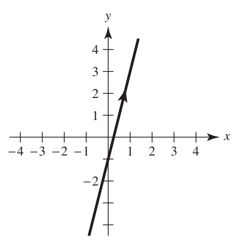
$x = t, y = \ln t, z = \dfrac{2t}{3}$

So, $z = \frac{2}{3}x$ and $y = \ln x$. Matches (a)

23. $x = \dfrac{t}{4} \Rightarrow t = 4x$

$y = t - 1$

$y = 4x - 1$

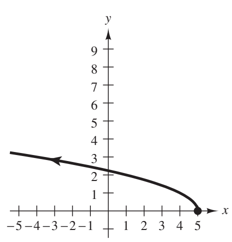


24. $x = 5 - t \Rightarrow t = 5 - x$

$y = \sqrt{t}$

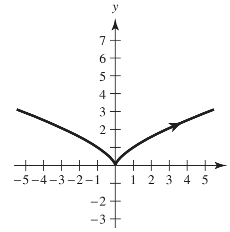
$y = \sqrt{5 - x}$

Domain: $t \ge 0$

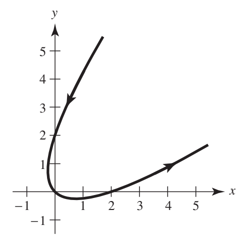


25. $x = t^3, y = t^2$

$y = x^{2/3}$

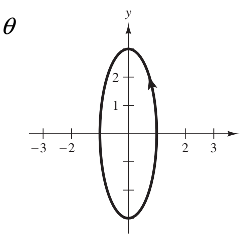


26. $x = t^2 + t, y = t^2 - t$



27. $x = \cos\theta, y = 3\sin\theta$

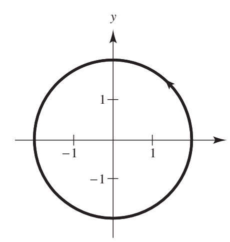
$x^2 + \dfrac{y^2}{9} = 1$, Ellipse



28. $x = 2\cos t$

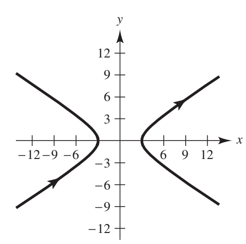
$y = 2\sin t$

$x^2 + y^2 = 4$, circle



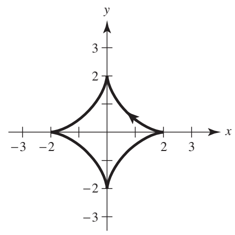
29. $x = 3\sec\theta, y = 2\tan\theta$

$\dfrac{x^2}{9} = \dfrac{y^2}{4} + 1$, Hyperbola



30. $x = 2\cos^3 t,\ y = 2\sin^3 t$

$$\left(\frac{x}{2}\right)^{2/3} + \left(\frac{y}{2}\right)^{2/3} = \cos^2 t + \sin^2 t$$
$$= 1$$
$$x^{2/3} + y^{2/3} = 2^{2/3}$$

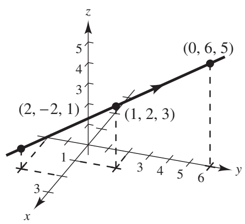


31. $x = -t + 1$

$y = 4t + 2$

$z = 2t + 3$

Line passing through the points: $(0, 6, 5), (1, 2, 3)$

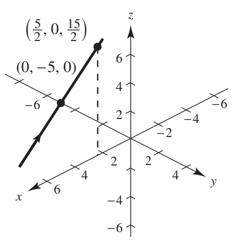


32. $x = t$

$y = 2t - 5$

$y = 3t$

Line passing through the points: $(0, -5, 0), \left(\frac{5}{2}, 0, \frac{15}{2}\right)$

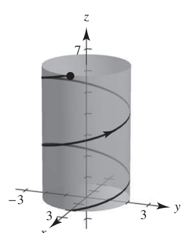


33. $x = 2\cos t,\ y = 2\sin t,\ z = t$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$z = t$

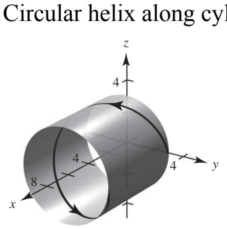
Circular helix



34. $x = t,\ y = 3\cos t,\ z = 3\sin t$

$$y^2 + z^2 = (3\cos t)^3 + (3\sin t)^3 = 9$$

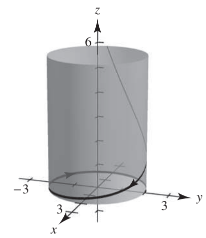
Circular helix along cylinder $y^2 + z^2 = 9$



35. $x = 2\sin t,\ y = 2\cos t,\ z = e^{-t}$

$x^2 + y^2 = 4$

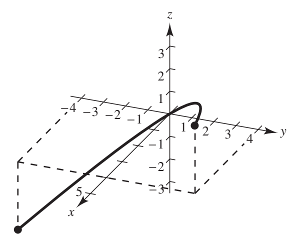
$z = e^{-t}$



36. $x = t^2,\ y = 2t,\ z = \frac{3}{2}t$

$x = \frac{y^2}{4},\ z = \frac{3}{4}y$

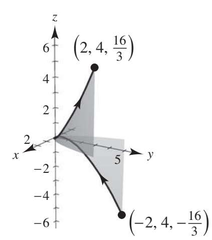
t	-2	-1	0	1	2
x	4	1	0	1	4
y	-4	-2	0	2	4
z	-3	$-\frac{3}{2}$	0	$\frac{3}{2}$	3



37. $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

$y = x^2,\ z = \frac{2}{3}x^3$

t	-2	-1	0	1	2
x	-2	-1	0	1	2
y	4	1	0	1	4
z	$-\frac{16}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{16}{3}$



38. $x = \cos t + t \sin t$

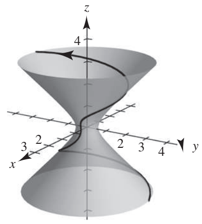
$y = \sin t - t \cos t$

$z = t$

$x^2 + y^2 = 1 + t^2 = 1 + z^2$

or $x^2 + y^2 - z^2 = 1$

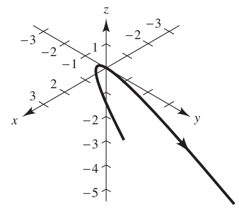
$z = t$



Helix along a hyperboloid of one sheet

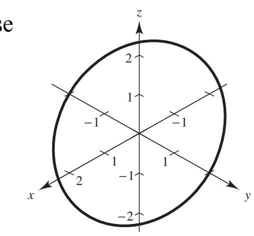
39. $\mathbf{r}(t) = -\frac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \frac{\sqrt{3}}{2}t^2\mathbf{k}$

Parabola

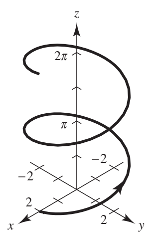


40. $\mathbf{r}(t) = -\sqrt{2}\sin t\mathbf{i} + 2\cos t\mathbf{j} + \sqrt{2}\sin t\mathbf{k}$

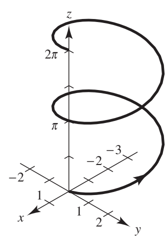
Ellipse



41.

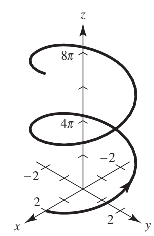


(a)



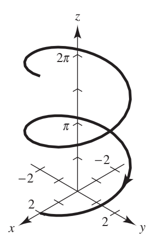
The helix is translated 2 units back on the x-axis.

(b)



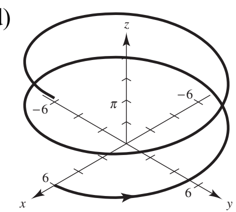
The height of the helix increases at a faster rate.

(c)



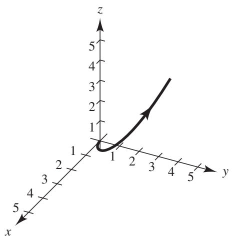
The orientation of the helix is reversed.

(d)

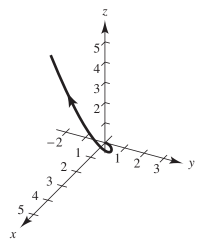


The radius of the helix is increased from 2 to 6.

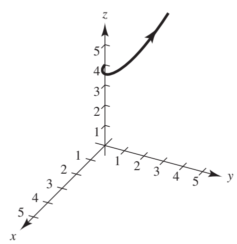
42. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^3\mathbf{h}$



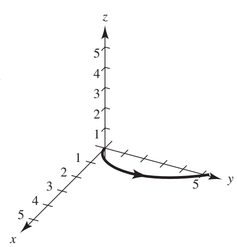
(a) $\mathbf{u}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$ has the roles of x and y interchanged. The graph is a reflection in the plane $x = y$.



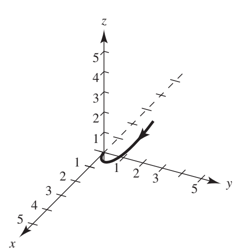
(b) $\mathbf{u}(t) = \mathbf{r}(t) + 4\mathbf{k}$ is an upward shift 4 units.



(c) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{8}t^3\mathbf{k}$ shrinks the z-value by a factor of 4. The curve rises more slowly.



(d) $\mathbf{u}(t) = \mathbf{r}(-t)$ reverses the orientation.



43. $\mathbf{u}(t) = \mathbf{r}(t) + 2\mathbf{k} = 3t^2\mathbf{i} + (t - 1)\mathbf{j} + (t + 2)\mathbf{k}$

44. $\mathbf{u}(t) = \mathbf{r}(t) + \mathbf{i} = (3t^2 + 1)\mathbf{i} + (t - 1)\mathbf{j} + t\mathbf{k}$

45. $\mathbf{u}(t) = 3t^2\mathbf{i} + 2(t - 1)\mathbf{j} + t\mathbf{k}$

46. $\mathbf{u}(t) = 3t^2\mathbf{i} + (t - 1)\mathbf{j} + 3t\mathbf{k}$

47. $y = x + 5$

Let $x = t$, then $y = t + 5$

$\mathbf{r}(t) = t\mathbf{i} + (t + 5)\mathbf{j}$

48. $2x - 3y + 5 = 0$

Let $x = t$, then $y = \frac{1}{3}(2t + 5)$.

$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}(2\mathbf{t} + 5)\mathbf{j}$

49. $y = (x - 2)^2$

Let $x = t$, then $y = (t - 2)^2$.

$\mathbf{r}(t) = t\mathbf{i} + (t - 2)^2\mathbf{j}$

50. $y = 4 - x^2$

Let $x = t$, then $y = 4 - t^2$.

$\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$

51. $x^2 + y^2 = 25$

Let $x = 5\cos t$, then $y = 5\sin t$.

$\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j}$

52. $(x - 2)^2 + y^2 = 4$

Let $x - 2 = 2\cos t$, $y = 2\sin t$.

$\mathbf{r}(t) = (2 + 2\cos t)\mathbf{i} + 2\sin t\mathbf{j}$

53. $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$

Let $x = 4\sec t$, $y = 2\tan t$.

$\mathbf{r}(t) = 4\sec t\mathbf{i} + 2\tan t\mathbf{j}$

54. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

Let $x = 3\cos t$ and $y = 4\sin t$

Then $\dfrac{x^2}{9} + \dfrac{y^2}{16} = \cos^2 t + \sin^2 t = 1$

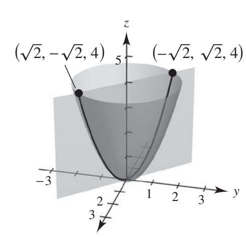
$\mathbf{r}(t) = 3\cos t\mathbf{i} + 4\sin t\mathbf{j}$

55. $z = x^2 + y^2, x + y = 0$

Let $x = t$, then $y = -x = -t$ and $z = x^2 + y^2 = 2t^2$.

So, $x = t, y = -t, z = 2t^2$.

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$

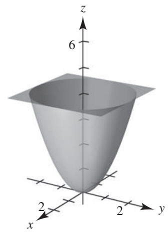


56. $z = x^2 + y^2, z = 4$

So, $x^2 + y^2 = 4$ or

$x = 2\cos t,\ y = 2\sin t, z = 4.$

$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 4\mathbf{k}$



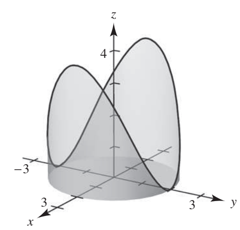
57. $x^2 + y^2 = 4, z = x^2$

$x = 2\sin t, y = 2\cos t$

$z = x^2 = 4\sin^2 t$

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	0	1	$\sqrt{2}$	2	$\sqrt{2}$	0
y	2	$\sqrt{3}$	$\sqrt{2}$	0	$-\sqrt{2}$	-2
z	0	1	2	4	2	0

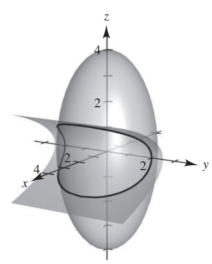
$\mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + 4\sin^2 t\mathbf{k}$



58. $4x^2 + 4y^2 + z^2 = 16, x = z^2$

If $z = t$, then $x = t^2$ and $y = \frac{1}{2}\sqrt{16 - 4t^4 - t^2}$.

t	-1.3	-1.2	-1	0	1	1.2
x	1.69	1.44	1	0	1	1.44
y	0.85	1.25	1.66	2	1.66	1.25
z	-1.3	-1.2	-1	0	1	1.2



$\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{2}\sqrt{16 - 4t^4 - t^2}\mathbf{j} + t\mathbf{k}$

59. $x^2 + y^2 + z^2 = 4, x + z = 2$

Let $x = 1 + \sin t$, then $z = 2 - x = 1 - \sin t$ and $x^2 + y^2 + z^2 = 4$.

$(1 + \sin t)^2 + y^2 + (1 - \sin t)^2 = 2 + 2\sin^2 t + y^2 = 4$

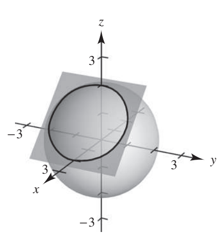
$y^2 = 2\cos^2 t, y = \pm\sqrt{2}\cos t$

$x = 1 + \sin t, y = \pm\sqrt{2}\cos t$

$z = 1 - \sin t$

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2}\cos t\mathbf{j} - (1 - \sin t)\mathbf{k}$ and

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2}\cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$

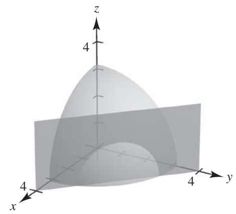


t	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	0	$\pm\frac{\sqrt{6}}{2}$	$\pm\sqrt{2}$	$\pm\frac{\sqrt{6}}{2}$	0
z	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0

60. $x^2 + y^2 + z^2 = 10, x + y = 4$

Let $x = 2 + \sin t$, then $y = 2 - \sin t$ and

$$\begin{aligned} z^2 &= 10 - x^2 - y^2 \\ &= 10 - (2 + \sin t)^2 - (2 - \sin t)^2 \\ &= 10 - (4 + 4\sin t + \sin^2 t) - (4 - 4\sin t + \sin^2 t) \\ &= 2 - 2\sin^2 t \end{aligned}$$



Taking square roots, $z = \sqrt{2(1 - \sin^2 t)} = \sqrt{2}\cos t$.

t	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	π
x	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	2
y	3	$\frac{5}{2}$	2	$\frac{3}{2}$	1	2
z	0	$\frac{\sqrt{6}}{2}$	$\sqrt{2}$	$\frac{\sqrt{6}}{2}$	0	$-\sqrt{2}$

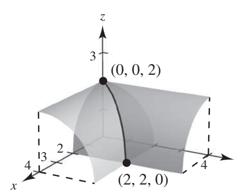
$\mathbf{r}(t) = (2 + \sin t)\mathbf{i} + (2 - \sin t)\mathbf{j} + \sqrt{2}\cos t\mathbf{k}$

61. $x^2 + z^2 = 4, y^2 + z^2 = 4$

Subtracting, you have $x^2 - y^2 = 0$ or $y = \pm x$.

So, in the first octant, if you let $x = t$, then $x = t,\ y = t,\ z = \sqrt{4 - t^2}$.

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

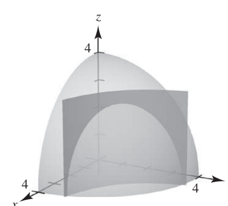


62. $x^2 + y^2 + z^2 = 16, xy = 4$ (first octant)

Let $x = t$, then $y = \dfrac{4}{t}$ and $x^2 + y^2 + z^2 = t^2 + \dfrac{16}{t^2} + z^2 = 16$.

$$z = \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}$$

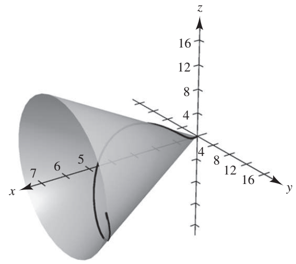
$$\left(\sqrt{8 - 4\sqrt{3}} \le t \le \sqrt{8 + 4\sqrt{3}}\right)$$



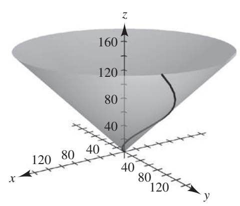
t	$\sqrt{8 + 4\sqrt{3}}$	1.5	2	2.5	3.0	3.5	$\sqrt{8 + 4\sqrt{3}}$
x	1.0	1.5	2	2.5	3.0	3.5	3.9
y	3.9	2.7	2	1.6	1.3	1.1	1.0
z	0	2.6	2.8	2.7	2.3	1.6	0

$\mathbf{r}(t) = t\mathbf{i} + \dfrac{4}{t}\mathbf{j} + \dfrac{1}{t}\sqrt{-t^4 + 16t^2 - 16}\mathbf{k}$

63. $y^2 + z^2 = (2t\cos t)^2 + (2t\sin t)^2 = 4t^2 = 4x^2$



64. $x^2 + y^2 = (e^{-t}\cos t)^2 + (e^{-t}\sin t)^2 = e^{-2t} = z^2$



65. $\lim_{t\to\pi}(t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}) = \pi\mathbf{i} - \mathbf{j}$

66. $\lim_{t\to 2}\left(3t\mathbf{i} + \frac{2}{t^2-1}\mathbf{j} + \frac{1}{t}\mathbf{k}\right) = 6\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{2}\mathbf{k}$

67. $\lim_{t\to 0}\left[t^2\mathbf{i} + 3t\mathbf{j} + \frac{1-\cos t}{t}\mathbf{k}\right] = 0$

because $\lim_{t\to 0}\frac{1-\cos t}{t} = \lim_{t\to 0}\frac{\sin t}{1} = 0.$

(L'Hôpital's Rule)

68. $\lim_{t\to 1}\left(\sqrt{t}\,\mathbf{i} + \frac{\ln t}{t^2-1}\mathbf{j} + \frac{1}{t-1}\mathbf{k}\right)$

does not exist because $\lim_{t\to 1}\frac{1}{t-1}$ does not exist.

69. $\lim_{t\to 0}\left[e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right] = \mathbf{i} + \mathbf{j} + \mathbf{k}$

because

$\lim_{t\to 0}\frac{\sin t}{t} = \lim_{t\to 0}\frac{\cos t}{1} = 1$ (L'Hôpital's Rule)

70. $\lim_{t\to\infty}\left(e^{-t}\mathbf{i} + \frac{1}{t}\mathbf{j} + t^{1/t}\mathbf{k}\right)$

Note that $\lim_{t\to\infty} t^{1/t} = 1$ because if $y = t^{1/t}$, then

$\ln y = \frac{1}{t}\ln t \to 0$ as $t \to \infty$.

Hence, $y \to 1$.

So, the limit is $0\mathbf{i} + 0\mathbf{j} + \mathbf{k} = \mathbf{k}$.

71. $\mathbf{r}(t) = \frac{1}{2t+1}\mathbf{i} + \frac{1}{t}\mathbf{j}$

The function is not continuous at $t = -\frac{1}{2}$ and $t = 0$. So,

$\mathbf{r}(t)$ is continuous on the intervals $\left(-\infty, -\frac{1}{2}\right), \left(-\frac{1}{2}, 0\right)$,

and $(0, \infty)$.

72. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t-1}\mathbf{j}$

Continuous on $[1, \infty)$

73. $\mathbf{r}(t) = t\mathbf{i} + \arcsin t\mathbf{j} + (t-1)\mathbf{k}$

Continuous on $[-1, 1]$

74. $\mathbf{r}(t) = \langle 2e^{-t}, e^{-t}, \ln(t-1)\rangle$

Continuous on $t - 1 > 0$ or $t > 1$: $(1, \infty)$.

75. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

Discontinuous at $t = \frac{\pi}{2} + n\pi$

Continuous on $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$

76. $\mathbf{r}(t) = \langle 8, \sqrt{t}, \sqrt[3]{t}\rangle$

Continuous on $[0, \infty)$

77. Yes. $\mathbf{r}(t)$ represents a line. Answers will vary.

78. Not necessarily. For example, the function $\mathbf{r}(t) = t^3\mathbf{i} + t^3\mathbf{j} + t^3\mathbf{k}$ represents a line.

79. Answers will vary. *Sample answer*:

$$\mathbf{r}(t) = \begin{cases} \mathbf{i} + \mathbf{j}, & t \ge 3 \\ -\mathbf{i} + \mathbf{j}, & t < 3 \end{cases}$$

80. (a) $x = -3\cos t + 1,\ y = 5\sin t + 2,\ z = 4$

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{25} = 1,\ z = 4$$

(b) $x = 4,\ y = -3\cos t + 1,\ z = 5\sin t + 2$

$$\frac{(y-1)^2}{9} + \frac{(z-2)^2}{25} = 1,\ x = 4$$

(c) $x = 3\cos t - 1,\ y = -5\sin t - 2,\ z = 4$

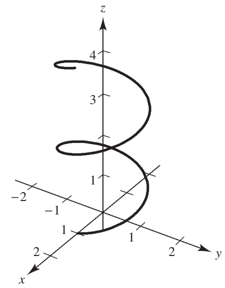
$$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1,\ z = 4$$

(d) $x = -3\cos 2t + 1,\ y = 5\sin 2t + 2,\ z = 4$

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{25} = 1,\ z = 4$$

(a) and (d) represent the same graph

81. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \dfrac{1}{\pi}t\mathbf{k},\ 0 \le t \le 4\pi$



82. (a) View from the negative x-axis: $(-20, 0, 0)$

(b) View from above the first octant: $(10, 20, 10)$

(c) View from the z-axis: $(0, 0, 20)$

(d) View from the positive x-axis: $(20, 0, 0)$

83. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}
\lim_{t\to c}\left[\mathbf{r}(t)\times\mathbf{u}(t)\right] &= \lim_{t\to c}\left\{\left[y_1(t)z_2(t) - y_2(t)z_1(t)\right]\mathbf{i} - \left[x_1(t)z_2(t) - x_2(t)z_1(t)\right]\mathbf{j} + \left[x_1(t)y_2(t) - x_2(t)y_1(t)\right]\mathbf{k}\right\}\\
&= \left[\lim_{t\to c} y_1(t)\lim_{t\to c} z_2(t) - \lim_{t\to c} y_2(t)\lim_{t\to c} z_1(t)\right]\mathbf{i} - \left[\lim_{t\to c} x_1(t)\lim_{t\to c} z_2(t) - \lim_{t\to c} x_2(t)\lim_{t\to c} z_1(t)\right]\mathbf{j}\\
&\quad + \left[\lim_{t\to c} x_1(t)\lim_{t\to c} y_2(t) - \lim_{t\to c} x_2(t)\lim_{t\to c} y_1(t)\right]\mathbf{k}\\
&= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right]\times\left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right]\\
&= \lim_{t\to c}\mathbf{r}(t)\times\lim_{t\to c}\mathbf{u}(t)
\end{aligned}$$

84. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}
\lim_{t\to c}\left[\mathbf{r}(t)\cdot\mathbf{u}(t)\right] &= \lim_{t\to c}\left[x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)\right]\\
&= \lim_{t\to c} x_1(t)\lim_{t\to c} x_2(t) + \lim_{t\to c} y_1(t)\lim_{t\to c} y_2(t) + \lim_{t\to c} z_1(t)\lim_{t\to c} z_2(t)\\
&= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right]\cdot\left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right]\\
&= \lim_{t\to c}\mathbf{r}(t)\cdot\lim_{t\to c}\mathbf{u}(t)
\end{aligned}$$

85. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Because $\mathbf{r}$ is continuous at $t = c$, then $\lim_{t\to c} \mathbf{r}(t) = \mathbf{r}(c)$.

$\mathbf{r}(c) = x(c)\mathbf{i} + y(c)\mathbf{j} + z(c)\mathbf{k} \Rightarrow x(c), y(c), z(c)$ are defined at c.

$$\|\mathbf{r}\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

$$\lim_{t\to c}\|\mathbf{r}\| = \sqrt{(x(c))^2 + (y(c))^2 + (z(c))^2} = \|\mathbf{r}(c)\|$$

So, $\|\mathbf{r}\|$ is continuous at c.

86. Let

$$f(t) = \begin{cases} 1, & \text{if } t \ge 0 \\ -1, & \text{if } t < 0 \end{cases}$$

and $\mathbf{r}(t) = f(t)\mathbf{i}$. Then $\mathbf{r}$ is not continuous at $c = 0$, whereas, $\|\mathbf{r}\| = 1$ is continuous for all t.

87. No, not necessarily. See Exercise 90.

88. Yes. See Exercise 89.

89. $\mathbf{r}(t) = t^2\mathbf{i} + (9t - 20)\mathbf{j} + t^2\mathbf{k}$

$\mathbf{u}(s) = (3s + 4)\mathbf{i} + s^2\mathbf{j} + (5s - 4)\mathbf{k}$.

Equating components:

$$t^2 = 3s + 4$$
$$9t - 20 = s^2$$
$$t^2 = 5s - 4$$

So, $3s + 4 = 5s - 4 \Rightarrow s = 4$

$9t - 20 = s^2 = 16 \Rightarrow t = 4$.

The paths intersect at the same time $t = 4$ at the point $(16, 16, 16)$. The particles collide.

90. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{u}(s) = (-2s + 3)\mathbf{i} + 8s\mathbf{j} + (12s + 2)\mathbf{k}$

Equating components

$$t = -2s + 3$$
$$t^2 = 8s$$
$$t^3 = 12s + 2$$
$$(-2s + 3)^2 = 8s$$
$$4s^2 - 12s + 9 = 8s$$
$$4s^2 - 20s + 9 = 0$$
$$(2s - 9)(2s - 1) = 0$$

For $s = \frac{1}{2}$, $t = -2\left(\frac{1}{2}\right) + 3 = 2$.

For $s = \frac{9}{2}$, $t = -2\left(\frac{9}{2}\right) + 3 = -6$ and $t^2 = 8\left(\frac{9}{2}\right) = 36$ and $t^3 = 12\left(\frac{9}{2}\right) = 54$. Impossible.

The paths intersect at $(2, 4, 8)$, but at different times $\left(t = 2 \text{ and } s = \frac{1}{2}\right)$. No collision.

Section 12.2 Differentiation and Integration of Vector-Valued Functions

1. $\mathbf{r}'(t_0)$ represents the vector that is tangent to the curve represented by $\mathbf{r}(t)$ at the point t_0.

2. You have $\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$ because

$$\frac{d}{dt}\left(\mathbf{R}(t) + \mathbf{C}\right) = \mathbf{r}(t).$$

3. $\mathbf{r}(t) = (1 - t^2)\mathbf{i} + t\mathbf{j}$, $t_0 = 3$

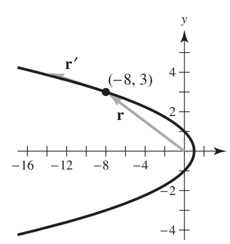
$x(t) = 1 - t^2$, $y(t) = t$

$x = 1 - y^2$

$\mathbf{r}(3) = -8\mathbf{i} + 3\mathbf{j}$

$\mathbf{r}'(t) = -2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(3) = -6\mathbf{i} + \mathbf{j}$



$\mathbf{r}'(t_0)$ is tangent to the curve at $t_0 = 3$.

4. $\mathbf{r}(t) = (1 + t)\mathbf{i} + t^3\mathbf{j},\ t_0 = 1$

$x(t) = 1 + t$

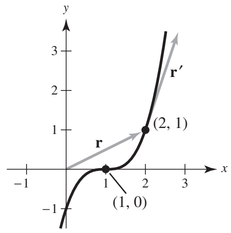
$y(t) = t^3$

$y = (x - 1)^3$

$\mathbf{r}(1) = 2\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

5. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ t_0 = \dfrac{\pi}{2}$

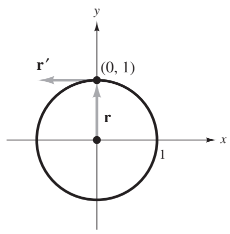
$x(t) = \cos t,\ y(t) = \sin t$

$x^2 + y^2 = 1$

$\mathbf{r}\left(\dfrac{\pi}{2}\right) = \mathbf{j}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{r}'\left(\dfrac{\pi}{2}\right) = -\mathbf{i}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

6. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4\cos t\mathbf{j},\ t_0 = \dfrac{\pi}{2}$

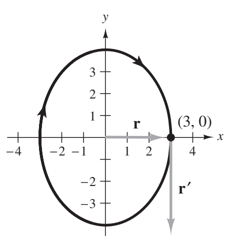
$x(t) = 3\sin t,\ y(t) = 4\cos t$

$\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{4}\right)^2 = 1$, ellipse

$\mathbf{r}\left(\dfrac{\pi}{2}\right) = 3i$

$\mathbf{r}'(t) = 3\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$\mathbf{r}'\left(\dfrac{\pi}{2}\right) = -4\mathbf{j}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

7. $\mathbf{r}(t) = \langle e^t, e^{2t}\rangle,\ t_0 = 0$

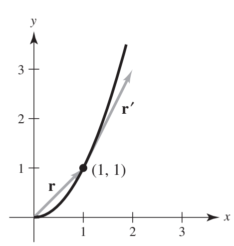
$x(t) = e^t,\ y(t) = e^{2t} = (e^t)^2$

$y = x^2,\ x > 0$

$\mathbf{r}(0) = \langle 1, 1\rangle$

$\mathbf{r}'(t) = \langle e^t, 2e^{2t}\rangle$

$\mathbf{r}'(0) = \langle 1, 2\rangle$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

8. $\mathbf{r}(t) = \langle e^{-t}, e^t\rangle,\ t_0 = 0$

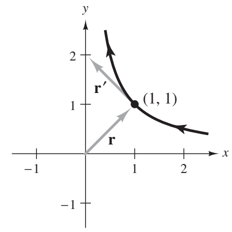
$x(t) = e^{-t} = \dfrac{1}{e^t},\ y(t) = e^t$

$y = \dfrac{1}{x},\ x > 0$

$\mathbf{r}(0) = \langle 1, 1\rangle$

$\mathbf{r}'(t) = \langle -e^{-t}, e^t\rangle$

$\mathbf{r}'(0) = \langle -1, 1\rangle$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

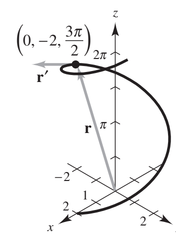
9. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ t_0 = \dfrac{3\pi}{2}$

$x^2 + y^2 = 4,\ z = t$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

$\mathbf{r}\left(\dfrac{3\pi}{2}\right) = -2\mathbf{j} + \dfrac{3\pi}{2}\mathbf{k}$

$\mathbf{r}'\left(\dfrac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$



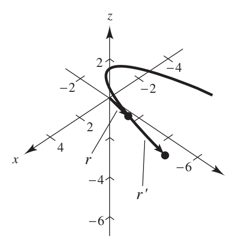
10. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k},\ t_0 = 2$

$y = x^2,\ z = \frac{3}{2}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}(2) = 2\mathbf{i} + 4\mathbf{j} + \frac{3}{2}\mathbf{k}$

$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j}$



11. $\mathbf{r}(t) = t^4\mathbf{i} - 5t\mathbf{j}$

$\mathbf{r}'(t) = 4t^3\mathbf{i} - 5\mathbf{j}$

12. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (1 - t^3)\mathbf{j}$

$\mathbf{r}'(t) = \dfrac{1}{2\sqrt{t}}\mathbf{i} - 3t^2\mathbf{j}$

13. $\mathbf{r}(t) = 3\cos^3 t\mathbf{i} + 2\sin^3 t\mathbf{j} + \mathbf{k}$

$$\begin{aligned}\mathbf{r}'(t) &= 9\cos^2 t(-\sin t)\mathbf{i} + 6\sin^2 t(\cos t)\mathbf{j}\\ &= -9\sin t\cos^2 t\mathbf{i} + 6\sin^2 t\cos t\mathbf{j}\end{aligned}$$

14. $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\sqrt{t}\mathbf{j} + \ln t^2\mathbf{k}$

$$\begin{aligned}\mathbf{r}'(t) &= \frac{2}{\sqrt{t}}\mathbf{i} + \left(2t\sqrt{t} + \frac{t^2}{2\sqrt{t}}\right)\mathbf{j} + \frac{2}{t}\mathbf{k}\\ &= \frac{2}{\sqrt{t}}\mathbf{i} + \frac{5t^{3/2}}{2}\mathbf{j} + \frac{2}{t}\mathbf{k}\end{aligned}$$

15. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j} + 5te^{t}\mathbf{k}$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i} + \left(5e^{t} + 5te^{t}\right)\mathbf{k}$

16. $\mathbf{r}(t) = \left\langle t^3, \cos 3t, \sin 3t\right\rangle$

$\mathbf{r}'(t) = \left\langle 3t^2, -3\sin 3t, 3\cos 3t\right\rangle$

17. $\mathbf{r}(t) = \left\langle t\sin t, t\cos t, t\right\rangle$

$\mathbf{r}'(t) = \left\langle \sin t + t\cos t, \cos t - t\sin t, 1\right\rangle$

18. $\mathbf{r}(t) = \left\langle \arcsin t, \arccos t, 0\right\rangle$

$$\mathbf{r}'(t) = \left\langle \frac{1}{\sqrt{1 - t^2}}, -\frac{1}{\sqrt{1 - t^2}}, 0\right\rangle$$

19. $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + t\mathbf{j}$

(b) $\mathbf{r}''(t) = 6t\mathbf{i} + \mathbf{j}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = 3t^2(6t) + t = 18t^3 + t$

20. $\mathbf{r}(t) = \left(t^2 + t\right)\mathbf{i} + \left(t^2 - t\right)\mathbf{j}$

(a) $\mathbf{r}'(\mathbf{t}) = (2t + 1)\mathbf{i} + (2t - 1)\mathbf{j}$

(b) $\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{j}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (2t + 1)(2) + (2t - 1)(2) = 8t$

21. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

(b) $\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (-4\sin t)(-4\cos t) + 4\cos t(-4\sin t) = 0$

22. $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -8\sin t\mathbf{i} + 3\cos t\mathbf{j}$

(b) $\mathbf{r}''(t) = -8\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (-8\sin t)(-8\cos t) + 3\cos t(-3\sin t) = 55\sin t\cos t$

23. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = t\mathbf{i} - \mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

(b) $\mathbf{r}''(t) = \mathbf{i} + t\mathbf{k}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = t(1) + (-1)(0) + \frac{1}{2}t^2(t) = t + \frac{1}{2}t^3$

(d)
$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & -1 & \frac{1}{2}t^2 \\ 1 & 0 & t \end{bmatrix}$$
$$= (-t)\mathbf{i} - \left(t^2 - \frac{1}{2}t^2\right)\mathbf{j} + \mathbf{k}$$
$$= -t\mathbf{i} - \frac{1}{2}t^2\mathbf{j} + \mathbf{k}$$

24. $\mathbf{r}(t) = t^3\mathbf{i} + (2t^2 + 3)\mathbf{j} + (3t - 5)\mathbf{k}$

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t\mathbf{j} + 3\mathbf{k}$

(b) $\mathbf{r}''(t) = 6t\mathbf{i} + 4\mathbf{j}$

(c) $$\begin{aligned}\mathbf{r}'(t)\cdot\mathbf{r}''(t) &= 3t^2(6t) + 4t(4) + 3(0)\\ &= 18t^3 + 16t\end{aligned}$$

(d) $$\begin{aligned}\mathbf{r}'(t)\times\mathbf{r}''(t) &= \begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3t^2 & 4t & 3\\ 6t & 4 & 0\end{bmatrix}\\ &= (0 - 12)\mathbf{i} - (0 - 18t)\mathbf{j} + (12t^2 - 24t^2)\mathbf{k}\\ &= -12\mathbf{i} + 18t\mathbf{j} - 12t^2\mathbf{k}\end{aligned}$$

25. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

(a) $$\begin{aligned}\mathbf{r}'(t) &= \langle -\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t, 1\rangle\\ &= \langle t\cos t, t\sin t, 1\rangle\end{aligned}$$

(b) $\mathbf{r}''(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 0\rangle$

(c) $$\begin{aligned}\mathbf{r}'(t)\cdot\mathbf{r}''(t) &= (t\cos t)(\cos t - t\sin t) + t\sin t(\sin t + t\cos t) + 1(0)\\ &= t\cos^2 t - t^2\cos t\sin t + t\sin^2 t + t^2\sin t\cos t\\ &= t(\cos^2 t + \sin^2 t) = t\end{aligned}$$

(d) $$\begin{aligned}\mathbf{r}'(t)\times\mathbf{r}''(t) &= \begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t\cos t & t\sin t & 1\\ \cos t - t\sin t & \sin t + t\cos t & 0\end{bmatrix}\\ &= (-\sin t - t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j} + (t\cos t\sin t + t^2\cos^2 t - t\sin t\cos t + t^2\sin^2 t)\mathbf{k}\\ &= (-\sin t - t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j} + t^2\mathbf{k}\\ &= \langle -\sin t - t\cos t, \cos t - t\sin t, t^2\rangle\end{aligned}$$

26. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

(a) $\mathbf{r}'(t) = \langle -e^{-t}, 2t, \sec^2 t\rangle$

(b) $\mathbf{r}''(t) = \langle e^{-t}, 2, 2\sec^2 t\tan t\rangle$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = -e^{-2t} + 4t + 2\sec^4 t\tan t$

(d) $$\begin{aligned}\mathbf{r}'(t)\times\mathbf{r}''(t) &= \begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -e^{-t} & 2t & \sec^2 t\\ e^{-t} & 2 & 2\sec^2 t\tan t\end{bmatrix}\\ &= \left[2t(2\sec^2 t\tan t) - 2\sec^2 t\right]\mathbf{i} - \left[-2e^{-t}\sec^2 t\tan t - e^{-t}\sec^2 t\right]\mathbf{j} + \left[-2e^{-t} - 2te^{-t}\right]\mathbf{k}\\ &= \left[4t\sec^2 t\tan t - 2\sec^2 t\right]\mathbf{i} + \left[2e^{-t}\sec^2 t\tan t + e^{-t}\sec^2 t\right]\mathbf{j} - \left[2e^{-t} + 2te^{-t}\right]\mathbf{k}\end{aligned}$$

27. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(0) = \mathbf{0}$

Smooth on $(-\infty, 0), (0, \infty)$

(Note: $t = 0$ corresponds to a cusp at $(x, y) = (0, 0)$.)

28. $\mathbf{r}(t) = 5t^2\mathbf{i} - t^4\mathbf{j}$

$\mathbf{r}'(t) = 25t^4\mathbf{i} - 4t^3\mathbf{j}$

$\mathbf{r}'(0) = \mathbf{0}$

Smooth on $(-\infty, 0), (0, \infty)$

29. $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}$

$\mathbf{r}'(\theta) = -6\cos^2\theta\sin\theta\mathbf{i} + 9\sin^2\theta\cos\theta\mathbf{j}$

$\mathbf{r}'\left(\frac{n\pi}{2}\right) = \mathbf{0}$

Smooth on $\left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right)$, n any integer.

30. $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$

$\mathbf{r}'(\theta) = (1 + \cos\theta)\mathbf{i} + \sin\theta\mathbf{j}$

$\mathbf{r}'((2n-1)\pi) = \mathbf{0}$, n any integer

Smooth on $((2n-1)\pi, (2n+1)\pi)$

31. $\mathbf{r}(t) = \frac{2t}{8+t^3}\mathbf{i} + \frac{2t^2}{8+t^3}\mathbf{j}$

$\mathbf{r}'(t) = \frac{16 - 4t^3}{(t^3+8)^2}\mathbf{i} + \frac{32t - 2t^4}{(t^3+8)^2}\mathbf{j}$

$\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t.

$\mathbf{r}$ is not continuous when $t = -2$.

Smooth on $(-\infty, -2), (-2, \infty)$

32. $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$

$\mathbf{r}'(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + 3\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all t: $(-\infty, \infty)$

33. $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - 3\mathbf{j} + \sec^2 t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t \neq \frac{\pi}{2} + n\pi = \frac{2n+1}{2}\pi$.

Smooth on intervals of form $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$,

n is an integer.

34. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (t^2 - 1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$

$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} + 2t\mathbf{j} + \frac{1}{4}\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t > 0$: $(0, \infty)$

35. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$, $\mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$, $\mathbf{u}'(t) = 4\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

(b) $$\begin{aligned}\frac{d}{dt}[3\mathbf{r}(t) - \mathbf{u}(t)] &= 3\mathbf{r}'(t) - \mathbf{u}'(t)\\ &= 3(\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}) - (4\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\\ &= (3-4)\mathbf{i} + (9-2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}\\ &= -\mathbf{i} + (9-2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}\end{aligned}$$

(c) $$\begin{aligned}\frac{d}{dt}(5t)\mathbf{u}(t) &= (5t)\mathbf{u}'(t) + 5\mathbf{u}(t)\\ &= 5t(4\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) + 5(4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k})\\ &= (20t + 20t)\mathbf{i} + (10t^2 + 5t^2)\mathbf{j} + (15t^3 + 5t^3)\mathbf{k}\\ &= 40t\mathbf{i} + 15t^2\mathbf{j} + 20t^3\mathbf{k}\end{aligned}$$

(d) $$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)] &= \mathbf{r}(t)\cdot\mathbf{u}'(t) + \mathbf{r}'(t)\cdot\mathbf{u}(t)\\ &= [(t)(4) + (3t)(2t) + (t^2)(3t^2)] + [(1)(4t) + (3)(t^2) + (2t)(t^3)]\\ &= (4t + 6t^2 + 3t^4) + (4t + 3t^2 + 2t^4)\\ &= 8t + 9t^2 + 5t^4\end{aligned}$$

(e) $$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{u}(t)] &= \mathbf{r}(t)\times\mathbf{u}'(t) + \mathbf{r}'(t)\times\mathbf{u}(t)\\ &= [7t^3\mathbf{i} + (4t^2 - 3t^3)\mathbf{j} + (2t^2 - 12t)\mathbf{k}] + [t^3\mathbf{i} + (8t^2 - t^3)\mathbf{j} + (t^2 - 12t)\mathbf{k}]\\ &= 8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}\end{aligned}$$

(f) $\frac{d}{dt}\mathbf{r}(2t) = 2\mathbf{r}'(2t)$

$$= 2\left[\mathbf{i} + 3\mathbf{j} + 2(2t)\mathbf{k}\right]$$
$$= 2\mathbf{i} + 6\mathbf{j} + 8t\mathbf{k}$$

36. $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}, \mathbf{u}(t) = \frac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}, \mathbf{u}'(t) = -\frac{1}{t^2}\mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$

(b) $\frac{d}{dt}\left[3\mathbf{r}(t) - \mathbf{u}(t)\right] = 3\mathbf{r}'(t) - \mathbf{u}'(t)$

$$= 3(\mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}) - \left(-\frac{1}{t^2}\mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}\right)$$
$$= \left(3 + \frac{1}{t^2}\right)\mathbf{i} + (6\cos t - 2\cos t)\mathbf{j} + (-6\sin t + 2\sin t)\mathbf{k}$$
$$= \left(3 + \frac{1}{t^2}\right)\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}$$

(c) $\frac{d}{dt}\left[(5t)\mathbf{u}(t)\right] = (5t)\mathbf{u}'(t) + 5\mathbf{u}(t)$

$$= 5t\left(-\frac{1}{t^2}\mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}\right) + 5\left(\frac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}\right)$$
$$= \left(-\frac{5}{t}\mathbf{i} + 10t\cos t\mathbf{j} - 10t\sin t\mathbf{k}\right) + \left(\frac{5}{t}\mathbf{i} + 10\sin t\mathbf{j} + 10\cos t\mathbf{k}\right)$$
$$= 10(t\cos t + \sin t)\mathbf{j} + 10(\cos t - t\sin t)\mathbf{k}$$

(d) $\frac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{u}(t)\right] = \mathbf{r}(t)\cdot\mathbf{u}'(t) + \mathbf{r}'(t)\cdot\mathbf{u}(t)$

$$= \left[(t)\left(-\frac{1}{t^2}\right) + (2\sin t)(2\cos t) + (2\cos t)(-2\sin t)\right] + \left[(1)\left(\frac{1}{t}\right) + (2\cos t)(2\sin t) + (-2\sin t)(2\cos t)\right]$$
$$= \left(-\frac{1}{t} + 4\sin t\cos t - 4\sin t\cos t\right) + \left(\frac{1}{t} + 4\sin t\cos t - 4\sin t\cos t\right)$$
$$= 0(t \neq 0)$$

(e) $\frac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{u}(t)\right] = \mathbf{r}(t)\times\mathbf{u}'(t) + \mathbf{r}'(t)\times\mathbf{u}(t)$

$$= \left[-4\mathbf{i} + \left(-\frac{2}{t^2}\cos t + 2t\sin t\right)\mathbf{j} + \left(2t\cos t + \frac{2}{t^2}\sin t\right)\mathbf{k}\right]$$
$$+ \left[4\mathbf{i} + \left(-\frac{2}{t}\sin t - 2\cos t\right)\mathbf{j} + \left(2\sin t - \frac{2}{t}\cos t\right)\mathbf{k}\right]$$
$$= 2\left[\left(t - \frac{1}{t}\right)\sin t - \left(\frac{1}{t^2} + 1\right)\cos t\right]\mathbf{j} + 2\left[\left(1 + \frac{1}{t^2}\right)\sin t + \left(t - \frac{1}{t}\right)\cos t\right]\mathbf{k}$$

(f) $\frac{d}{dt}\mathbf{r}(2t) = 2\mathbf{r}'(2t)$

$$= 2(\mathbf{i} + 2\cos(2t)\mathbf{j} - 2\sin(2t)\mathbf{k})$$
$$= 2\mathbf{i} + 4\cos(2t)\mathbf{j} - 4\sin(2t)\mathbf{k}$$

37. $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}, \mathbf{u}(t) = t^4\mathbf{k}$

(a) $\mathbf{r}(t) \cdot \mathbf{u}(t) = t^7$

(i) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 7t^6$

(ii) Alternate Solution:

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) = (t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}) \cdot (4t^3\mathbf{k}) + (\mathbf{i} + 4t\mathbf{j} + 3t^2\mathbf{k}) \cdot (t^4\mathbf{k}) = 4t^6 + 3t^6 = 7t^6$$

(b) $$\mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^2 & t^3 \\ 0 & 0 & t^4 \end{vmatrix} = 2t^6\mathbf{i} - t^5\mathbf{j}$$

(i) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 12t^5\mathbf{i} - 5t^4\mathbf{j}$

(ii) Alternate Solution: $$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) \times \mathbf{r}'(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^2 & t^3 \\ 0 & 0 & 4t^3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4t & 3t^2 \\ 0 & 0 & t^4 \end{vmatrix} = 12t^5\mathbf{i} - 5t^4\mathbf{j}$$

38. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$

(a) $\mathbf{r}(t) \cdot \mathbf{u}(t) = \sin t + t^2$

(i) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \cos t + 2t$

(ii) Alternate Solution:

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) \\ &= (\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}) \cdot \mathbf{k} + (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}) \cdot (\mathbf{j} + t\mathbf{k}) = t + \cos t + t = 2t + \cos t \end{aligned}$$

(b) $$\mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & t \\ 0 & 1 & t \end{vmatrix} = (t\sin t - t)\mathbf{i} - (t\cos t)\mathbf{j} + \cos t\mathbf{k}$$

(i) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = (t\cos t + \sin t - 1)\mathbf{i} - (\cos t - t\sin t)\mathbf{j} - \sin t\mathbf{k}$

(ii) Alternate Solution:

$$\begin{aligned} D_t[\mathbf{r}(t) \times \mathbf{u}(t)] &= \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & t \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ 0 & 1 & t \end{vmatrix} = (\sin t + t\cos t - 1)\mathbf{i} + (t\sin t - \cos t)\mathbf{j} - \sin t\mathbf{k} \end{aligned}$$

39. $\int (2t\mathbf{i} + \mathbf{j} + 9\mathbf{k})\,dt = t^2\mathbf{i} + t\mathbf{j} + 9t\mathbf{k} + \mathbf{C}$

40. $\int \left(4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k}\right) dt = t^4\mathbf{i} + 3t^2\mathbf{j} - \frac{8}{3}t^{3/2}\mathbf{k} + \mathbf{C}$

41. $\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k}\right) dt = \ln|t|\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

42. $\int \left[\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right] dt = (t\ln|t| - t)\mathbf{i} + \ln t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

(Integration by parts)

43. $\int \left(\mathbf{i} + 4t^3\mathbf{j} + 5^t\mathbf{k}\right) dt = t\mathbf{i} + t^4\mathbf{j} + \frac{5t}{\ln 5}\mathbf{k} + \mathbf{C}$

44. $\int \left[\sec^2 t\mathbf{i} + \frac{1}{1+t^2}\mathbf{j}\right] dt = \tan t\mathbf{i} + \arctan t\mathbf{j} + \mathbf{C}$

45. $\int \left(e^t\mathbf{i} + \mathbf{j} + t\cos t\mathbf{k}\right) dt = e^t\mathbf{i} + t\mathbf{j} + (\cos t + t\sin t)\mathbf{k} + \mathbf{C}$

(Note: Integration by parts needed for $\int t\cos t\,dt$.)

46. $\int \left(e^{-t} \sin t\,\mathbf{i} + \cot t\,\mathbf{j}\right) dt = \left[-\frac{1}{2}(\cos t + \sin t)e^{-t}\right]\mathbf{i} + \ln\left|\sin t\right|\mathbf{j} + \mathbf{k} + \mathbf{C}$

(Note: Integration by parts needed for the first integral. Also, you could combine the constants $\mathbf{k} + \mathbf{C}$.)

47. $\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k})\, dt = \left[4t^2\mathbf{i}\right]_0^1 + \left[\frac{t^2}{2}\mathbf{j}\right]_0^1 - \left[t\mathbf{k}\right]_0^1 = 4\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$

48. $\int_{-1}^1 \left(t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\mathbf{k}\right) dt = \left[\frac{t^2}{2}\mathbf{i}\right]_{-1}^1 + \left[\frac{t^4}{4}\mathbf{j}\right]_{-1}^1 + \left[\frac{3}{4}t^{4/3}\mathbf{k}\right]_{-1}^1 = \mathbf{0}$

49.
$$\begin{aligned}\int_0^{\pi/2} (5 \cos t\mathbf{i} + 6 \sin t\mathbf{j} + \mathbf{k})\, dt &= \left[5 \sin t\mathbf{i} - 6 \cos t\mathbf{j} + t\mathbf{k}\right]_0^{\pi/2}\\ &= \left(5\mathbf{i} + \frac{\pi}{2}\mathbf{k}\right) - (-6\mathbf{j})\\ &= 5\mathbf{i} + 6\mathbf{j} + \frac{\pi}{2}\mathbf{k}\end{aligned}$$

50. $\int_0^{\pi/4}\left[(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2 \sin t \cos t)\mathbf{k}\right] dt = \left[\sec t\mathbf{i} + \ln\left|\sec t\right|\mathbf{j} + \sin^2 t\mathbf{k}\right]_0^{\pi/4} = \left(\sqrt{2} - 1\right)\mathbf{i} + \ln\sqrt{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$

51. $\int_0^2 \left(t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k}\right) dt = \left[\frac{t^2}{2}\mathbf{i}\right]_0^2 + \left[e^t\mathbf{j}\right]_0^2 - \left[(t-1)\mathbf{e}^t\mathbf{k}\right]_0^2 = 2\mathbf{i} + \left(e^2 - 1\right)\mathbf{j} - \left(e^2 + 1\right)\mathbf{k}$

52. $\left\|t\mathbf{i} + t^2\mathbf{j}\right\| = \sqrt{t^2 + t^4} = t\sqrt{1 + t^2}$ for $t \ge 0$

$\int_0^3 \left\|t\mathbf{i} + t^2\mathbf{j}\right\| dt = \int_0^3 t\sqrt{1 + t^2}\, dt = \left[\frac{1}{3}\left(1 + t^2\right)^{3/2}\right]_0^3 = \frac{1}{3}\left(10^{3/2} - 1\right)$

53.
$$\begin{aligned}\mathbf{r}(t) &= \int \left(4e^{2t}\mathbf{i} + 3e^t\mathbf{j}\right) dt = 2e^{2t}\mathbf{i} + 3e^t\mathbf{j} + \mathbf{C}\\ \mathbf{r}(0) &= 2\mathbf{i} + 3\mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = -3\mathbf{j}\\ \mathbf{r}(t) &= 2e^{2t}\mathbf{i} + 3\left(e^t - 1\right)\mathbf{j}\end{aligned}$$

54.
$$\begin{aligned}\mathbf{r}(t) &= \int \left(3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}\right) dt = t^3\mathbf{j} + 4t^{3/2}\mathbf{k} + \mathbf{C}\\ \mathbf{r}(0) &= C = \mathbf{i} + 2\mathbf{j}\\ \mathbf{r}(t) &= \mathbf{i} + \left(2 + t^3\right)\mathbf{j} + 4t^{3/2}\mathbf{k}\end{aligned}$$

55.
$$\begin{aligned}\mathbf{r}'(t) &= \int -32\mathbf{j}\, dt = -32t\mathbf{j} + \mathbf{C}_1\\ \mathbf{r}'(0) &= \mathbf{C}_1 = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}\\ \mathbf{r}'(t) &= 600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}\\ \mathbf{r}(t) &= \int\left[600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}\right]dt\\ &= 600\sqrt{3}t\mathbf{i} + \left(600t - 16t^2\right)\mathbf{j} + \mathbf{C}\\ \mathbf{r}(0) &= \mathbf{C} = \mathbf{0}\\ \mathbf{r}(t) &= 600\sqrt{3}t\mathbf{i} + \left(600t - 16t^2\right)\mathbf{j}\end{aligned}$$

56.
$$\begin{aligned}\mathbf{r}''(t) &= -4 \cos t\mathbf{j} - 3 \sin t\mathbf{k}\\ \mathbf{r}'(t) &= -4 \sin t\mathbf{j} + 3 \cos t\mathbf{k} + \mathbf{C}_1\\ \mathbf{r}'(0) &= 3\mathbf{k} = 3\mathbf{k} + \mathbf{C}_1 \Rightarrow \mathbf{C}_1 = \mathbf{0}\\ \mathbf{r}(t) &= 4 \cos t\mathbf{j} + 3 \sin t\mathbf{k} + \mathbf{C}_2\\ \mathbf{r}(0) &= 4j + \mathbf{C}_2 = 4\mathbf{j} \Rightarrow \mathbf{C}_2 = 0\\ \mathbf{r}(t) &= 4 \cos t\mathbf{j} + 3 \sin t\mathbf{k}\end{aligned}$$

57.
$$\mathbf{r}(t) = \int\left(te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}\right)dt = -\frac{1}{2}e^{-t^2}\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = -\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{C} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}\mathbf{r}(t) &= \left(1 - \frac{1}{2}e^{-t^2}\right)\mathbf{i} + \left(e^{-t} - 2\right)\mathbf{j} + (t + 1)\mathbf{k}\\ &= \left(\frac{2 - e^{-t^2}}{2}\right)\mathbf{i} + \left(e^{-t} - 2\right)\mathbf{j} + (t + 1)\mathbf{k}\end{aligned}$$

58. $\mathbf{r}(t) = \int \left(\frac{1}{\sqrt{1-t^2}}\mathbf{i} + \frac{1}{t^2}\mathbf{j} + \frac{1}{t}\mathbf{k}\right) dt = \arcsin t\mathbf{i} - \frac{1}{t}\mathbf{j} + \ln t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(1) = \frac{\pi}{2}\mathbf{i} - \mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = \left(2 - \frac{\pi}{2}\right)\mathbf{i} + \mathbf{j}$

$\mathbf{r}(t) = \left(\arcsin t + 2 - \frac{\pi}{2}\right)\mathbf{i} + \left(1 - \frac{1}{t}\right)\mathbf{j} + \ln t\,\mathbf{k}$

59. At $t = t_0$, the graph of $\mathbf{u}(t)$ is increasing in the x, y, and z directions simultaneously.

60. *Sample answer*: $\mathbf{f}(t) = t\mathbf{i} + 3t\mathbf{j}$, $\mathbf{g}(t) = 4\mathbf{i} - t\mathbf{j}$

$\mathbf{f}(t) \cdot \mathbf{g}(t) = 4t - 3t^2$

$\int_0^1 \left[\mathbf{f}(t) \cdot \mathbf{g}(t)\right] dt = \int_0^1 \left(4t - 3t^2\right) dt = \left[2t^2 - t^3\right]_0^1 = 1$

$\int_0^1 \mathbf{f}(t)\, dt = \int_0^1 (t\mathbf{i} + 3t\mathbf{j})\, dt = \left[\frac{t^2}{2}\mathbf{i} + \frac{3t^2}{2}\mathbf{j}\right]_0^1 = \frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$

$\int_0^1 \mathbf{g}(t)\, dt = \int_0^1 (4\mathbf{i} - t\mathbf{j})\, dt = \left[4t\mathbf{i} - \frac{t^2}{2}\mathbf{j}\right]_0^1 = 4\mathbf{i} - \frac{1}{2}\mathbf{j}$

$\left(\int_0^1 \mathbf{f}(t)\, dt\right) \cdot \left(\int_0^1 \mathbf{g}(t)\, dt\right) = 2 - \frac{3}{4} \neq 1$

61. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $c\mathbf{r}(t) = cx(t)\mathbf{i} + cy(t)\mathbf{j} + cz(t)\mathbf{k}$ and

$$\frac{d}{dt}\left[c\mathbf{r}(t)\right] = cx'(t)\mathbf{i} + cy'(t)\mathbf{j} + cz'(t)\mathbf{k}$$
$$= c\left[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}\right] = c\mathbf{r}'(t).$$

62. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$$\mathbf{r}(t) \pm \mathbf{u}(t) = \left[x_1(t) \pm x_2(t)\right]\mathbf{i} + \left[y_1(t) \pm y_2(t)\right]\mathbf{j} + \left[z_1(t) \pm z_2(t)\right]\mathbf{k}$$
$$\frac{d}{dt}\left[\mathbf{r}(t) \pm \mathbf{u}(t)\right] = \left[x_1'(t) \pm x_2'(t)\right]\mathbf{i} + \left[y_1'(t) \pm y_2'(t)\right]\mathbf{j} + \left[z_1'(t) \pm z_2'(t)\right]\mathbf{k}$$
$$= \left[x_1'(t)\mathbf{i} + y_1'(t)\mathbf{j} + z_1'(t)\mathbf{k}\right] \pm \left[x_2'(t)\mathbf{i} + y_2'(t)\mathbf{j} + z_2'(t)\mathbf{k}\right] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$$

63. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $w(t)\mathbf{r}(t) = w(t)x(t)\mathbf{i} + w(t)y(t)\mathbf{j} + w(t)z(t)\mathbf{k}$.

$$\frac{d}{dt}\left[w(t)\mathbf{r}(t)\right] = \left[w(t)x'(t) + w'(t)x(t)\right]\mathbf{i} + \left[w(t)y'(t) + w'(t)y(t)\right]\mathbf{j} + \left[w(t)z'(t) + w'(t)z(t)\right]\mathbf{k}$$
$$= w(t)\left[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}\right] + w'(t)\left[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\right] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$$

64. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$$\mathbf{r}(t) \times \mathbf{u}(t) = \left[y_1(t)z_2(t) - z_1(t)y_2(t)\right]\mathbf{i} - \left[x_1(t)z_2(t) - z_1(t)x_2(t)\right]\mathbf{j} + \left[x_1(t)y_2(t) - y_1(t)x_2(t)\right]\mathbf{k}$$
$$\frac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \left[y_1(t)z_2'(t) + y_1'(t)z_2(t) - z_1(t)y_2'(t) - z_1'(t)y_2(t)\right]\mathbf{i} - \left[x_1(t)z_2'(t) + x_1'(t)z_2(t) - z_1(t)x_2'(t) - z_1'(t)x_2(t)\right]\mathbf{j}$$
$$+ \left[x_1(t)y_2'(t) + x_1'(t)y_2(t) - y_1(t)x_2'(t) - y_1'(t)x_2(t)\right]\mathbf{k}$$
$$= \left\{\left[y_1(t)z_2'(t) - z_1(t)y_2'(t)\right]\mathbf{i} - \left[x_1(t)z_2'(t) - z_1(t)x_2'(t)\right]\mathbf{j} + \left[x_1(t)y_2'(t) - y_1(t)x_2'(t)\right]\mathbf{k}\right\}$$
$$+ \left\{\left[y_1'(t)z_2(t) - z_1'(t)y_2(t)\right]\mathbf{i} - \left[x_1'(t)z_2(t) - z_1'(t)x_2(t)\right]\mathbf{j} + \left[x_1'(t)y_2(t) - y_1'(t)x_2(t)\right]\mathbf{k}\right\}$$
$$= \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$$

65. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}(w(t)) = x(w(t))\mathbf{i} + y(w(t))\mathbf{j} + z(w(t))\mathbf{k}$ and

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{r}(w(t))\right] &= x'(w(t))w'(t)\mathbf{i} + y'(w(t))w'(t)\mathbf{j} + z'(w(t))w'(t)\mathbf{k} \quad \text{(Chain Rule)}\\ &= w'(t)\left[x'(w(t))\mathbf{i} + y'(w(t))\mathbf{j} + z'(w(t))\mathbf{k}\right] = w'(t)\mathbf{r}'(w(t)).\end{aligned}$$

66. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

$$\begin{aligned}\mathbf{r}(t)\times\mathbf{r}'(t) &= \left[y(t)z'(t) - z(t)y'(t)\right]\mathbf{i} - \left[x(t)z'(t) - z(t)x'(t)\right]\mathbf{j} + \left[x(t)y'(t) - y(t)x'(t)\right]\mathbf{k}\\ \frac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] &= \left[y(t)z''(t) + y'(t)z'(t) - z(t)y''(t) - z'(t)y'(t)\right]\mathbf{i} - \left[x(t)z''(t) + x'(t)z'(t) - z(t)x''(t) - z'(t)x'(t)\right]\mathbf{j}\\ &\quad + \left[x(t)y''(t) + x'(t)y'(t) - y(t)x''(t) - y'(t)x'(t)\right]\mathbf{k}\\ &= \left[y(t)z''(t) - z(t)y''(t)\right]\mathbf{i} - \left[x(t)z''(t) - z(t)x''(t)\right]\mathbf{j} + \left[x(t)y''(t) - y(t)x''(t)\right]\mathbf{k} = \mathbf{r}(t)\times\mathbf{r}''(t)\end{aligned}$$

67. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$, $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$, and $\mathbf{v}(t) = x_3(t)\mathbf{i} + y_3(t)\mathbf{j} + z_3(t)\mathbf{k}$. Then:

$$\mathbf{r}(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] = x_1(t)\left[y_2(t)z_3(t) - z_2(t)y_3(t)\right] - y_1(t)\left[x_2(t)z_3(t) - z_2(t)x_3(t)\right] + z_1(t)\left[x_2(t)y_3(t) - y_2(t)x_3(t)\right]$$

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{r}(t)\cdot(\mathbf{u}(t)\times\mathbf{v}(t))\right] &= x_1(t)y_2(t)z_3'(t) + x_1(t)y_2'(t)z_3(t) + x_1'(t)y_2(t)z_3(t) - x_1(t)y_3(t)z_2'(t)\\ &\quad - x_1(t)y_3'(t)z_2(t) - x_1'(t)y_3(t)z_2(t) - y_1(t)x_2(t)z_3'(t) - y_1(t)x_2'(t)z_3(t) - y_1'(t)x_2(t)z_3(t)\\ &\quad + y_1(t)z_2(t)x_3'(t) + y_1(t)z_2'(t)x_3(t) + y_1'(t)z_2(t)x_3(t) + z_1(t)x_2(t)y_3'(t) + z_1(t)x_2'(t)y_3(t)\\ &\quad + z_1'(t)x_2(t)y_3(t) - z_1(t)y_2(t)x_3'(t) - z_1(t)y_2'(t)x_3(t) - z_1'(t)y_2(t)x_3(t)\\ &= \left\{x_1'(t)\left[y_2(t)z_3(t) - y_3(t)z_2(t)\right] + y_1'(t)\left[-x_2(t)z_3(t) + z_2(t)x_3(t)\right] + z_1'(t)\left[x_2(t)y_3(t) - y_2(t)x_3(t)\right]\right\}\\ &\quad + \left\{x_1(t)\left[y_2'(t)z_3(t) - y_3(t)z_2'(t)\right] + y_1(t)\left[-x_2'(t)z_3(t) + z_2'(t)x_3(t)\right] + z_1(t)\left[x_2'(t)y_3(t) - y_2'(t)x_3(t)\right]\right\}\\ &\quad + \left\{x_1(t)\left[y_2(t)z_3'(t) - y_3'(t)z_2(t)\right] + y_1(t)\left[-x_2(t)z_3'(t) + z_2(t)x_3'(t)\right] + z_1(t)\left[x_2(t)y_3'(t) - y_2(t)x_3'(t)\right]\right\}\\ &= \mathbf{r}'(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] + \mathbf{r}(t)\cdot\left[\mathbf{u}'(t)\times\mathbf{v}(t)\right] + \mathbf{r}(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}'(t)\right]\end{aligned}$$

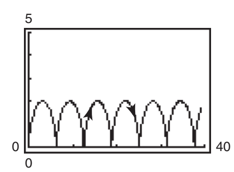
68. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If $\mathbf{r}(t)\cdot\mathbf{r}(t)$ is constant, then:

$$\begin{aligned}x^2(t) + y^2(t) + z^2(t) &= C\\ \frac{d}{dt}\left[x^2(t) + y^2(t) + z^2(t)\right] &= D_t[C]\\ 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) &= 0\\ 2\left[x(t)x'(t) + y(t)y'(t) + z(t)z'(t)\right] &= 0\\ 2\left[\mathbf{r}(t)\cdot\mathbf{r}'(t)\right] &= 0.\end{aligned}$$

So, $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$.

69. $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$

(a)



The curve is a cycloid.

(b) $\mathbf{r}'(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$

$\mathbf{r}''(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$

$$\begin{aligned}\left\|\mathbf{r}'(t)\right\| &= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\\ &= \sqrt{2 - 2\cos t}\end{aligned}$$

Minimum of $\left\|\mathbf{r}'(t)\right\|$ is 0, $(t = 0)$.

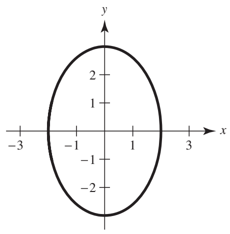
Maximum of $\left\|\mathbf{r}'(t)\right\|$ is 2, $(t = \pi)$.

$\left\|\mathbf{r}''(t)\right\| = \sqrt{\sin^2 t + \cos^2 t} = 1$

Minimum and maximum of $\left\|\mathbf{r}''(t)\right\|$ is 1.

70. $r(t) = 2\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) Ellipse



(b) $\mathbf{r}'(t) = -2\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$\mathbf{r}''(t) = -2\cos t\mathbf{i} - 3\sin t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 9\cos^2 t}$

Minimum of $\|\mathbf{r}'(t)\|$ is 2, $(t = \pi/2)$.

Maximum of $\|\mathbf{r}'(t)\|$ is 3, $(t = 0)$.

$\|r''(t)\| = \sqrt{4\cos^2 t + 9\sin^2 t}$

Minimum of $\|r''(t)\|$ is 2, $(t = 0)$.

Maximum of $\|r''(t)\|$ is 3, $\left(t = \dfrac{\pi}{2}\right)$.

71. $\mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{j}$

$\mathbf{r}'(t) = (e^t\cos t + e^t\sin t)\mathbf{i} + (e^t\cos t - e^t\sin t)\mathbf{j}$

$\mathbf{r}''(t) = (-e^t\sin t + e^t\cos t + e^t\sin t + e^t\cos t)\mathbf{i} + (e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t)\mathbf{j} = 2e^t\cos t\mathbf{i} - 2e^t\sin t\mathbf{j}$

$\mathbf{r}(t)\cdot\mathbf{r}''(t) = 2e^{2t}\sin t\cos t - 2e^{2t}\sin t\cos t = 0$

So, $\mathbf{r}(t)$ is always perpendicular to $\mathbf{r}''(t)$.

72. (a) $t = \dfrac{\pi}{4}$: both components positive

$t = \dfrac{5\pi}{6}$: x-component negative, y-component positive

$t = \dfrac{5\pi}{4}$: x-component positive, y-component negative

(b) No. There is a cusp when $t = 0$, at $(1, 0)$.

73. True

74. False. The definite integral is a vector, not a real number.

75. False. Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$.

$\|\mathbf{r}(t)\| = \sqrt{2}$

$\dfrac{d}{dt}\big[\|\mathbf{r}(t)\|\big] = 0$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 1$

76. False.

$\dfrac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)] = \mathbf{r}(t)\cdot\mathbf{u}'(t) + \mathbf{r}'(t)\cdot\mathbf{u}(t)$

(See Theorem 2.2, part 4)

Section 12.3 Velocity and Acceleration

1. The direction of the velocity vector provides the direction of motion at time t. The magnitude of the velocity vector provides the speed of the object.

2. (a) The acceleration vector points toward the sun. The comet's speed increases as it gets closer to the vertex of the parabola and decreases as it moves farther from the vertex.

(b) The acceleration vector points downward due to gravity.

3. $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}, (3, 0)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{3^2 + 1^2} = \sqrt{10}$

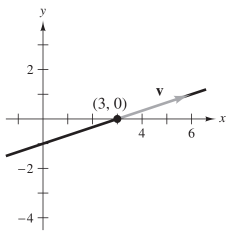
$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

(b) At $(3, 0), t = 1$.

$\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}, \mathbf{a}(1) = \mathbf{0}$

(c) $x = 3t,\ y = t - 1$

$y = \dfrac{x}{3} - 1$, line



4. $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}, (1, 3)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{1 + (-2t)^2} = \sqrt{1 + 4t^2}$

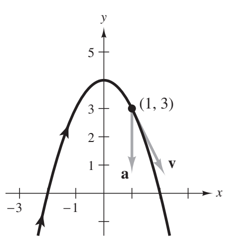
$\mathbf{a}(t) = \mathbf{r}''(t) = -2\mathbf{j}$

(b) At $(1, 3), t = 1$.

$\mathbf{v}(1) = \mathbf{i} - 2\mathbf{j}, \mathbf{a}(1) = -2\mathbf{j}$

(c) $x = t,\ y = 4 - t^2$

$y = 4 - x^2$, parabola



5. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}, (4, 2)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{4t^2 + 1}$

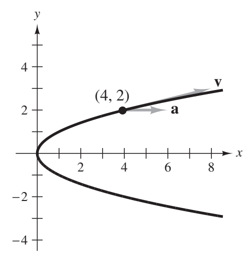
$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}$

(b) At $(4, 2), t = 2$.

$\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}, \mathbf{a}(2) = 2\mathbf{i}$

(c) $x = t^2,\ y = t$

$x = y^2$, parabola



6. $\mathbf{r}(t) = \left(\dfrac{1}{4}t^3 + 1\right)\mathbf{i} + t\mathbf{j}, (3, 2)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \dfrac{3}{4}t^2\mathbf{i} + \mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{\dfrac{9}{16}t^4 + 1}$

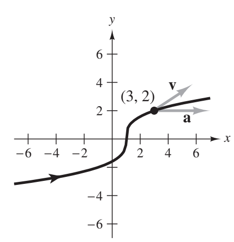
$\mathbf{a}(t) = \dfrac{3}{2}t\mathbf{i}$

(b) At $(3, 2), t = 2$.

$\mathbf{v}(2) = 3\mathbf{i} + \mathbf{j}, \mathbf{a}(2) = 3\mathbf{i}$

(c) $x = \dfrac{1}{4}t^3 + 1,\ y = t$

$x = \dfrac{1}{4}y^3 + 1 \Rightarrow y = \sqrt[3]{4(x - 1)}$



7. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\ \left(\sqrt{2}, \sqrt{2}\right)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$\text{Speed} = \left\|\mathbf{v}(t)\right\| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$

$\mathbf{a}(t) = -2\cos t\mathbf{i} - 2\sin t\,\mathbf{j}$

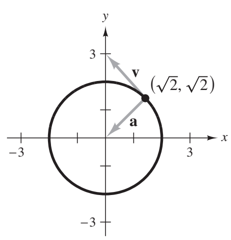
(b) At $\left(\sqrt{2}, \sqrt{2}\right)$, $t = \dfrac{\pi}{4}$.

$\mathbf{v}\left(\dfrac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

$\mathbf{a}\left(\dfrac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$

(c) $x = 2\cos t,\ y = 2\sin t$

$x^2 + y^2 = 4$, circle



8. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j},\ (3, 0)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = -3\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$\text{Speed} = \left\|\mathbf{v}(t)\right\| = \sqrt{9\sin^2 t + 4\cos^2 t}$

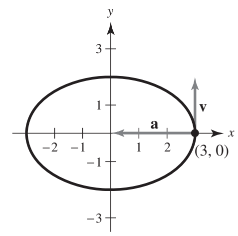
$\mathbf{a}(t) = -3\cos t\mathbf{i} - 2\sin t\mathbf{j}$

(b) At $(3, 0)$, $t = 0$.

$\mathbf{v}(0) = 2\mathbf{j},\ \mathbf{a}(0) = -3\mathbf{i}$

(c) $x = 3\cos t,\ y = 2\sin t$

$\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$, ellipse



9. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle,\ (\pi, 2)$

(a) $\mathbf{v}(t) = \mathbf{v}'(t) = \langle 1 - \cos t, \sin t\rangle$

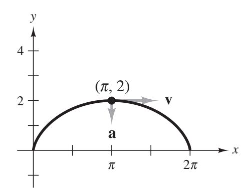
$\text{Speed} = \left\|\mathbf{v}(t)\right\| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$

$\mathbf{a}(t) = \langle \sin t, \cos t\rangle$

(b) At $(\pi, 2)$, $t = \pi$.

$\mathbf{v}(\pi) = \langle 2, 0\rangle,\ \mathbf{a}(\pi) = \langle 0, -1\rangle$

(c) $x = t - \sin t,\ y = 1 - \cos t$



10. $\mathbf{r}(t) = \left\langle e^{-t}, e^t\right\rangle,\ (1, 1)$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle -e^{-t}, e^t\right\rangle$

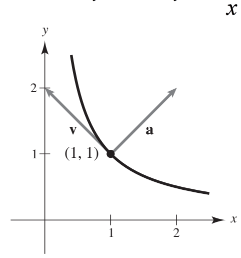
$\text{Speed} = \left\|\mathbf{v}(t)\right\| = \sqrt{\left(-e^{-t}\right)^2 + \left(e^t\right)^2} = \sqrt{e^{-2t} + e^{2t}}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle e^{-t}, e^t\right\rangle$

(b) At $(1, 1)$, $t = 0$.

$\mathbf{v}(0) = \langle -1, 1\rangle,\ \mathbf{a}(0) = \langle 1, 1\rangle$

(c) $x = e^{-t},\ y = e^t,\ y = \dfrac{1}{x}$



11. $\mathbf{r}(t) = t\mathbf{i} + 5t\mathbf{j} + 3t\mathbf{k},\ t = 1$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

$\text{Speed} = \left\|\mathbf{v}(t)\right\| = \sqrt{1^2 + 5^2 + 3^2} = \sqrt{35}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

(b) $\mathbf{v}(1) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

$\mathbf{a}(1) = \mathbf{0}$

12. $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} - 2t\mathbf{k},\ t = 3$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{4^2 + 4^2 + (-2)^2} = \sqrt{36} = 6$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

(b) $\mathbf{v}(3) = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$\mathbf{a}(3) = \mathbf{0}$

13. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k},\ t = 4$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{1 + 4t^2 + t^2} = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{j} + \mathbf{k}$

(b) $\mathbf{v}(4) = \mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$

$\mathbf{a}(4) = 2\mathbf{j} + \mathbf{k}$

14. $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \frac{1}{4}t^2\mathbf{k},\ t = 2$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j} + \frac{1}{2}t\mathbf{k}$

$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{9 + 1 + \frac{1}{4}t^2} = \sqrt{10 + \frac{1}{4}t^2}$

$\mathbf{a}(t) = \frac{1}{2}\mathbf{k}$

(b) $\mathbf{v}(2) = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{a}(2) = \frac{1}{2}\mathbf{k}$

15. $\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + \sqrt{9 - t^2}\mathbf{k},\ t = 0$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} - \mathbf{j} - \frac{t}{\sqrt{9 - t^2}}\mathbf{k}$

$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{1 + 1 + \frac{t^2}{9 - t^2}} = \sqrt{\frac{18 - t^2}{9 - t^2}}$

$\mathbf{a}(t) = \mathbf{r}''(t) = -\frac{9}{(9 - t^2)^{3/2}}\mathbf{k}$

(b) $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$

$\mathbf{a}(0) = -\frac{9}{9^{3/2}}\mathbf{k} = -\frac{1}{3}\mathbf{k}$

16. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k},\ t = 4$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j} + 3\sqrt{t}\,\mathbf{k}$

$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{4t^2 + 1 + 9t}$

$\mathbf{a}(t) = 2\mathbf{i} + \frac{3}{2\sqrt{t}}\mathbf{k}$

(b) $\mathbf{v}(4) = 8\mathbf{i} + \mathbf{j} + 6\mathbf{k}$

$\mathbf{a}(4) = 2\mathbf{i} + \frac{3}{4}\mathbf{k}$

17. $\mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle,\ t = \pi$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 4, -3\sin t, 3\cos t\rangle$

$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{4^2 + (-3\sin t)^2 + (3\cos t)^2} = \sqrt{16 + 9} = 5$

$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, -3\cos t, -3\sin t\rangle$

(b) $\mathbf{v}(\pi) = \langle 4, 0, -3\rangle$

$\mathbf{a}(\pi) = \langle 0, 3, 0\rangle$

18. $\mathbf{r}(t) = \langle 2\cos t, \sin 3t, t^2\rangle,\ t = \frac{\pi}{4}$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin t, 3\cos 3t, 2t\rangle$

$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{(-2\sin t)^2 + (3\cos 3t)^2 + 4t^2} = \sqrt{4\sin^2 t + 9\cos^2 3t + 4t^2}$

$\mathbf{a}(t) = \langle -2\cos t, -9\sin 3t, 2\rangle$

(b) $\mathbf{v}\left(\frac{\pi}{4}\right) = \left\langle -2\left(\frac{\sqrt{2}}{2}\right), 3\left(\frac{-\sqrt{2}}{2}\right), 2\left(\frac{\pi}{4}\right)\right\rangle$

$= \left\langle -\sqrt{2}, -\frac{3}{2}\sqrt{2}, \frac{\pi}{2}\right\rangle$

$\mathbf{a}\left(\frac{\pi}{4}\right) = \left\langle -2\left(\frac{\sqrt{2}}{2}\right), -9\left(\frac{\sqrt{2}}{2}\right), 2\right\rangle$

$= \left\langle -\sqrt{2}, -\frac{9}{2}\sqrt{2}, 2\right\rangle$

19. $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle, t = 0$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, e^t \rangle$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + e^{2t}} = e^t\sqrt{3}$$

$$\begin{aligned}\mathbf{a}(t) &= \mathbf{r}''(t)\\ &= \langle e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t, e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t, e^t \rangle\\ &= \langle -2e^t \sin t, 2e^t \cos t, e^t \rangle\end{aligned}$$

(b) $\mathbf{v}(0) = \langle 1, 1, 1 \rangle$

$\mathbf{a}(0) = \langle 0, 2, 1 \rangle$

20. $\mathbf{r}(t) = \left\langle \ln t, \dfrac{1}{t^2}, t^4 \right\rangle, t = \sqrt{3}$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \dfrac{1}{t}, -\dfrac{2}{t^3}, 4t^3 \right\rangle$

$$\text{speed} = \|\mathbf{v}(t)\| = \sqrt{\frac{1}{t^2} + \frac{4}{t^6} + 16t^6}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle -\frac{1}{t^2}, \frac{6}{t^4}, 12t^2 \right\rangle$$

(b) $\mathbf{v}(\sqrt{3}) = \left\langle \dfrac{1}{\sqrt{3}}, -\dfrac{2}{3\sqrt{3}}, 12\sqrt{3} \right\rangle$

$\mathbf{a}(\sqrt{3}) = \left\langle -\dfrac{1}{3}, \dfrac{2}{3}, 36 \right\rangle$

21. $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v}(0) = \mathbf{0}, \mathbf{r}(0) = \mathbf{0}$

$\mathbf{v}(t) = \int (\mathbf{i} + \mathbf{j} + \mathbf{k})\, dt = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = \mathbf{0}, \mathbf{v}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\mathbf{r}(t) = \int (t\mathbf{i} + t\mathbf{j} + t\mathbf{k})\, dt = \dfrac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}, \mathbf{r}(t) = \dfrac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}),$

$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

22. $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}, \mathbf{v}(0) = 4\mathbf{j}, \mathbf{r}(0) = \mathbf{0}$

$\mathbf{v}(t) = \int (2\mathbf{i} + 3\mathbf{k})\, dt = 2t\mathbf{i} + 3t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = 4\mathbf{j} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k}$

$\mathbf{r}(t) = \int (2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k})\, dt = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 0 \Rightarrow \mathbf{r}(t) = t^2\mathbf{i} + 4t\mathbf{j} + \frac{3}{2}t^2\mathbf{k}$

$\mathbf{r}(2) = 4\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$

23. $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}, \mathbf{v}(1) = 5\mathbf{j}, \mathbf{r}(1) = \mathbf{0}$

$$\mathbf{v}(t) = \int (t\mathbf{j} + t\mathbf{k})\,dt = \frac{t^2}{2}\mathbf{j} + \frac{t^2}{2}\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(1) = \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k} + \mathbf{C} = 5\mathbf{j} \Rightarrow \mathbf{C} = \frac{9}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$$

$$\mathbf{v}(t) = \left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k}$$

$$\mathbf{r}(t) = \int\left[\left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k}\right]dt = \left(\frac{t^3}{6} + \frac{9}{2}t\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t\right)\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(1) = \frac{14}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = -\frac{14}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\mathbf{r}(t) = \left(\frac{t^3}{6} + \frac{9}{2}t - \frac{14}{3}\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t + \frac{1}{3}\right)\mathbf{k}$$

$$\mathbf{r}(2) = \frac{17}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

24. $\mathbf{a}(t) = -32\mathbf{k}, \mathbf{v}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \mathbf{r}(0) = 5\mathbf{j} + 2\mathbf{k}$

$$\mathbf{v}(t) = \int -32\mathbf{k}\,dt = -32t\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{C} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 3\mathbf{i} - 2\mathbf{j} + (1 - 32t)\mathbf{k}$$

$$\mathbf{r}(t) = \int\left[3\mathbf{i} - 2\mathbf{j} + (1 - 32t)\mathbf{k}\right]dt = 3t\mathbf{i} - 2t\mathbf{j} + \left(t - 16t^2\right)\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{C} = 5\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{r}(t) = 3t\mathbf{i} + (5 - 2t)\mathbf{j} + \left(2 + t - 16t^2\right)\mathbf{k}$$

$$\mathbf{r}(2) = 6\mathbf{i} + \mathbf{j} - 60\mathbf{k}$$

25. $\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \mathbf{v}(0) = \mathbf{j} + \mathbf{k}, \mathbf{r}(0) = \mathbf{i}$

$$\mathbf{v}(t) = \int(-\cos t\mathbf{i} - \sin t\mathbf{j})\,dt = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{j} + \mathbf{C} = \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{k}$$

$$\mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}(t) = \int(-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k})\,dt = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{i} \Rightarrow \mathbf{C} = \mathbf{0}$$

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}(2) = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j} + 2\mathbf{k}$$

26. $\mathbf{a}(t) = e^t\mathbf{i} - 8\mathbf{k}, \mathbf{v}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \mathbf{r}(0) = \mathbf{0}$

$$\mathbf{v}(t) = \int\left(e^t\mathbf{i} - 8\mathbf{k}\right)dt = e^t\mathbf{i} - 8t\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{i} + \mathbf{C} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\mathbf{v}(t) = \left(e^t + 1\right)\mathbf{i} + 3\mathbf{j} + (1 - 8t)\mathbf{k}$$

$$\mathbf{r}(t) = \int\left[\left(e^t + 1\right)\mathbf{i} + 3\mathbf{j} + (1 - 8t)\mathbf{k}\right]dt$$

$$= \left(e^t + t\right)\mathbf{i} + 3t\mathbf{j} + \left(t - 4t^2\right)\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = -\mathbf{i}$$

$$\mathbf{r}(t) = \left(e^t + t - 1\right)\mathbf{i} + 3t\mathbf{j} + \left(t - 4t^2\right)\mathbf{k}$$

27. $\mathbf{r}(t) = 140(\cos 22°)t\mathbf{i} + \left(2.5 + 140(\sin 22°)t - 16t^2\right)\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 140(\cos 22°)\mathbf{i} + \left(140(\sin 22°) - 32t\right)\mathbf{j}$

The maximum height occurs when

$$y'(t) = 140(\sin 22°) - 32t = 0 \Rightarrow t = \frac{140 \sin 22°}{32} = \frac{35}{8}\sin 22° \approx 1.639.$$

The maximum height is

$$y = 2.5 + 140(\sin 22°)\left(\frac{35}{8}\sin 22°\right) - 16\left(\frac{35}{8}\sin 22°\right)^2 \approx 45.5 \text{ feet.}$$

When $x = 375$, $t = \dfrac{375}{140 \cos 22°} \approx 2.889.$

For this value of t, $y \approx 20.47$ feet.

So the ball clears the 10-foot fence.

28. $\mathbf{r}(t) = (900 \cos 45°)t\mathbf{i} + \left[3 + (900 \sin 45°)t - 16t^2\right]\mathbf{j} = 450\sqrt{2}t\mathbf{i} + \left(3 + 450\sqrt{2}t - 16t^2\right)\mathbf{j}$

The maximum height occurs when $y'(t) = 450\sqrt{2} - 32t = 0$, which implies that $t = \left(225\sqrt{2}\right)/16.$

The maximum height reached by the projectile is

$$y = 3 + 450\sqrt{2}\left(\frac{225\sqrt{2}}{16}\right) - 16\left(\frac{225\sqrt{2}}{16}\right)^2 = \frac{50{,}649}{8} = 6331.125 \text{ feet.}$$

The range is determined by setting $y(t) = 3 + 450\sqrt{2}t - 16t^2 = 0$ which implies that

$$t = \frac{-450\sqrt{2} - \sqrt{405{,}192}}{-32} \approx 39.779 \text{ seconds}$$

$$\text{Range: } x = 450\sqrt{2}\left(\frac{-450\sqrt{2} - \sqrt{405{,}192}}{-32}\right) \approx 25{,}315.500 \text{ feet}$$

29. $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \dfrac{1}{2}gt^2\right]\mathbf{j} = \dfrac{v_0}{\sqrt{2}}t\mathbf{i} + \left(3 + \dfrac{v_0}{\sqrt{2}}t - 16t^2\right)\mathbf{j}$

$\dfrac{v_0}{\sqrt{2}}t = 300$ when $3 + \dfrac{v_0}{\sqrt{2}}t - 16t^2 = 3.$

$$t = \frac{300\sqrt{2}}{v_0},\ \frac{v_0}{\sqrt{2}}\left(\frac{300\sqrt{2}}{v_0}\right) - 16\left(\frac{300\sqrt{2}}{v_0}\right)^2 = 0,\ 300 - \frac{300^2(32)}{v_0^2} = 0$$

$$v_0^2 = 300(32),\ v_0 = \sqrt{9600} = 40\sqrt{6},\ v_0 = 40\sqrt{6} \approx 97.98 \text{ ft/sec}$$

The maximum height is reached when the derivative of the vertical component is zero.

$$y(t) = 3 + \frac{tv_0}{\sqrt{2}} - 16t^2 = 3 + \frac{40\sqrt{6}}{\sqrt{2}}t - 16t^2 = 3 + 40\sqrt{3}t - 16t^2$$

$$y'(t) = 40\sqrt{3} - 32t = 0$$

$$t = \frac{40\sqrt{3}}{32} = \frac{5\sqrt{3}}{4}$$

$$\text{Maximum height: } y\left(\frac{5\sqrt{3}}{4}\right) = 3 + 40\sqrt{3}\left(\frac{5\sqrt{3}}{4}\right) - 16\left(\frac{5\sqrt{3}}{4}\right)^2 = 78 \text{ feet}$$

30. $50 \text{ mi/h} = \dfrac{220}{3} \text{ ft/sec}$

$$\mathbf{r}(t) = \left(\frac{220}{3}\cos 15°\right)t\mathbf{i} + \left[5 + \left(\frac{220}{3}\sin 15°\right)t - 16t^2\right]\mathbf{j}$$

The ball is 90 feet from where it is thrown when $x = \dfrac{220}{3}\cos 15° t = 90 \Rightarrow t = \dfrac{27}{22\cos 15°} \approx 1.2706$ seconds.

The height of the ball at this time is $y = 5 + \left(\dfrac{220}{3}\sin 15°\right)\left(\dfrac{27}{22\cos 15°}\right) - 16\left(\dfrac{27}{22\cos 15°}\right)^2 \approx 3.286$ feet.

31. $x(t) = t(v_0 \cos\theta)$ or $t = \dfrac{x}{v_0\cos\theta}$

$y(t) = t(v_0\sin\theta) - 16t^2 + h$

$$y = \frac{x}{v_0\cos\theta}(v_0\sin\theta) - 16\left(\frac{x^2}{v_0^2\cos^2\theta}\right) + h = (\tan\theta)x - \left(\frac{16}{v_0^2}\sec^2\theta\right)x^2 + h$$

32. $y = x - 0.005x^2$

From Exercise 31 you know that $\tan\theta$ is the coefficient of x. So, $\tan\theta = 1$, $\theta = (\pi/4)\text{ rad} = 45°$. Also

$\dfrac{16}{v_0^2}\sec^2\theta =$ negative of coefficient of x^2

$\dfrac{16}{v_0^2}(2) = 0.005$ or $v_0 = 80$ ft/sec

$\mathbf{r}(t) = \left(40\sqrt{2}t\right)\mathbf{i} + \left(40\sqrt{2}t - 16t^2\right)\mathbf{j}$. Position function

When $40\sqrt{2}t = 60$,

$$t = \frac{60}{40\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

$$\mathbf{v}(t) = 40\sqrt{2}\mathbf{i} + \left(40\sqrt{2} - 32t\right)\mathbf{j}$$

$$\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right) = 40\sqrt{2}\mathbf{i} + \left(40\sqrt{2} - 24\sqrt{2}\right)\mathbf{j}$$

$$= 8\sqrt{2}(5\mathbf{i} + 2\mathbf{j}).\text{ Direction}$$

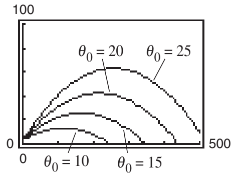
$$\text{Speed} = \left\|\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right)\right\| = 8\sqrt{2}\sqrt{25 + 4} = 8\sqrt{58}\text{ ft/sec}$$

33. $100 \text{ mi/h} = \left(100\frac{\text{mi}}{\text{hr}}\right)\left(5280\frac{\text{ft}}{\text{mi}}\right)\Big/\left(3600\frac{\text{sec}}{\text{hr}}\right) = \frac{440}{3}\text{ft/sec}$

(a) $\mathbf{r}(t) = \left(\frac{440}{3}\cos\theta_0\right)t\mathbf{i} + \left[3 + \left(\frac{440}{3}\sin\theta_0\right)t - 16t^2\right]\mathbf{j}$

Graphing these curves together with $y = 10$ shows that $\theta_0 = 20°$.

(b)



(c) You want

$x(t) = \left(\frac{440}{3}\cos\theta\right)t \geq 400$ and $y(t) = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 \geq 10.$

From $x(t)$, the minimum angle occurs when $t = 30/(11\cos\theta)$. Substituting this for t in $y(t)$ yields:

$$
\begin{aligned}
3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{30}{11\cos\theta}\right) - 16\left(\frac{30}{11\cos\theta}\right)^2 &= 10\\
400\tan\theta - \frac{14{,}400}{121}\sec^2\theta &= 7\\
\frac{14{,}400}{121}\left(1 + \tan^2\theta\right) - 400\tan\theta + 7 &= 0\\
14{,}400\tan^2\theta - 48{,}400\tan\theta + 15{,}247 &= 0\\
\tan\theta &= \frac{48{,}400 \pm \sqrt{48{,}400^2 - 4(14{,}400)(15{,}247)}}{2(14{,}400)}\\
\theta &= \tan^{-1}\left(\frac{48{,}400 - \sqrt{1{,}464{,}332{,}800}}{28{,}800}\right) \approx 19.38°
\end{aligned}
$$

34. $h = 7$ feet, $\theta = 35°$, 30 yards $= 90$ feet

$\mathbf{r}(t) = (v_0\cos 35°)t\mathbf{i} + \left[7 + (v_0\sin 35°)t - 16t^2\right]\mathbf{j}$

(a) $v_0\cos 35°t = 90$ when $7 + (v_0\sin 35°)t - 16t^2 = 4$

$$
\begin{aligned}
t &= \frac{90}{v_0\cos 35°}\\
7 + (v_0\sin 35°)\left(\frac{90}{v_0\cos 35°}\right) - 16\left(\frac{90}{v_0\cos 35°}\right)^2 &= 4\\
90\tan 35° + 3 &= \frac{129{,}600}{v_0^2\cos^2 35°}\\
v_0^2 &= \frac{129{,}600}{\cos^2 35°(90\tan 35° + 3)}\\
v_0 &\approx 54.088 \text{ ft/sec}
\end{aligned}
$$

(b) The maximum height occurs when $y'(t) = v_0\sin 35° - 32t = 0.$

$t = \frac{v_0\sin 35}{32} \approx 0.969 \text{ sec}$

At this time, the height is $y(0.969) \approx 22.0$ ft.

(c) $x(t) = 90 \Rightarrow (v_0\cos 35°)t = 90$

$t = \frac{90}{54.088\cos 35°} \approx 2.0 \text{ sec}$

35. $\mathbf{r}(t) = (v\cos\theta)t\mathbf{i} + \left[(v\sin\theta)t - 16t^2\right]\mathbf{j}$

(a) You want to find the minimum initial speed v as a function of the angle θ. Because the bale must be thrown to the position $(16, 8)$, you have

$$16 = (v\cos\theta)t$$
$$8 = (v\sin\theta)t - 16t^2.$$

$t = 16/(v\cos\theta)$ from the first equation. Substituting into the second equation and solving for v, you obtain:

$$8 = (v\sin\theta)\left(\frac{16}{v\cos\theta}\right) - 16\left(\frac{16}{v\cos\theta}\right)^2$$
$$1 = 2\left(\frac{\sin\theta}{\cos\theta}\right) - 512\left(\frac{1}{v^2\cos^2\theta}\right)$$
$$512\left(\frac{1}{v^2\cos^2\theta}\right) = 2\left(\frac{\sin\theta}{\cos\theta}\right) - 1$$
$$\frac{1}{v^2} = \left(2\frac{\sin\theta}{\cos\theta} - 1\right)\frac{\cos^2\theta}{512} = \frac{2\sin\theta\cos\theta - \cos^2\theta}{512}$$
$$v^2 = \frac{512}{2\sin\theta\cos\theta - \cos^2\theta}$$

You minimize $f(\theta) = \dfrac{512}{2\sin\theta\cos\theta - \cos^2\theta}$.

$$f'(\theta) = -512\left(\frac{2\cos^2\theta - 2\sin^2\theta + 2\sin\theta\cos\theta}{\left(2\sin\theta\cos\theta - \cos^2\theta\right)^2}\right)$$
$$f'(\theta) = 0 \Rightarrow 2\cos(2\theta) + \sin(2\theta) = 0$$
$$\tan(2\theta) = -2$$
$$\theta \approx 1.01722 \approx 58.28°$$

Substituting into the equation for v, $v \approx 28.78$ ft/sec.

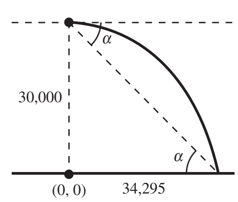
(b) If $\theta = 45°$,

$$16 = (v\cos\theta)t = v\frac{\sqrt{2}}{2}t$$
$$8 = (v\sin\theta)t - 16t^2 = v\frac{\sqrt{2}}{2}t - 16t^2$$

From part (a), $v^2 = \dfrac{512}{2\left(\sqrt{2}/2\right)\left(\sqrt{2}/2\right) - \left(\sqrt{2}/2\right)^2} = \dfrac{512}{1/2} = 1024 \Rightarrow v = 32$ ft/sec.

36. Place the origin directly below the plane. Then $\theta = 0$, $v_0 = 792$ and

$$\mathbf{r}(t) = (v_0\cos\theta)t\mathbf{i} + \left(30{,}000 + (v_0\sin\theta)t - 16t^2\right)\mathbf{j} = 792t\mathbf{i} + \left(30{,}000 - 16t^2\right)\mathbf{j}$$
$$\mathbf{v}(t) = 792\mathbf{i} - 32t\mathbf{j}.$$



At time of impact, $30{,}000 - 16t^2 = 0 \Rightarrow t^2 = 1875 \Rightarrow t \approx 43.3$ seconds.

$$\mathbf{r}(43.3) = 34{,}294.6\mathbf{i}$$
$$\mathbf{v}(43.3) = 792\mathbf{i} - 1385.6\mathbf{j}$$
$$\left\|\mathbf{v}(43.3)\right\| = 1596 \text{ ft/sec} = 1088 \text{ mi/h}$$
$$\tan\alpha = \frac{30{,}000}{34{,}294.6} \approx 0.8748 \Rightarrow \alpha \approx 0.7187(41.18°)$$

37. $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[(v_0 \sin\theta)t - 16t^2\right]\mathbf{j}$

$(v_0 \sin\theta)t - 16t^2 = 0$ when $t = 0$ and $t = \dfrac{v_0 \sin\theta}{16}$.

The range is

$$x = (v_0 \cos\theta)t = (v_0 \cos\theta)\frac{v_0 \sin\theta}{16} = \frac{v_0^2}{32}\sin 2\theta.$$

So,

$$x = \frac{1200^2}{32}\sin(2\theta) = 3000 \Rightarrow \sin 2\theta = \frac{1}{15} \Rightarrow \theta \approx 1.91°.$$

38. From Exercise 37, the range is

$$x = \frac{v_0^2}{32}\sin 2\theta$$

So, $x = 200 = \dfrac{v_0^2}{32}\sin(24°)$

$\Rightarrow v_0^2 = 6400/\sin(24°)$

$\Rightarrow v_0 \approx 125.4$ ft/sec

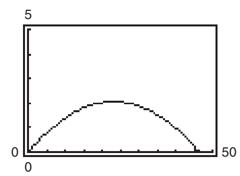
39. (a) $\theta = 10°$, $v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 10°)t\mathbf{i} + \left[0 + (66 \sin 10°)t - 16t^2\right]\mathbf{j}$

$\mathbf{r}(t) \approx (65t)\mathbf{i} + (11.46t - 16t^2)\mathbf{j}$

Maximum height: 2.052 feet

Range: 46.557 feet



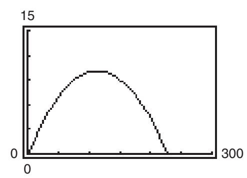
(b) $\theta = 10°$, $v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 10°)t\mathbf{i} + \left[0 + (146 \sin 10°)t - 16t^2\right]\mathbf{j}$

$\mathbf{r}(t) \approx (143.78t)\mathbf{i} + (25.35t - 16t^2)\mathbf{j}$

Maximum height: 10.043 feet

Range: 227.828 feet



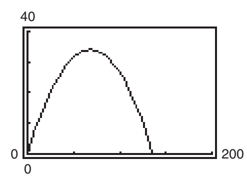
(c) $\theta = 45°$, $v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 45°)t\mathbf{i} + \left[0 + (66 \sin 45°)t - 16t^2\right]\mathbf{j}$

$\mathbf{r}(t) \approx (46.67t)\mathbf{i} + (46.67t - 16t^2)\mathbf{j}$

Maximum height: 34.031 feet

Range: 136.125 feet



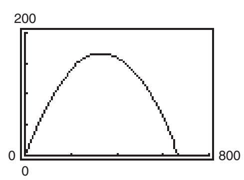
(d) $\theta = 45°$, $v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 45°)t\mathbf{i} + \left[0 + (146 \sin 45°)t - 16t^2\right]\mathbf{j}$

$\mathbf{r}(t) \approx (103.24t)\mathbf{i} + (103.24t - 16t^2)\mathbf{j}$

Maximum height: 166.531 feet

Range: 666.125 feet



(e) $\theta = 60°$, $v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 60°)t\mathbf{i} + \left[0 + (66 \sin 60°)t - 16t^2\right]\mathbf{j}$

$\mathbf{r}(t) \approx (33t)\mathbf{i} + (57.16t - 16t^2)\mathbf{j}$

Maximum height: 51.047 feet

Range: 117.888 feet

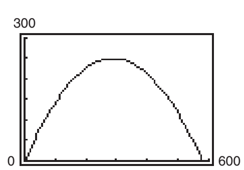
(f) $\theta = 60°$, $v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 60°)t\mathbf{i} + \left[0 + (146 \sin 60°)t - 16t^2\right]\mathbf{j}$

$\mathbf{r}(t) \approx (73t)\mathbf{i} + (126.44t - 16t^2)\mathbf{j}$

Maximum height: 249.797 feet

Range: 576.881 feet



40. (a) $\mathbf{r}(t) = t(v_0 \cos\theta)\mathbf{i} + (tv_0 \sin\theta - 16t^2)\mathbf{j}$

$t(v_0 \sin\theta - 16t) = 0$ when $t = \dfrac{v_0 \sin\theta}{16}$.

Range: $x = v_0 \cos\theta\left(\dfrac{v_0 \sin\theta}{32}\right) = \left(\dfrac{v_0^2}{32}\right)\sin 2\theta$

The range will be maximum when

$$\frac{dx}{dt} = \left(\frac{v_0^2}{32}\right)2\cos 2\theta = 0$$

or

$$2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4} \text{ rad.}$$

(b) $y(t) = tv_0 \sin\theta - 16t^2$

$\dfrac{dy}{dt} = v_0 \sin\theta - 32t = 0$ when $t = \dfrac{v_0 \sin\theta}{32}$.

Maximum height:

$$y\left(\frac{v_0 \sin\theta}{32}\right) = \frac{v_0^2 \sin^2\theta}{32} - 16\frac{v_0^2 \sin^2\theta}{32^2} = \frac{v_0^2 \sin^2\theta}{64}$$

Maximum height when $\sin\theta = 1$, or $\theta = \dfrac{\pi}{2}$.

41. $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - 4.9t^2\right]\mathbf{j}$

$= (100\cos 30°)t\mathbf{i} + \left[1.5 + (100 \sin 30°)t - 4.9t^2\right]\mathbf{j}$

The projectile hits the ground when

$-4.9t^2 + 100\left(\frac{1}{2}\right)t + 1.5 = 0 \Rightarrow t \approx 10.234$ seconds.

So the range is $(100\cos 30°)(10.234) \approx 886.3$ meters.

The maximum height occurs when $dy/dt = 0$.

$100 \sin 30 = 9.8t \Rightarrow t \approx 5.102$ sec

The maximum height is

$y = 1.5 + (100 \sin 30°)(5.102) - 4.9(5.102)^2$

≈ 129.1 meters.

42. $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - 4.9t^2\right]\mathbf{j}$

$= (v_0 \cos 8°)t\mathbf{i} + \left[(v_0 \sin 8°)t - 4.9t^2\right]\mathbf{j}$

$x = 50$ when $(v_0 \cos 8°)t = 50 \Rightarrow t = \dfrac{50}{v_0 \cos 8°}$. For this value of t, $y = 0$:

$$(v_0 \sin 8°)\left(\frac{50}{v_0 \cos 8°}\right) - 4.9\left(\frac{50}{v_0 \cos 8°}\right)^2 = 0$$

$$50 \tan 8° = \frac{(4.9)(2500)}{v_0^2 \cos^2 8°} \Rightarrow v_0^2 = \frac{(4.9)50}{\tan 8° \cos^2 8°} \approx 1777.698 \Rightarrow v_0 \approx 42.2 \text{ m/sec}$$

43. To find the range, set $y(t) = h + (v_0 \sin\theta)t - \dfrac{1}{2}gt^2 = 0$ then $0 = \left(\dfrac{1}{2}g\right)t^2 - (v_0 \sin\theta)t - h$. By the Quadratic Formula, (discount the negative value)

$$t = \frac{v_0 \sin\theta + \sqrt{(-v_0 \sin\theta)^2 - 4[(1/2)g](-h)}}{2[(1/2)g]} = \frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gh}}{g} \text{ second.}$$

At this time,

$$x(t) = v_0 \cos\theta\left(\frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gh}}{g}\right)$$

$$= \frac{v_0 \cos\theta}{g}\left(v_0 \sin\theta + \sqrt{v_0^2\left(\sin^2\theta + \frac{2gh}{v_0^2}\right)}\right)$$

$$= \frac{v_0^2 \cos\theta}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2gh}{v_0^2}}\right) \text{ feet.}$$

44. $h = 5.75,\ v_0 = 41,\ \theta = 42.5°,\ g = 32$

From Exercise 43, the total time of travel is

$$t = \frac{41 \sin 42.5° + \sqrt{41^2 \sin^2 42.5 + 2(32)(5.75)}}{32} \approx 1.92 \text{ seconds}$$

The total horizontal distance is

$$x(t) = \frac{41^2 \cos 42.5°}{32}\left(\sin 42.5° + \sqrt{\sin^2 42.5° + \frac{2(32)(5.75)}{41^2}}\right)$$

≈ 58 feet.

45. $\mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$

$\mathbf{v}(t) = b(\omega - \omega \cos \omega t)\mathbf{i} + b\omega \sin \omega t\mathbf{j} = b\omega(1 - \cos \omega t)\mathbf{i} + b\omega \sin \omega t\mathbf{j}$

$\mathbf{a}(t) = \left(b\omega^2 \sin \omega t\right)\mathbf{i} + \left(b\omega^2 \cos \omega t\right)\mathbf{j} = b\omega^2\left[\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}\right]$

$\|\mathbf{v}(t)\| = \sqrt{2}b\omega\sqrt{1 - \cos(\omega t)}$

$\|\mathbf{a}(t)\| = b\omega^2$

(a) $\|\mathbf{v}(t)\| = 0$ when $\omega t = 0, 2\pi, 4\pi, \ldots$.

(b) $\|\mathbf{v}(t)\|$ is maximum when $\omega t = \pi, 3\pi, \ldots$, then $\|\mathbf{v}(t)\| = 2b\omega$.

46. $\mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$

$\mathbf{v}(t) = b\omega\left[(1 - \cos \omega t)\mathbf{i} + (\sin \omega t)\mathbf{j}\right]$

Speed $= \|\mathbf{v}(t)\| = b\omega\sqrt{1 - 2\cos \omega t + \cos^2 \omega t + \sin^2 \omega t} = \sqrt{2}\, b\omega\sqrt{1 - \cos \omega t}$.

The speed has a maximum value of $2b\omega$ when $\omega t = \pi, 3\pi, \ldots$

$60 \text{ mi/h} = 88 \text{ ft/sec} = 88 \text{ rad/sec}$ (since $b = 1$).

So, the maximum speed of a point on the tire is twice the speed of the car:
$2(88) \text{ ft/sec} = 120 \text{ mi/h}$

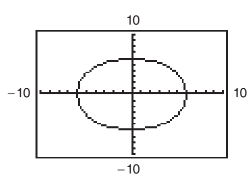
47. $\mathbf{v}(t) = -b\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$

$\mathbf{r}(t) \cdot \mathbf{v}(t) = -b^2\omega \sin(\omega t)\cos(\omega t) + b^2\omega \sin(vt)\cos(\omega t) = 0$

So, $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are orthogonal.

48. (a) Speed $= \|\mathbf{v}\| = \sqrt{b^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)} = \sqrt{b^2\omega^2\left[\sin^2(\omega t) + \cos^2(\omega t)\right]} = b\omega$

(b)



The graphing utility draws the circle faster for greater values of ω.

49. $\mathbf{a}(t) = -b\omega^2 \cos(\omega t)\mathbf{i} - b\omega^2 \sin(\omega t)\mathbf{j} = -b\omega^2\left[\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\right] = -\omega^2\mathbf{r}(t)$

$\mathbf{a}(t)$ is a negative multiple of a unit vector from $(0, 0)$ to $(\cos \omega t, \sin \omega t)$ and so $\mathbf{a}(t)$ is directed toward the origin.

50. $\|\mathbf{a}(t)\| = b\omega^2\|\cos(\omega t)\mathbf{i} + \sin(vt)\mathbf{j}\| = b\omega^2$

51. $\|\mathbf{a}(t)\| = \omega^2 b,\ b = \dfrac{1}{2}$

$$\mathbf{F} = m\mathbf{a} \Rightarrow \frac{1}{4} = m(32) \Rightarrow m = \frac{1}{4(32)}$$

$$2 = m(\omega^2 b) = \frac{1}{4(32)}\left(\frac{1}{2}\omega^2\right) \Rightarrow \omega^2 = 512 \Rightarrow \omega = 16\sqrt{2}$$

$$\|\mathbf{v}(t)\| = b\omega = \frac{1}{2}\left(16\sqrt{2}\right) = 8\sqrt{2} \text{ ft/sec}$$

52. $\|\mathbf{v}(t)\| = 30 \text{ mi/h} = 44 \text{ ft/sec}$

$$\omega = \frac{\|\mathbf{v}(t)\|}{b} = \frac{44}{300} \text{ rad/sec}$$

$$\|a(t)\| = b\omega^2$$

$$\|\mathbf{F}\| = m(b\omega^2) = \frac{3400}{32}(300)\left(\frac{44}{300}\right)^2 = \frac{2057}{3}\text{lb}$$

Let n be normal to the road.

$$\|\mathbf{n}\|\cos\theta = 3400$$

$$\|\mathbf{n}\|\sin\theta = \frac{2057}{3}$$

Dividing, $\tan\theta = \dfrac{121}{600}$

$$\theta \approx 11.4°$$

53. The particle could be changing direction. For example, let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. Then the speed is $\|\mathbf{r}(t)\| = 1$, but $\mathbf{a}(t)$ is not constant.

54. Let $\mathbf{r}_2(t) = 4\mathbf{r}_1(t)$. Then $\mathbf{r}_2'(t) = 4\mathbf{r}_1'(t)$ and $\|\mathbf{v}_2(t)\| = \|\mathbf{r}_2'(t)\| = 4\|\mathbf{r}_1'(t)\| = 4\|\mathbf{v}_1(t)\|$.

55. No. This is true for uniform circular motion only.

56. $\mathbf{r}(t) = a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}$

(a) $\mathbf{r}'(t) = \mathbf{v}(t) = -a\omega\sin\omega t\mathbf{i} + b\omega\cos\omega t\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t}$

(b)
$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) &= -a\omega^2\cos\omega t\mathbf{i} - b\omega^2\sin\omega t\mathbf{j}\\ &= \omega^2(-a\cos\omega t\mathbf{i} - b\sin\omega t\mathbf{j})\\ &= -\omega^2\mathbf{r}(t)\end{aligned}$$

57.
$$\begin{aligned}\mathbf{a}(t) &= \sin t\mathbf{i} - \cos t\mathbf{j}\\ \mathbf{v}(t) &= \int \mathbf{a}(t)\,dt = -\cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{C}_1\\ \mathbf{v}(0) &= -\mathbf{i} = -\mathbf{i} + \mathbf{C}_1 \Rightarrow \mathbf{C}_1 = \mathbf{0}\\ \mathbf{v}(t) &= -\cos t\mathbf{i} - \sin t\mathbf{j}\\ \mathbf{r}(t) &= \int \mathbf{v}(t)\,dt = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{C}_2\\ \mathbf{r}(0) &= \mathbf{j} = \mathbf{j} + \mathbf{C}_2 \Rightarrow \mathbf{C}_2 = \mathbf{0}\\ \mathbf{r}(t) &= -\sin t\mathbf{i} + \cos t\mathbf{j}\end{aligned}$$

The path is a circle.

58. The angle at time t_1 is obtuse.

The angle at time t_2 is acute.

The speed is decreasing at time t_1 because the projectile is reaching its maximum height.

The speed is increasing at time t_2 because the object is accelerating due to gravity.

59. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ Position vector

$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ Velocity vector

$\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$ Acceleration vector

$$\begin{aligned}\text{Speed} = \|\mathbf{v}(t)\| &= \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\\ &= C,\ C \text{ is a constant.}\\ \frac{d}{dt}\left[x'(t)^2 + y'(t)^2 + z'(t)^2\right] &= 0\\ 2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t) &= 0\\ 2\left[x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)\right] &= 0\\ \mathbf{v}(t)\cdot\mathbf{a}(t) &= 0\end{aligned}$$

Orthogonal

60. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$y(t) = m(x(t)) + b$, m and b are constants.

$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$

$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$

$\mathbf{s}(t) = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = C$, C is a constant.

So, $x'(t) = \dfrac{C}{\sqrt{1 + m^2}}$

$x''(t) = 0$

$\mathbf{a}(t) = x''(t)\mathbf{i} + mx''(t)\mathbf{j} = \mathbf{0}$.

61. True

62. False. For example, $\mathbf{r}(t) = t^3\mathbf{i}$. Then $\mathbf{v}(t) = 3t^2\mathbf{i}$ and $\mathbf{a}(t) = 6t\mathbf{i}$. $\mathbf{v}(t)$ is not orthogonal to $\mathbf{a}(t)$.

63. False. Consider $\mathbf{r}(t) = \langle t^2, -t^2\rangle$. Then $\mathbf{v}(t) = \langle 2t, -2t\rangle$ and $\|\mathbf{v}(t)\| = \sqrt{8t^2}$.

Section 12.4 Tangent Vectors and Normal Vectors

1. The unit tangent vector points in the direction of motion.

2. The principal unit normal vector points in the direction in which the curve is turning.

3. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}$, $t = 1$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j}$, $\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4} = 2\sqrt{t^2 + 1}$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

4. $\mathbf{r}(t) = t^3\mathbf{i} + 2t^2\mathbf{j}$, $t = 1$

$\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{9t^4 + 16t^2}$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{1}{\sqrt{9 + 16}}(3\mathbf{i} + 4\mathbf{j}) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

5. $\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j}$, $t = \dfrac{\pi}{3}$

$\mathbf{r}'(t) = -5\sin t\mathbf{i} + 5\cos t\,\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{25\sin^2 t + 25\cos^2 t} = 5$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = -\sin\frac{\pi}{3}\mathbf{i} + \cos\frac{\pi}{3}\mathbf{j} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

6. $\mathbf{r}(t) = 6\sin t\mathbf{i} - 2\cos t\mathbf{j}$, $t = \dfrac{\pi}{6}$

$\mathbf{r}'(t) = 6\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = 6\left(\frac{\sqrt{3}}{2}\right)\mathbf{i} + 2\left(\frac{1}{2}\right)\mathbf{j} = 3\sqrt{3}\mathbf{i} + \mathbf{j}$$

$$\left\|\mathbf{r}'\left(\frac{\pi}{6}\right)\right\| = \sqrt{27 + 1} = \sqrt{28}$$

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{28}}\left(3\sqrt{3}\mathbf{i} + \mathbf{j}\right)$$

7. $\mathbf{r}(t) = 3t\mathbf{i} - \ln t\mathbf{j}$, $t = e$

$\mathbf{r}'(t) = 3\mathbf{i} - \dfrac{1}{t}\mathbf{j}$

$\mathbf{r}'(e) = 3\mathbf{i} - \dfrac{1}{e}\mathbf{j}$

$$\mathbf{T}(e) = \frac{\mathbf{r}'(e)}{\|\mathbf{r}'(e)\|}$$

$$= \frac{3\mathbf{i} - \frac{1}{e}\mathbf{j}}{\sqrt{9 + \frac{1}{e^2}}} = \frac{3e\mathbf{i} - \mathbf{j}}{\sqrt{9e^2 + 1}} \approx 0.9926\mathbf{i} - 0.1217\mathbf{j}$$

8. $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\mathbf{j}$, $t = 0$

$\mathbf{r}'(t) = (e^t\cos t - e^t\sin t)\mathbf{i} + e^t\mathbf{j}$

$\mathbf{r}'(0) = \mathbf{i} + \mathbf{j}$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

9. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $P(0, 0, 0)$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$

When $t = 0$, $\mathbf{r}'(0) = \mathbf{i} + \mathbf{k}$, $[t = 0 \text{ at } (0, 0, 0)]$.

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = 1$, $b = 0$, $c = 1$

Parametric equations: $x = t$, $y = 0$, $z = t$

10. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{4}{3}\mathbf{k},\ P\left(1, 1, \frac{4}{3}\right)$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

When $t = 1$, $\mathbf{r}'(t) = \mathbf{r}'(1) = 2\mathbf{i} + \mathbf{j}\ \left[t = 1\ at\left(1, 1, \frac{4}{3}\right)\right]$.

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = \frac{\sqrt{5}}{5}(2\mathbf{i} + \mathbf{j})$$

Direction numbers: $a = 2,\ b = 1,\ c = 0$

Parametric equations: $x = 2t + 1,\ y = t + 1,\ z = \frac{4}{3}$

11. $\mathbf{r}(t) = \cos t\mathbf{i} + 3\sin t\mathbf{j} + (3t - 4)\mathbf{k},\ P(1, 0, -4)$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + 3\cos t\mathbf{j} + 3\mathbf{k}$

$t = 0$ at $P(1, 0, -4)$

$\mathbf{r}'(0) = 3\mathbf{j} + 3\mathbf{k}$

$\|\mathbf{r}'(0)\| = \sqrt{9 + 9} = 3\sqrt{2}$

$$\mathbf{T}(0) = \frac{1}{3\sqrt{2}}(3\mathbf{j} + 3\mathbf{k}) = \frac{\sqrt{2}}{2}(\mathbf{j} + \mathbf{k})$$

Direction numbers: $a = 0, b = 1, c = 1$

Parametric equations: $x = 1, y = 3t, z = -4 + 3t$ or $x = 1, y = t, z = -4 + t$

12. $\mathbf{r}(t) = \left\langle t, t, \sqrt{4 - t^2}\right\rangle, P\left(1, 1, \sqrt{3}\right)$

$$\mathbf{r}'(t) = \left\langle 1, 1, -\frac{t}{4 - t^2}\right\rangle$$

When $t = 1$, $\mathbf{r}'(1) = \left\langle 1, 1, -\frac{1}{\sqrt{3}}\right\rangle,\ t = 1$ at $\left(1, 1, \sqrt{3}\right)\Big]$.

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\sqrt{21}}{7}\left\langle 1, 1, -\frac{1}{\sqrt{3}}\right\rangle$$

Direction numbers: $a = 1, b = 1, c = -\frac{1}{\sqrt{3}}$

Parametric equations: $x = t + 1,\ y = t + 1,\ z = -\frac{1}{\sqrt{3}}t + \sqrt{3}$

13. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle, P\left(\sqrt{2}, \sqrt{2}, 4\right)$

$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

When $t = \frac{\pi}{4}$, $\mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$,

$\left[t = \frac{\pi}{4}\text{ at }\left(\sqrt{2}, \sqrt{2}, 4\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\mathbf{r}'(\pi/4)}{\|\mathbf{r}'(\pi/4)\|} = \frac{1}{2}\left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$$

Direction numbers: $a = -\sqrt{2}, b = \sqrt{2}, c = 0$

Parametric equations: $x = -\sqrt{2}t + \sqrt{2},\ y = \sqrt{2}t + \sqrt{2},\ z = 4$

14. $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t\rangle, P\left(1, \sqrt{3}, 1\right)$

$\mathbf{r}'(t) = \langle 2\cos t, -2\sin t, 8\sin t\cos t\rangle$

When $t = \frac{\pi}{6}$, $\mathbf{r}'\left(\frac{\pi}{6}\right) = \left\langle \sqrt{3}, -1, 2\sqrt{3}\right\rangle$,

$\left[t = \frac{\pi}{6}\text{ at }\left(1, \sqrt{3}, 1\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{\mathbf{r}'(\pi/6)}{\|\mathbf{r}'(\pi/6)\|} = \frac{1}{4}\left\langle \sqrt{3}, -1, 2\sqrt{3}\right\rangle$$

Direction numbers: $a = \sqrt{3},\ b = -1,\ c = 2\sqrt{3}$

Parametric equations: $x = \sqrt{3}t + 1,\ y = -t + \sqrt{3},\ z = 2\sqrt{3}t + 1$

15. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}, t = 2$

$\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + t\mathbf{j}}{\sqrt{1 + t^2}}$$

$$\mathbf{T}'(t) = \frac{-t}{\left(t^2 + 1\right)^{3/2}}\mathbf{i} + \frac{1}{\left(t^2 + 1\right)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(2) = \frac{-2}{5^{3/2}}\mathbf{i} + \frac{1}{5^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} = \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j}) = \frac{-2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

16. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{6}{t}\mathbf{j}, t = 3$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{6}{t^2}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1 + \left(36/t^4\right)}}\left(i - \frac{6}{t^2}\mathbf{j}\right)$$

$$= \frac{t^2}{\sqrt{t^4 + 36}}\left(\mathbf{i} - \frac{6}{t^2}\mathbf{j}\right)$$

$$\mathbf{T}'(t) = \frac{72t}{\left(t^4 + 36\right)^{3/2}}\mathbf{i} + \frac{12t^3}{\left(t^4 + 36\right)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(2) = \frac{144}{52^{3/2}}\mathbf{i} + \frac{96}{52^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} + \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$$

17. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, t = 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}}{\sqrt{1 + 4t^2 + \frac{1}{t^2}}} = \frac{t\mathbf{i} + 2t^2\mathbf{j} + \mathbf{k}}{\sqrt{4t^4 + t^2 + 1}}$$

$$\mathbf{T}'(t) = \frac{1 - 4t^4}{\left(4t^4 + t^2 + 1\right)^{3/2}}\mathbf{i} + \frac{2t^3 + 4t}{\left(4t^4 + t^2 + 1\right)^{3/2}}\mathbf{j} + \frac{-8t^3 - t}{\left(4t^4 + t^2 + 1\right)^{3/2}}\mathbf{k}$$

$$\mathbf{T}'(1) = \frac{-3}{6^{3/2}}\mathbf{i} + \frac{6}{6^{3/2}}\mathbf{j} + \frac{-9}{6^{3/2}}\mathbf{k} = \frac{3}{6^{3/2}}[-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}]$$

$$\mathbf{N}(1) = \frac{-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{14}} = \frac{-\sqrt{14}}{14}\mathbf{i} + \frac{2\sqrt{14}}{14}\mathbf{j} - \frac{3\sqrt{14}}{14}\mathbf{k}$$

18. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k},\ t = 0$

$$\mathbf{r}'(t) = \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{\left(e^t + e^{-t}\right)^2} = e^t + e^{-t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}}{e^t + e^{-t}}$$

$$\mathbf{T}'(t) = \frac{\sqrt{2}\left(e^{-t} - e^t\right)}{\left(e^t + e^{-t}\right)^2}\mathbf{i} + \frac{2}{\left(e^t + e^{-t}\right)^2}\mathbf{j} + \frac{2}{\left(e^t + e^{-t}\right)^2}\mathbf{k}$$

$$\mathbf{T}'(0) = \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

19. $\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin t\mathbf{j} + \mathbf{k}, t = \dfrac{3\pi}{4}$

$$\mathbf{r}'(t) = -6\sin t\mathbf{i} + 6\cos t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j},\ \|\mathbf{T}'(t)\| = 1$$

$$\mathbf{N}\left(\frac{3\pi}{4}\right) = \frac{\mathbf{T}'(3\pi/4)}{\|\mathbf{T}'(3\pi/4)\|} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

20. $\mathbf{r}(t) = \cos 3t\mathbf{i} + 2\sin 3t\mathbf{j} + \mathbf{k},\ t = \pi$

$\mathbf{r}'(t) = -3\sin 3t\mathbf{i} + 6\cos 3t\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-3\sin 3t\mathbf{i} + 6\cos 3t\mathbf{j}}{\sqrt{9\sin^2 3t + 36\cos^2 3t}}$$

The normal vector is perpendicular to $\mathbf{T}(t)$ and points toward the z-axis:

$$\mathbf{N}(t) = \frac{-6\cos 3t\mathbf{i} - 3\sin 3t\mathbf{j}}{\sqrt{9\sin^2 3t + 36\cos^2 3t}}$$

$$\mathbf{N}(\pi) = \frac{6\mathbf{i}}{\sqrt{36}} = \mathbf{i}$$

21. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j},\ t_0 = 2$

$x = t,\ y = \dfrac{1}{t} \Rightarrow xy = 1$

$\mathbf{r}'(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j}$

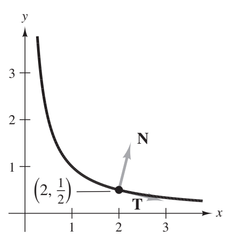
$\mathbf{T}(t) = \dfrac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{N}(t) = \dfrac{\mathbf{i} + t^2\mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{r}(2) = 2\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

$\mathbf{T}(2) = \dfrac{\sqrt{17}}{17}(4\mathbf{i} - \mathbf{j})$

$\mathbf{N}(2) = \dfrac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$



22. $\mathbf{r}(t) = t\mathbf{i} - t^3\mathbf{j},\ t = 1$

$x = t,\ y = -t^3 \Rightarrow y = -x^3$

$\mathbf{r}(1) = \mathbf{i} - \mathbf{j}$

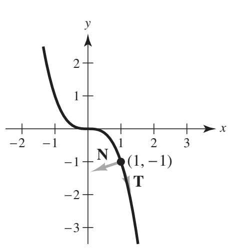
$\mathbf{r}'(t) = \mathbf{i} - 3t^2\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} - 3\mathbf{j}$

$\|\mathbf{r}'(1)\| = \sqrt{1 + 9} = \sqrt{10}$

$\mathbf{T}(1) = \dfrac{\sqrt{10}}{10}(\mathbf{i} - 3\mathbf{j})$

$\mathbf{N}(1) = -\dfrac{\sqrt{10}}{10}(3\mathbf{i} + \mathbf{j})$



23. $\mathbf{r}(t) = (2t + 1)\mathbf{i} - t^2\mathbf{j},\ t_0 = 2$

$x = 2t + 1,$

$y = -t^2 = -\left(\dfrac{x-1}{2}\right)^2$

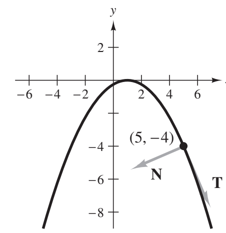
$\mathbf{r}(2) = 5\mathbf{i} - 4\mathbf{j}$

$\mathbf{r}'(t) = 2\mathbf{i} - 2t\mathbf{j}$

$\mathbf{T}(t) = \dfrac{2\mathbf{i} - 2t\mathbf{j}}{\sqrt{4 + 4t^2}} = \dfrac{\mathbf{i} - t\mathbf{j}}{\sqrt{1 + t^2}}$

$\mathbf{T}(2) = \dfrac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$

$\mathbf{N}(2) = \dfrac{-2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$, perpendicular to $\mathbf{T}(2)$



24. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\ t = \dfrac{7\pi}{6}$

$x^2 + y^2 = 4\cos^2 t + 4\sin^2 t = 4$, circle

$\mathbf{r}\left(\dfrac{7\pi}{6}\right) = -\sqrt{3}\mathbf{i} - \mathbf{j}$

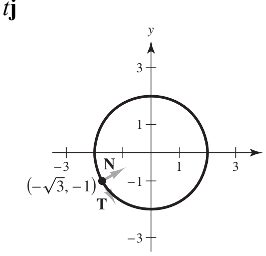
$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 2$

$\mathbf{T}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{T}\left(\dfrac{7\pi}{6}\right) = \dfrac{1}{2}\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{j}$

$\mathbf{N}\left(\dfrac{7\pi}{6}\right) = \dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$



25. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j},\ t = 1$

$\mathbf{v}(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j},\ \mathbf{v}(1) = \mathbf{i} - \mathbf{j},$

$\mathbf{a}(t) = \dfrac{2}{t^3}\mathbf{j},\ \mathbf{a}(1) = 2\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{t^2}{\sqrt{t^4 + 1}}\left(\mathbf{i} - \frac{1}{t^2}\mathbf{j}\right) = \frac{1}{\sqrt{t^4 + 1}}\left(t^2\mathbf{i} - \mathbf{j}\right)$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{2t}{\left(t^4 + 1\right)^{3/2}}\mathbf{i} + \dfrac{2t^3}{\left(t^4 + 1\right)^{3/2}}\mathbf{j}}{\dfrac{2t}{\left(t^4 + 1\right)}}$$

$$= \frac{1}{\sqrt{t^4 + 1}}\left(\mathbf{i} + t^2\mathbf{j}\right)$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$$

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = -\sqrt{2}$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \sqrt{2}$

26. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j},\ t = 1$

$\mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{j},\ \mathbf{v}(1) = 2\mathbf{i} + 2\mathbf{j}$

$\mathbf{a}(t) = 2\mathbf{i},\ \mathbf{a}(1) = 2\mathbf{i}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{4t^2+4}}(2t\mathbf{i} + 2\mathbf{j}) = \frac{1}{\sqrt{t^2+1}}(t\mathbf{i} + \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{1}{(t^2+1)^{3/2}}\mathbf{i} + \dfrac{-t}{(t^2+1)^{3/2}}\mathbf{j}}{\dfrac{1}{t^2+1}}$$

$$= \frac{1}{\sqrt{t^2+1}}(\mathbf{i} + t\mathbf{j})$$

$$\mathbf{N}(1) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \sqrt{2}$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \sqrt{2}$

27. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-2t}\mathbf{j},\ t = 0$

$\mathbf{v}(t) = e^t\mathbf{i} - 2e^{-2t}\mathbf{j},\ \mathbf{v}(0) = \mathbf{i} - 2\mathbf{j}$

$\mathbf{a}(t) = e^t\mathbf{i} + 4e^{-2t}\mathbf{j},\ \mathbf{a}(0) = \mathbf{i} + 4\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{e^t\mathbf{i} - 2e^{-2t}\mathbf{j}}{\sqrt{4e^{-4t} + e^{2t}}}$$

$$\mathbf{T}(0) = \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$$

$$\mathbf{N}(0) = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{1}{\sqrt{5}}(1 - 8) = \frac{-7\sqrt{5}}{5}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{1}{\sqrt{5}}(2 + 4) = \frac{6\sqrt{5}}{5}$$

28. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j},\ t = 0$

$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j},\ \mathbf{v}(0) = \mathbf{i} - \mathbf{j}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j},\ \mathbf{a}(0) = \mathbf{i} + \mathbf{j}$

$\|\mathbf{v}(0)\| = \sqrt{2}$

$$\mathbf{T}(0) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = (\mathbf{i} + \mathbf{j})\cdot\frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j}) = 0$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = (\mathbf{i} + \mathbf{j})\cdot\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j}) = \sqrt{2}$$

29. $\mathbf{r}(t) = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j},\ t = \dfrac{\pi}{2}$

$\mathbf{v}(t) = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j}$

$\mathbf{a}(t) = e^t(-2\sin t)\mathbf{i} + e^t(2\cos t)\mathbf{j}$

At $t = \dfrac{\pi}{2}$, $\mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = \dfrac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$.

Motion along $\mathbf{r}$ is counterclockwise. So,

$$\mathbf{N} = \frac{1}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}) = -\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j}).$$

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \sqrt{2}e^{\pi/2}$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \sqrt{2}e^{\pi/2}$

30. $\mathbf{r}(t) = 4\cos 3t\mathbf{i} + 4\sin 3t\mathbf{j},\ t = \pi$

$\mathbf{v}(t) = -12\sin 3t\mathbf{i} + 12\cos 3t\mathbf{j}$

$\mathbf{a}(t) = -36\cos 3t\mathbf{i} - 36\sin 3t\mathbf{j}$

$\mathbf{a}(\pi) = 36\mathbf{i}$

At $t = \pi$, $\mathbf{v}\pi = -12\mathbf{j}$ and $\mathbf{T}(\pi) = -\mathbf{j}$

Movement is counterclockwise around a circle. So, $\mathbf{N}(\pi) = \mathbf{i}$.

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = 36$

31. $\mathbf{r}(t) = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$

$\mathbf{v}(t) = -a\omega\sin\omega t\mathbf{i} + a\omega\cos\omega t\mathbf{j}$

$\mathbf{a}(t) = -a\omega^2\cos\omega t\mathbf{i} - a\omega^2\sin\omega t\mathbf{j}$

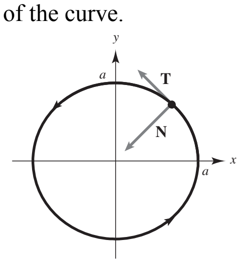
$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = -\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos\omega t\mathbf{i} - \sin\omega t\mathbf{j}$$

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = a\omega^2$

32. $\mathbf{T}(t)$ points in the direction that $\mathbf{r}$ is moving. $\mathbf{N}(t)$ points in the direction that $\mathbf{r}$ is turning, toward the concave side of the curve.



33. Speed: $\|\mathbf{v}(t)\| = a\omega$

The speed is constant because $a_{\mathbf{T}} = 0$.

34. If the angular velocity ω is halved,

$$a_{\mathbf{N}} = a\left(\frac{\omega}{2}\right)^2 = \frac{a\omega^2}{4}.$$

$a_{\mathbf{N}}$ is changed by a factor of $\frac{1}{4}$.

35. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k},\ t = 1$

$$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \frac{\sqrt{14}}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \mathbf{T}(1)$$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'}{\|\mathbf{T}'\|}$ is undefined.

$a_{\mathbf{T}}, a_{\mathbf{N}}$ are not defined.

36. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k},\ t = \dfrac{\pi}{3}$

$$\mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j},\ \|\mathbf{a}\| = 1$$

$$\|\mathbf{v}(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{5}}(-\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k})$$

$$a_{\mathbf{T}} = \mathbf{a}(t) \cdot \mathbf{T}(t) = 0$$

$$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{1 - 0} = 1$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{5}}\left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}\right)$$

$$\mathbf{a}\left(\frac{\pi}{3}\right) = -\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} = a_{\mathbf{T}}\mathbf{T} + a_n\mathbf{N} = \mathbf{N}$$

37. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k},\ t = 1$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1 + 5t^2}}(\mathbf{i} + 2t\mathbf{j} + t\mathbf{k})$$

$$\mathbf{T}(1) = \frac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\dfrac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1 + 5t^2)^{3/2}}}{\dfrac{\sqrt{5}}{1 + 5t^2}} = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1 + 5t^2}}$$

$$\mathbf{N}(1) = \frac{\sqrt{30}}{30}(-5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{5\sqrt{6}}{6}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{\sqrt{30}}{6}$$

38. $\mathbf{r}(t) = (2t - 1)\mathbf{i} + t^2\mathbf{j} - 4t\mathbf{k},\ t = 2$

$$\mathbf{v}(t) = 2\mathbf{i} + 2t\mathbf{j} - 4\mathbf{k},$$

$$\|\mathbf{v}(t)\| = \sqrt{20 + 4t^2} = 2\sqrt{5 + t^2}$$

$$\mathbf{a}(t) = 2\mathbf{j},\ \|\mathbf{a}\| = 2$$

$$\mathbf{T}(t) = \frac{2\mathbf{i} + 2t\mathbf{j} - 4\mathbf{k}}{2\sqrt{5 + t^2}} = \frac{1}{\sqrt{5 + t^2}}(\mathbf{i} + t\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{a}(2) = 2\mathbf{j},\ \mathbf{T}(2) = \frac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$

$$a_{\mathbf{T}} = \mathbf{a}(2) \cdot \mathbf{T}(2) = \frac{4}{3}$$

$$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{4 - \frac{16}{9}} = \frac{2}{3}\sqrt{5}$$

$$\mathbf{a} = 2\mathbf{j} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} = \frac{4}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) + \frac{2}{3}\sqrt{5}\mathbf{N}$$

$$\mathbf{N} = \frac{3}{2\sqrt{5}}\left[2\mathbf{j} - \frac{4}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})\right] = \frac{-2\sqrt{5}}{15}\mathbf{i} + \frac{\sqrt{5}}{3}\mathbf{j} + \frac{4\sqrt{5}}{15}\mathbf{k}$$

39. $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} + e^t\mathbf{k},\ t = 0$

$\mathbf{v}(t) = \left(e^t \cos t + e^t \sin t\right)\mathbf{i} + \left(-e^t \sin t + e^t \cos t\right)\mathbf{j} + e^t\mathbf{k}$

$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = 2e^t \cos t\mathbf{i} - 2e^t \sin t\mathbf{j} + e^t\mathbf{k}$

$\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k}$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$= \frac{1}{\sqrt{3}}\left[(\cos t + \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} + \mathbf{k}\right]$$

$$\mathbf{T}(0) = \frac{1}{\sqrt{3}}[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{2}}\left[(-\sin t + \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}\right]$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{3}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

40. $\mathbf{r}(t) = e^t\mathbf{i} + 2t\mathbf{j} + e^{-t}\mathbf{k},\ t = 0$

$\mathbf{v}(t) = e^t\mathbf{i} + 2\mathbf{j} - e^{-t}\mathbf{k},\ \|\mathbf{v}(t)\| = \sqrt{e^{2t} + 4 + e^{-2t}}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{k},\ \|\mathbf{a}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

$\mathbf{v}(0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k},\ \mathbf{a}(0) = \mathbf{i} + \mathbf{k}$

$$\mathbf{T}(0) = \frac{\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{6}}$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{2}$

$\mathbf{a} = \mathbf{i} + \mathbf{k} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} = \sqrt{2}\mathbf{N}$

$$\Rightarrow \mathbf{N} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$$

41. If $a_{\mathbf{N}} = 0$, then the motion is in a straight line.

42. If $a_{\mathbf{T}} = 0$, then the speed is constant.

43. $\mathbf{r}(t) = 3t\mathbf{i} + 4t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j},\ \|\mathbf{v}(t)\| = \sqrt{9 + 16} = 5$

$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{0}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$\mathbf{T}'(t) = 0 \Rightarrow \mathbf{N}(t)$ does not exist.

The path is a line. The speed is constant (5).

44. (a)(i) The vector **s** represents the unit tangent vector because it points in the direction of motion.

(ii) The vector **t** represents the unit tangent vector because it points in the direction of motion.

(b)(i) The vector **z** represents the unit normal vector because it points in the direction that the curve is bending.

(ii) The vector **z** represents the unit normal vector because it points in the direction that the curve is bending.

45. $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$

The graph is a cycloid.

(a) $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$

$$\mathbf{v}(t) = \langle \pi - \pi \cos \pi t, \pi \sin \pi t \rangle$$

$$\mathbf{a}(t) = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t \rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \langle 1 - \cos \pi t, \sin \pi t \rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \langle \sin \pi t, -1 + \cos \pi t \rangle$$

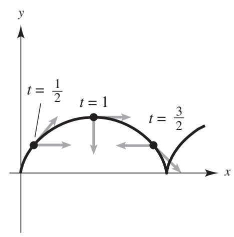
$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \left[\pi^2 \sin \pi t(1 - \cos \pi t) + \pi^2 \cos \pi t \sin \pi t \right] = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \left[\pi^2 \sin^2 \pi t + \pi^2 \cos \pi t(-1 + \cos \pi t) \right] = \frac{\pi^2(1 - \cos \pi t)}{\sqrt{2(1 - \cos \pi t)}} = \frac{\pi^2 \sqrt{2(1 - \cos \pi t)}}{2}$$

When $t = \frac{1}{2}$: $a_{\mathbf{T}} = \frac{\pi^2}{\sqrt{2}} = \frac{\sqrt{2}\pi^2}{2}$, $a_{\mathbf{N}} = \frac{\sqrt{2}\pi^2}{2}$

When $t = 1$: $a_{\mathbf{T}} = 0$, $a_{\mathbf{N}} = \pi^2$

When $t = \frac{3}{2}$: $a_{\mathbf{T}} = -\frac{\sqrt{2}\pi^2}{2}$, $a_{\mathbf{N}} = \frac{\sqrt{2}\pi^2}{2}$



(b) Speed: $s = \|\mathbf{v}(t)\| = \pi\sqrt{2(1 - \cos \pi t)}$

$$\frac{ds}{dt} = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}} = a_{\mathbf{T}}$$

When $t = \frac{1}{2}$: $a_{\mathbf{T}} = \frac{\sqrt{2}\pi^2}{2} > 0 \Rightarrow$ the speed in increasing.

When $t = 1$: $a_{\mathbf{T}} = 0 \Rightarrow$ the height is maximum.

When $t = \frac{3}{2}$: $a_{\mathbf{T}} = -\frac{\sqrt{2}\pi^2}{2} < 0 \Rightarrow$ the speed is decreasing.

46. (a) $\mathbf{r}(t) = \langle \cos \pi t + \pi t \sin \pi t, \sin \pi t - \pi t \cos \pi t \rangle$

$$\mathbf{v}(t) = \langle -\pi \sin \pi t + \pi \sin \pi t + \pi^2 t \cos \pi t, \pi \cos \pi t - \pi \cos \pi t + \pi^2 t \sin \pi t \rangle = \langle \pi^2 t \cos \pi t, \pi^2 t \sin \pi t \rangle$$

$$\mathbf{a}(t) = \langle \pi^2 \cos \pi t - \pi^3 t \sin \pi t, \pi^2 \sin \pi t + \pi^3 t \cos \pi t \rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \langle \cos \pi t, \sin \pi t \rangle$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \cos \pi t(\pi^2 \cos \pi t - \pi^3 t \sin \pi t) + \sin \pi t(\pi^2 \sin \pi t + \pi^3 t \cos \pi t) = \pi^2$$

$$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{\pi^4(1 + \pi^2 t^2) - \pi^4} = \pi^3 t$$

When $t = 1$, $a_{\mathbf{T}} = \pi^2$, $a_{\mathbf{N}} = \pi^3$. When $t = 2$, $a_{\mathbf{T}} = \pi^2$, $a_{\mathbf{N}} = 2\pi^3$.

(b) Because $a_{\mathbf{T}} = \pi^2 > 0$ for all values of t, the speed is increasing when $t = 1$ and $t = 2$.

47. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \dfrac{t}{2}\mathbf{k},\ t_0 = \dfrac{\pi}{2}$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \dfrac{1}{2}\mathbf{k}$

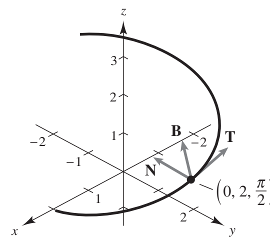
$\mathbf{T}(t) = \dfrac{2\sqrt{17}}{17}\left(-2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \dfrac{1}{2}\mathbf{k}\right)$

$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{r}\left(\dfrac{\pi}{2}\right) = 2\mathbf{j} + \dfrac{\pi}{4}\mathbf{k}$

$\mathbf{T}\left(\dfrac{\pi}{2}\right) = \dfrac{2\sqrt{17}}{17}\left(-2\mathbf{i} + \dfrac{1}{2}\mathbf{k}\right) = \dfrac{\sqrt{17}}{17}(-4\mathbf{i} + \mathbf{k})$

$\mathbf{N}\left(\dfrac{\pi}{2}\right) = -\mathbf{j}$



$$\mathbf{B}\left(\frac{\pi}{2}\right) = \mathbf{T}\left(\frac{\pi}{2}\right) \times \mathbf{N}\left(\frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4\sqrt{17}}{17} & 0 & \frac{\sqrt{17}}{17} \\ 0 & -1 & 0 \end{vmatrix} = \frac{\sqrt{17}}{17}\mathbf{i} + \frac{4\sqrt{17}}{17}\mathbf{k} = \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{k})$$

48. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^3}{3}\mathbf{k},\ t_0 = 1$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$

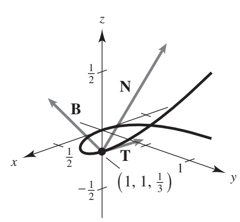
$\mathbf{T}(t) = \dfrac{1}{\sqrt{1 + 4t^2 + t^4}}\left(\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}\right)$

$\mathbf{N}(t) = \dfrac{1}{\sqrt{1 + 4t^2 + t^4}\sqrt{1 + t^2 + t^4}}\left[\left(-2t - t^3\right)\mathbf{i} + \left(1 - t^4\right)\mathbf{j} + \left(t + 2t^3\right)\mathbf{k}\right]$

$\mathbf{r}(1) = \mathbf{i} + \mathbf{j} + \dfrac{1}{3}\mathbf{k}$

$\mathbf{T}(1) = \dfrac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$\mathbf{N}(1) = \dfrac{1}{\sqrt{6}\sqrt{3}}(-3\mathbf{i} + 3\mathbf{k}) = \dfrac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{k})$



$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

49. $\mathbf{r}(t) = \mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k},\ t_0 = \dfrac{\pi}{4}$

$\mathbf{r}'(t) = \cos t\mathbf{j} - \sin t\mathbf{k},$

$\|\mathbf{r}'(t)\| = 1$

$\mathbf{r}'\left(\dfrac{\pi}{4}\right) = \mathbf{T}\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}\mathbf{j} - \dfrac{\sqrt{2}}{2}\mathbf{k}$

$\mathbf{T}'(t) = -\sin t\mathbf{j} - \cos t\mathbf{k},$

$\mathbf{N}\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}\mathbf{j} - \dfrac{\sqrt{2}}{2}\mathbf{k}$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{T}\left(\frac{\pi}{4}\right) \times \mathbf{N}\left(\frac{\pi}{4}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} = -\mathbf{i}$$

50. $\mathbf{r}(t) = 2e^t\mathbf{i} + e^t\cos t\mathbf{j} + e^t\sin t\mathbf{k},\ t_0 = 0$

$\mathbf{r}'(t) = 2e^t\mathbf{i} + \left(e^t\cos t - e^t\sin t\right)\mathbf{j} + \left(e^t\sin t + e^t\cos t\right)\mathbf{k}$

$\mathbf{r}'(0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{T}(0) = \dfrac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\begin{aligned}\|\mathbf{r}'(t)\|^2 &= 4e^{2t} + e^{2t}\cos^2 t + e^{2t}\sin^2 t - 2e^{2t}\cos t\sin t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^{2t}\sin t\cos t \\ &= 4e^{2t} + 2e^{2t}\left(\cos^2 t + \sin^2 t\right) = 6e^{2t}\end{aligned}$$

$\|\mathbf{r}'(t)\| = \sqrt{6}e^t$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{1}{\sqrt{6}}\left[2\mathbf{i} + (\cos t - \sin t)\mathbf{j} + (\sin t + \cos t)\mathbf{k}\right]$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{6}}\left[(-\sin t - \cos t)\mathbf{j} + (\cos t - \sin t)\mathbf{k}\right]$

$\mathbf{T}'(0) = \dfrac{1}{\sqrt{6}}[-\mathbf{j} + \mathbf{k}] \Rightarrow \mathbf{N}(0) = \dfrac{-\sqrt{2}}{2}\mathbf{j} + \dfrac{\sqrt{2}}{2}\mathbf{k}$

$$\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$$

51. $\mathbf{r}(t) = 4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 2t\mathbf{k},\ t_0 = \dfrac{\pi}{3}$

$\mathbf{r}'(t) = 4\cos t\mathbf{i} - 4\sin t\mathbf{j} + 2\mathbf{k},$

$\|\mathbf{r}'(t)\| = \sqrt{16\cos^2 t + 16\sin^2 t + 4} = \sqrt{20} = 2\sqrt{5}$

$\mathbf{r}'\left(\dfrac{\pi}{3}\right) = 2\mathbf{i} - 2\sqrt{3}\mathbf{j} + 2\mathbf{k}$

$\mathbf{T}\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2\sqrt{5}}\left(2\mathbf{i} - 2\sqrt{3}\mathbf{j} + 2\mathbf{k}\right) = \dfrac{\sqrt{5}}{5}\mathbf{i} - \dfrac{\sqrt{15}}{5}\mathbf{j} + \dfrac{\sqrt{5}}{5}\mathbf{k} = \dfrac{\sqrt{5}}{5}\left(\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k}\right)$

$\mathbf{T}'(t) = \dfrac{1}{2\sqrt{5}}(-4\sin t\mathbf{i} - 4\cos t\mathbf{j})$

$\mathbf{N}\left(\dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{j}$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right) \times \mathbf{N}\left(\frac{\pi}{3}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\sqrt{5}}{5} & -\dfrac{\sqrt{15}}{5} & \dfrac{\sqrt{5}}{5} \\ -\dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} & 0 \end{vmatrix} = \frac{\sqrt{5}}{10}\mathbf{i} - \frac{\sqrt{15}}{10}\mathbf{j} - \frac{4\sqrt{5}}{10}\mathbf{k} = \frac{\sqrt{5}}{10}\left(\mathbf{i} - \sqrt{3}\mathbf{j} - 4\mathbf{k}\right)$$

52. $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j} + t\mathbf{k},\ t = \dfrac{\pi}{4}$

$\mathbf{r}'(t) = -6\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j} + \mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{37}$

$\mathbf{r}'\left(\dfrac{\pi}{4}\right) = -6\mathbf{i} + \mathbf{k}$

$\mathbf{T}(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{1}{\sqrt{37}}\left(-6\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j} + \mathbf{k}\right)$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{37}}(-12\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j})$

$\mathbf{T}\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{37}}(-6\mathbf{i} + \mathbf{k})$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos 2t\mathbf{i} - \sin 2t\mathbf{j}$

$\mathbf{N}\left(\dfrac{\pi}{4}\right) = -j$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{T}\left(\frac{\pi}{4}\right) \times \mathbf{N}\left(\frac{\pi}{4}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{-6}{\sqrt{37}} & 0 & \dfrac{1}{\sqrt{37}} \\ 0 & -1 & 0 \end{vmatrix} = \frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k}$$

53. $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j}$

$\mathbf{v}(t) = 3\mathbf{i} + 4t\mathbf{j}$

$\mathbf{a}(t) = 4\mathbf{j}$

$\mathbf{v} \cdot \mathbf{v} = 9 + 16t^2$

$\mathbf{v} \cdot \mathbf{a} = 16t$

$$(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v} = (9 + 16t^2)4\mathbf{j} - (16t)(3\mathbf{i} + 4t\mathbf{j}) = -48t\mathbf{i} + 36\mathbf{j}$$

$$\mathbf{N} = \frac{(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}}{\|(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}\|} = \frac{1}{\sqrt{9 + 16t^2}}(-4t\mathbf{i} + 3\mathbf{j})$$

54. $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j}$

$\mathbf{v}(t) = -6\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j}$

$\mathbf{a}(t) = -12\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j}$

$\mathbf{v} \cdot \mathbf{v} = 36\sin^2 2t + 36\cos^2 2t = 36$

$\mathbf{v} \cdot \mathbf{a} = 0$

$$\mathbf{N} = \frac{(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}}{\|(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}\|} = \frac{36(-12\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j})}{\|36(-12\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j})\|} = -\cos 2t\mathbf{i} - \sin 2t\mathbf{j}$$

55. $\mathbf{r}(t) = 2t\mathbf{i} + 4t\mathbf{j} + t^2\mathbf{k}$

$\mathbf{v}(t) = 2\mathbf{i} + 4\mathbf{j} + 2t\mathbf{k}$

$\mathbf{a}(t) = 2\mathbf{k}$

$\mathbf{v} \cdot \mathbf{v} = 4 + 16 + 4t^2 = 20 + 4t^2$

$\mathbf{v} \cdot \mathbf{a} = 4t$

$$(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v} = (20 + 4t^2)2\mathbf{k} - 4t(2\mathbf{i} + 4\mathbf{j} + 2t\mathbf{k}) = -8t\mathbf{i} - 16t\mathbf{j} + 40\mathbf{k}$$

$$\mathbf{N} = \frac{(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}}{\|(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}\|} = \frac{1}{\sqrt{5t^2 + 25}}(-t\mathbf{i} - 2t\mathbf{j} + 5\mathbf{k})$$

56. $\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j} + 3t\mathbf{k}$

$\mathbf{v}(t) = -5\sin t\mathbf{i} + 5\cos t\mathbf{j} + 3\mathbf{k}$

$\mathbf{a}(t) = -5\cos t\mathbf{i} - 5\sin t\mathbf{j}$

$\mathbf{v} \cdot \mathbf{v} = 25 + 9 = 34$

$\mathbf{v} \cdot \mathbf{a} = 0$

$$(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v} = 34(-5\cos t\mathbf{i} - 5\sin t\mathbf{j})$$

$$\mathbf{N} = \frac{(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}}{\|(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}\|} = \frac{34(-5\cos t\mathbf{i} - 5\sin t\mathbf{j})}{\|34(-5\cos t\mathbf{i} - 5\sin t\mathbf{j})\|} = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

57. From Theorem 12.3 you have:

$$\mathbf{r}(t) = (v_0 t\cos\theta)\mathbf{i} + (h + v_0 t\sin\theta - 16t^2)\mathbf{j}$$

$$\mathbf{v}(t) = v_0\cos\theta\mathbf{i} + (v_0\sin\theta - 32t)\mathbf{j}$$

$$\mathbf{a}(t) = -32\mathbf{j}$$

$$\mathbf{T}(t) = \frac{(v_0\cos\theta)\mathbf{i} + (v_0\sin\theta - 32t)\mathbf{j}}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

$$\mathbf{N}(t) = \frac{(v_0\sin\theta - 32t)\mathbf{i} - v_0\cos\theta\mathbf{j}}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}} \quad \text{(Motion is clockwise.)}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{-32(v_0\sin\theta - 32t)}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{32v_0\cos\theta}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

Maximum height when $v_0\sin\theta - 32t = 0$; (vertical component of velocity)

At maximum height, $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 32$.

58. $\theta = 45°,\ v_0 = 150$

$$v_0 \cos\theta = 150 \cdot \frac{\sqrt{2}}{2} = 75\sqrt{2}$$

$$v_0 \sin\theta - 32t = 150 \cdot \frac{\sqrt{2}}{2} - 32t = 75\sqrt{2} - 32t$$

$$a_{\mathbf{T}} = \frac{-32\left(75\sqrt{2} - 32t\right)}{\sqrt{11250 + \left(75\sqrt{2} - 32t\right)^2}} = \frac{16\left(32t - 75\sqrt{2}\right)}{\sqrt{256t^2 - 1200\sqrt{2}t + 5625}}$$

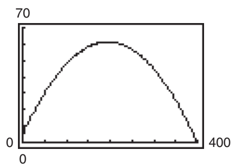
$$a_{\mathbf{N}} = \frac{32\left(75\sqrt{2}\right)}{\sqrt{11250 + \left(75\sqrt{2} - 32t\right)^2}} = \frac{1200\sqrt{2}}{\sqrt{256t^2 - 1200\sqrt{2}t + 5625}}$$

At the maximum height, $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 32$.

59. (a) $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$= (120 \cos 30°)t\mathbf{i} + \left[5 + (120 \sin 30°)t - 16t^2\right]\mathbf{j} = 60\sqrt{3}t\mathbf{i} + \left[5 + 60t - 16t^2\right]\mathbf{j}$

(b)



Maximum height $\approx$ 61.25 feet

range $\approx$ 398.2 feet

(c) $\mathbf{v}(t) = 60\sqrt{3}\mathbf{i} + (60 - 32t)\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{3600(3) + (60 - 32t)^2} = 8\sqrt{16t^2 - 60t + 225}$

$\mathbf{a}(t) = -32\mathbf{j}$

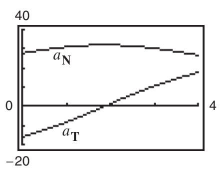
(d)

t	0.5	1.0	1.5	2.0	2.5	3.0
Speed	112.85	107.63	104.61	104.0	105.83	109.98

(e) From Exercise 61, using $\mathbf{v}_0 = 120$ and $\theta = 30°$,

$$a_{\mathbf{T}} = \frac{-32(60 - 32t)}{\sqrt{\left(60\sqrt{3}\right)^2 + (60 - 32t)^2}}$$

$$a_{\mathbf{N}} = \frac{32\left(60\sqrt{3}\right)}{\sqrt{\left(60\sqrt{3}\right)^2 + (60 - 32t)^2}}$$

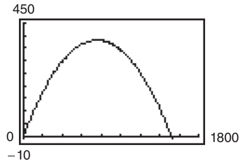


At $t = 1.875$, $a_{\mathbf{T}} = 0$ and the projectile is at its maximum height. When $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ have opposite signs, the speed is decreasing.

60. (a) $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$= (220 \cos 45°)t\mathbf{i} + \left[4 + (220 \sin 45°)t - 16t^2\right]\mathbf{j}$

$= 110\sqrt{2}t\mathbf{i} + \left[4 + 110\sqrt{2}t - 16t^2\right]\mathbf{j}$

(b) 

Maximum height $\approx$ 382.125 at $t \approx 4.86$

Range $\approx$ 1516.4

(c) $\mathbf{v}(t) = 110\sqrt{2}\mathbf{i} + \left[110\sqrt{2} - 32t\right]\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{\left(110\sqrt{2}\right)^2 + \left(110\sqrt{2} - 32t\right)^2}$

$\mathbf{a}(t) = -32\mathbf{j}$

(d)

t	0.5	1.0	1.5	2.0	2.5	3.0
Speed	208.99	198.67	189.13	180.51	172.94	166.58

61. $\mathbf{r}(t) = \langle 10 \cos 10\pi t, 10 \sin 10\pi t, 4 + 4t\rangle, 0 \le t \le \frac{1}{20}$

(a) $\mathbf{r}'(t) = \langle -100\pi \sin(10\pi t), 100\pi \cos(10\pi t), 4\rangle$

$\|\mathbf{r}'(t)\| = \sqrt{(100\pi)^2 \sin^2(10\pi t) + (100\pi)^2 \cos^2(10\pi t) + 16}$

$= \sqrt{(100\pi)^2 + 16} = 4\sqrt{625\pi^2 + 1} \approx 314$ mi/h

(b) $a_\mathbf{T} = 0$ and $a_\mathbf{N} = 1000\pi^2$

$a_\mathbf{T} = 0$ because the speed is constant

62. 600 mi/h $=$ 880 ft/sec

$\mathbf{r}(t) = 880t\mathbf{i} + \left(-16t^2 + 36{,}000\right)\mathbf{j}$

$\mathbf{v}(t) = 880\mathbf{i} - 32t\mathbf{j}$

$\mathbf{a}(t) = -32\mathbf{j}$

$$\mathbf{T}(t) = \frac{880\mathbf{i} - 32t\mathbf{j}}{16\sqrt{4t^2 + 3025}} = \frac{55\mathbf{i} - 2t\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

Motion along $\mathbf{r}$ is clockwise, therefore

$$\mathbf{N}(t) = \frac{-2t\mathbf{i} - 55\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

$$a_\mathbf{T} = \mathbf{a} \cdot \mathbf{T} = \frac{64t}{\sqrt{4t^2 + 3025}}$$

$$a_\mathbf{N} = \mathbf{a} \cdot \mathbf{N} = \frac{1760}{\sqrt{4t^2 + 3025}}$$

63. $\mathbf{r}(t) = (a \cos \omega t)\mathbf{i} + (a \sin \omega t)\mathbf{j}$

From Exercise 31, you know $\mathbf{a} \cdot \mathbf{T} = 0$ and $\mathbf{a} \cdot \mathbf{N} = a\omega^2$.

(a) Let $\omega_0 = 2\omega$. Then

$$\mathbf{a} \cdot \mathbf{N} = a\omega_0^2 = a(2\omega)^2 = 4a\omega^2$$

or the centripetal acceleration is increased by a factor of 4 when the velocity is doubled.

(b) Let $a_0 = a/2$. Then

$$\mathbf{a} \cdot \mathbf{N} = a_0\omega^2 = \left(\frac{a}{2}\right)\omega^2 = \left(\frac{1}{2}\right)a\omega^2$$

or the centripetal acceleration is halved when the radius is halved.

64. $\mathbf{r}(t) = (r\cos\omega t)\mathbf{i} + (r\sin\omega t)\mathbf{j}$

$\mathbf{v}(t) = (-r\omega\sin\omega t)\mathbf{i} + (r\omega\cos\omega t)\mathbf{j}$

$\|\mathbf{v}(t)\| = r\omega\sqrt{1} = r\omega = v$

$\mathbf{a}(t) = (-r\omega^2\cos\omega t)\mathbf{i} + (r\omega\cos\omega t)\mathbf{j}$

$\|\mathbf{a}(t)\| = r\omega^2$

(a) $F = m\|\mathbf{a}(t)\| = m(r\omega^2) = \dfrac{m}{r}(r^2\omega^2) = \dfrac{mv^2}{r}$

(b) By Newton's Law:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}, v^2 = \frac{GM}{r}, v = \sqrt{\frac{GM}{r}}$$

65. $\mathbf{v} = \sqrt{\dfrac{GM}{r}} = \sqrt{\dfrac{9.56\times10^4}{4000+255}} \approx 4.74$ mi/sec

66. $\mathbf{v} = \sqrt{\dfrac{GM}{r}} = \sqrt{\dfrac{9.56\times10^4}{4000+3460}} \approx 4.69$ mi/sec

67. $v = \sqrt{\dfrac{9.56\times10^4}{4000+385}} \approx 4.67$ mi/sec

68. Let x = distance from the satellite to the center of the earth $(x = r + 4000)$. Then:

$$v = \frac{2\pi x}{t} = \frac{2\pi x}{24(3600)} = \sqrt{\frac{9.56\times10^4}{x}}$$

$$\frac{4\pi^2x^2}{(24)^2(3600)^2} = \frac{9.56\times10^4}{x}$$

$$x^3 = \frac{(9.56\times10^4)(24)^2(3600)^2}{4\pi^2} \Rightarrow x \approx 26{,}245 \text{ mi}$$

$$v \approx \frac{2\pi(26{,}245)}{24(3600)} \approx 1.92 \text{ mi/sec} \approx 6871 \text{ mi/h}$$

69. False. These vectors are perpendicular for an object traveling at a constant speed but not for an object traveling at a variable speed.

70. True.

71. (a) $\mathbf{r}(t) = \cosh(bt)\mathbf{i} + \sinh(bt)\mathbf{j}, b > 0$

$x = \cosh(bt), y = \sinh(bt)$

$x^2 - y^2 = \cosh^2(bt) - \sinh^2(bt) = 1$, hyperbola

(b) $\mathbf{v}(t) = b\sinh(bt)\mathbf{i} + b\cosh(bt)\mathbf{j}$

$\mathbf{a}(t) = b^2\cosh(bt)\mathbf{i} + b^2\sinh(bt)\mathbf{j} = b^2\mathbf{r}(t)$

72. Let $\mathbf{T}(t) = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$ be the unit tangent vector.

Then

$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{\text{dt}}$$

$$= \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{dt} = -(\sin\phi\mathbf{i} - \cos\phi\mathbf{j})\frac{d\phi}{dt} = \mathbf{M}\frac{d\phi}{dt}.$$

$$\mathbf{M} = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$$

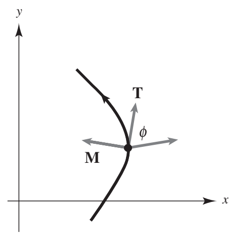
$$= \cos[\phi + (\pi/2)]\mathbf{i} + \sin[\phi + (\pi/2)]\mathbf{j}$$

and is rotated counterclockwise through an angle of $\pi/2$ from $\mathbf{T}$.

If $d\phi/dt > 0$, then the curve bends to the left and $\mathbf{M}$ has the same direction as $\mathbf{T}'$. So, $\mathbf{M}$ has the same direction as

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|},$$

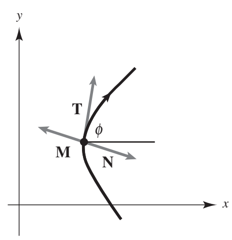
which is toward the concave side of the curve.



If $d\phi/dt < 0$, then the curve bends to the right and $\mathbf{M}$ has the opposite direction as $\mathbf{T}'$. Thus,

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

again points to the concave side of the curve.



73. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$y(t) = m(x(t)) + b$, m and b are constants.

$$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$$

$$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = |x'(t)|\sqrt{1 + m^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\pm(\mathbf{i} + m\mathbf{j})}{\sqrt{1 + m^2}}, \text{ constant}$$

So, $\mathbf{T}'(t) = \mathbf{0}$.

74. Using $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$, $\mathbf{T}\times\mathbf{T} = \mathbf{0}$, and $\|\mathbf{T}\times\mathbf{N}\| = 1$, you have:

$$\mathbf{v}\times\mathbf{a} = \|\mathbf{v}\|\mathbf{T}\times(a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N})$$

$$= \|\mathbf{v}\|a_{\mathbf{T}}(\mathbf{T}\times\mathbf{T}) + \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T}\times\mathbf{N})$$

$$= \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T}\times\mathbf{N})$$

$$\|\mathbf{v}\times\mathbf{a}\| = \|\mathbf{v}\|a_{\mathbf{N}}\|\mathbf{T}\times\mathbf{N}\| = \|\mathbf{v}\|a_{\mathbf{N}}$$

So, $a_{\mathbf{N}} = \dfrac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|}$.

75. $\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a}$

$$= (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N})\cdot(a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N})$$

$$= a_{\mathbf{T}}^2\|\mathbf{T}\|^2 + 2a_{\mathbf{T}}a_{\mathbf{N}}\mathbf{T}\cdot\mathbf{N} + a_{\mathbf{N}}^2\|\mathbf{N}\|^2$$

$$= a_{\mathbf{T}}^2 + a_{\mathbf{N}}^2$$

$$a_{\mathbf{N}}^2 = \|\mathbf{a}\|^2 - a_{\mathbf{T}}^2$$

Because $a_{\mathbf{N}} > 0$, we have $a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$.

76. $F = ma = (1)\dfrac{dv}{dt} = \dfrac{dv}{dt}$ Force

$$x = at + bt^2 + ct^3$$

$$v = \frac{dx}{dt} = a + 2bt + 3ct^2$$

$$\frac{dv}{dt} = 2b + 6ct$$

$$F^2 = 4b^2 + 24bct + 36c^2t^2$$

$$= 4b^2 + 12c + (2bt + 3ct^2)$$

$$= 4b^2 + 12c + (v - a)$$

$$F = f(v) = \pm\sqrt{4b^2 - 12ac + 12cv}$$

The sign of the radical is the sign of $2b + 6ct$, which cannot change.

Section 12.5 Arc Length and Curvature

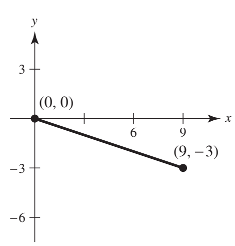
1. The curve bends more sharply at Q than at P.

2. t is the arc length parameter if $\|\mathbf{r}'(t)\| = 1$.

3. $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j}, [0, 3]$

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = -1, \frac{dz}{dt} = 0$$

$$s = \int_0^3 \sqrt{3^2 + (-1)^2}\, dt = \left[\sqrt{10}t\right]_0^3 = 3\sqrt{10}$$

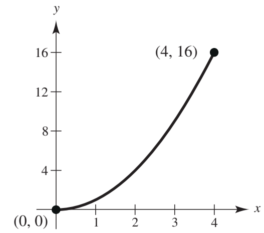


4. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, [0, 4]$

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = 2t, \frac{dz}{dt} = 0$$

$$s = \int_0^4 \sqrt{1 + 4t^2}\,dt$$

$$= \frac{1}{4}\left[2t\sqrt{1 + 4t^2} + \ln\left|2t + \sqrt{1 + 4t}\right|\right]_0^4 = \frac{1}{4}\left[8\sqrt{65} + \ln\left(8 + \sqrt{65}\right)\right] \approx 16.819$$

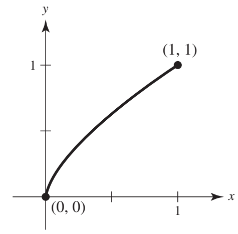


5. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}, [0, 1]$

$$\frac{dx}{dt} = 3t^2, \frac{dy}{dt} = 2t, \frac{dz}{dt} = 0$$

$$s = \int_0^1 \sqrt{9t^4 + 4t^2}\,dt = \int_0^1 \sqrt{9t^2 + 4}\,t\,dt$$

$$= \frac{1}{18}\int_0^1 \left(9t^2 + 4\right)^{1/2}(18t)\,dt = \frac{1}{27}\left[\left(9t^2 + 4\right)^{3/2}\right]_0^1 = \frac{1}{27}\left(13^{3/2} - 8\right) \approx 1.4397$$

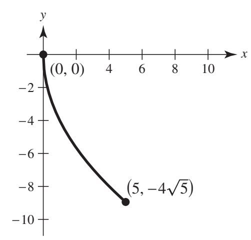


6. $\mathbf{r}(t) = t^2\mathbf{i} - 4t\mathbf{j}, [0, 5]$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = -4, \frac{dz}{dt} = 0$$

$$s = \int_0^5 \sqrt{4t^2 + 16}\,dt = \left[4\ln\left(\sqrt{t^2 + 4} + t\right) + t\sqrt{t^2 + 4}\right]_0^5$$

$$= \left(4\ln\left(\sqrt{29} + 5\right) + 5\sqrt{29}\right) - (4\ln 2) \approx 33.515$$

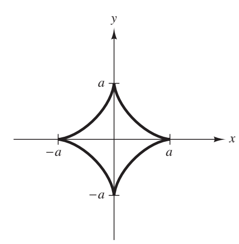


7. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j}, [0, 2\pi]$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t, \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$s = 4\int_0^{\pi/2} \sqrt{\left[-3a\cos^2 t\sin t\right]^2 + \left[3a\sin^2 t\cos t\right]^2}\,dt$$

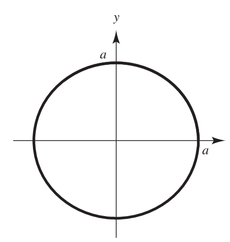
$$= 12a\int_0^{\pi/2} \sin t\cos t\,dt = 3a\int_0^{\pi/2} 2\sin 2t\,dt = \left[-3a\cos 2t\right]_0^{\pi/2} = 6a$$



8. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}, [0, 2\pi]$

$$\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = a\cos t$$

$$s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = \int_0^{2\pi} a\,dt = \left[at\right]_0^{2\pi} = 2\pi a$$



9. $\mathbf{r}(t) = 50\sqrt{2}t\mathbf{i} + \left[3 + 50\sqrt{2}t - 16t^2\right]\mathbf{j}$

$$\mathbf{v}(t) = 50\sqrt{2}\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j}$$

$$3 + 50\sqrt{2}t - 16t^2 = 0 \Rightarrow t \approx 4.4614$$

$$s = \int_0^{4.4614} \sqrt{\left(50\sqrt{2}\right)^2 + \left(50\sqrt{2} - 32t\right)^2}\,dt \approx 362.9 \text{ ft}$$

10. (a) $\mathbf{r}(t) = 40\sqrt{3}t\mathbf{i} + \left[4 + 40t - 16t^2\right]\mathbf{j}$

$\mathbf{v}(t) = 40\sqrt{3}\mathbf{i} + (40 - 32t)\mathbf{j}$

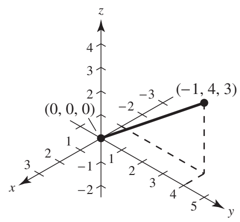
$40 - 32t = 0 \Rightarrow t = \dfrac{5}{4}$

(b) $s = \int_0^{2.596} \sqrt{\left(40\sqrt{3}\right)^2 + (40 - 32t)^2}\, dt \approx 190.15$ ft

11. $\mathbf{r}(t) = -t\mathbf{i} + 4t\mathbf{j} + 3t\mathbf{k}, [0, 1]$

$\dfrac{dx}{dt} = -1, \dfrac{dy}{dt} = 4, \dfrac{dz}{dt} = 3$

$s = \int_0^1 \sqrt{1 + 16 + 9}\, dt = \left[\sqrt{26}t\right]_0^1 = \sqrt{26}$

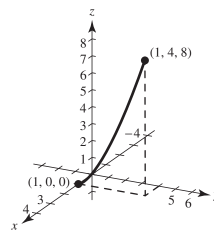


12. $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, [0, 2]$

$\dfrac{dx}{dt} = 0, \dfrac{dy}{dt} = 2t, \dfrac{dz}{dt} = 3t^2$

$s = \int_0^2 \sqrt{4t^2 + 9t^4}\, dt$

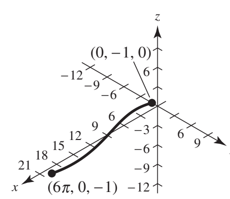
$= \int_0^2 \sqrt{4 + 9t^2}\, t\, dt = \dfrac{1}{27}\left(4 + 9t^2\right)^{3/2}\Big]_0^2 = \dfrac{1}{27}\left(40^{3/2} - 4^{3/2}\right) = \dfrac{1}{27}\left[80\sqrt{10} - 8\right]$



13. $\mathbf{r}(t) = \langle 4t, -\cos t, \sin t\rangle, \left[0, \dfrac{3\pi}{2}\right]$

$\dfrac{dx}{dt} = 4, \dfrac{dy}{dt} = \sin t, \dfrac{dz}{dt} = \cos t$

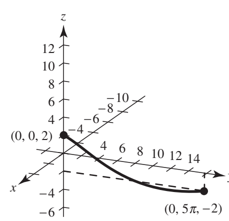
$s = \int_0^{3\pi/2} \sqrt{16 + \sin^2 t + \cos^2 t}\, dt = \int_0^{3\pi/2} \sqrt{17}\, dt = \left[\sqrt{17}t\right]_0^{3\pi/2} = \dfrac{3\pi}{2}\sqrt{17}$



14. $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t\rangle, [0, \pi]$

$\dfrac{dx}{dt} = 2\cos t, \dfrac{dy}{dt} = 5, \dfrac{dz}{dt} = -2\sin t$

$s = \int_0^{\pi} \sqrt{4\cos^2 t + 25 + 4\sin^2 t}\, dt = \int_0^{\pi} \sqrt{29}\, dt = \sqrt{29}\pi$

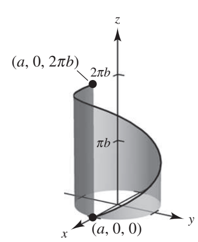


15. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}, [0, 2\pi]$

$\dfrac{dx}{dt} = -a\sin t, \dfrac{dy}{dt} = a\cos t, \dfrac{dz}{dt} = b$

$s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}\, dt$

$= \int_0^{2\pi} \sqrt{a^2 + b^2}\, dt = \left[\sqrt{a^2 + b^2}t\right]_0^{2\pi} = 2\pi\sqrt{a^2 + b^2}$

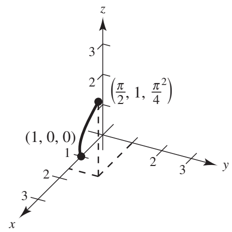


16. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2\rangle, \left[0, \dfrac{\pi}{2}\right]$

$$\frac{dx}{dt} = t\cos t,\ \frac{dy}{dt} = t\sin t,\ \frac{dz}{dt} = 2t$$

$$s = \int_0^{\pi/2} \sqrt{(t\cos t)^2 + (t\sin t)^2 + (2t)^2}\,dt$$

$$= \int_0^{\pi/2} \sqrt{5t^2}\,dt = \sqrt{5}\frac{t^2}{2}\Bigg]_0^{\pi/2} = \frac{\sqrt{5}\pi^2}{8}$$



17. $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j} + t^3\mathbf{k},\ [0, 2]$

(a) $\mathbf{r}(0) = \langle 0, 4, 0\rangle, \mathbf{r}(2) = \langle 2, 0, 8\rangle$

$$\text{distance} = \sqrt{2^2 + 4^2 + 8^2} = \sqrt{84}$$
$$= 2\sqrt{21} \approx 9.165$$

(b) $\mathbf{r}(0) = \langle 0, 4, 0\rangle$

$\mathbf{r}(0.5) = \langle 0.5, 3.75, 0.125\rangle$

$\mathbf{r}(1) = \langle 1, 3, 1\rangle$

$\mathbf{r}(1.5) = \langle 1.5, 1.75, 3.375\rangle$

$\mathbf{r}(2) = \langle 2, 0, 8\rangle$

$$\text{distance} \approx \sqrt{(0.5)^2 + (0.25)^2 + (0.125)^2}$$
$$+ \sqrt{(0.5)^2 + (0.75)^2 + (0.875)^2}$$
$$+ \sqrt{(0.5)^2 + (1.25)^2 + (2.375)^2}$$
$$+ \sqrt{(0.5)^2 + (1.75)^2 + (4.625)^2}$$
$$\approx 0.5728 + 1.2562 + 2.7300 + 4.9702$$
$$\approx 9.529$$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain 9.57057.

18. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$

(a) $s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du = \int_0^t \sqrt{(-2\sin u)^2 + (2\cos u)^2 + (1)^2}\,du = \int_0^t \sqrt{5}\,du = \left[\sqrt{5}u\right]_0^t = \sqrt{5}t$

(b) $\dfrac{s}{\sqrt{5}} = t$

$$x = 2\cos\left(\frac{s}{\sqrt{5}}\right), y = 2\sin\left(\frac{s}{\sqrt{5}}\right), z = \frac{s}{\sqrt{5}}$$

$$\mathbf{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

(c) When $s = \sqrt{5}$: $x = 2\cos 1 \approx 1.081$

$y = 2\sin 1 \approx 1.683$

$z = 1$

$(1.081, 1.683, 1.000)$

When $s = 4$: $x = 2\cos\dfrac{4}{\sqrt{5}} \approx -0.433$

$y = 2\sin\dfrac{4}{\sqrt{5}} \approx 1.953$

$z = \dfrac{4}{\sqrt{5}} \approx 1.789$

$(-0.433, 1.953, 1.789)$

(d) $\|\mathbf{r}'(s)\| = \sqrt{\left(-\frac{2}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{2}{\sqrt{5}}\cos\left(\frac{\sqrt{s}}{\sqrt{5}}\right)\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$

19. $\mathbf{r}(s) = \left(1 + \frac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \frac{\sqrt{2}}{2}s\right)\mathbf{j}$

$\mathbf{r}'(s) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and $\|\mathbf{r}'(s)\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s)$

$\mathbf{T}'(s) = \mathbf{0} \Rightarrow K = \|\mathbf{T}'(s)\| = 0$ (The curve is a line.)

20. $\mathbf{r}(s) = (3 + s)\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(s) = \mathbf{i}$ and $\|\mathbf{r}'(s)\| = 1$

$\mathbf{T}(s) = \mathbf{r}'(s)$

$\mathbf{T}'(s) = \mathbf{0} \Rightarrow K = \|\mathbf{T}'(s)\| = 0$ (The curve is a line.)

21. $\mathbf{r}(s) = \cos\left(\frac{s}{2}\right)\mathbf{i} + \frac{\sqrt{3}}{2}s\mathbf{j} + \sin\left(\frac{s}{2}\right)\mathbf{k}$

$\mathbf{r}'(s) = -\frac{1}{2}\sin\left(\frac{s}{2}\right)\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \frac{1}{2}\cos\left(\frac{s}{2}\right)\mathbf{k}, \ \|\mathbf{r}'(s)\| = 1$

$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = -\frac{1}{2}\sin\left(\frac{s}{2}\right)\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \frac{1}{2}\cos\left(\frac{s}{2}\right)\mathbf{k}$

$\mathbf{T}'(s) = -\frac{1}{4}\cos\left(\frac{s}{2}\right)\mathbf{i} - \frac{1}{4}\sin\left(\frac{s}{2}\right)\mathbf{k}$

$K = \|\mathbf{T}'(s)\| = \sqrt{\frac{1}{16}\cos^2\left(\frac{s}{2}\right) + \frac{1}{16}\sin^2\left(\frac{s}{2}\right)} = \frac{1}{4}$

22. $\mathbf{r}(s) = \cos s\mathbf{i} + \sin s\mathbf{j} + 5\mathbf{k}$

$\mathbf{r}'(s) = -\sin s\mathbf{i} + \cos s\mathbf{j}, \ \|\mathbf{r}'(s)\| = 1$

$\mathbf{T}(s) = -\sin s\mathbf{i} + \cos s\mathbf{j}$

$\mathbf{T}'(s) = -\cos s\mathbf{i} - \sin s\mathbf{j}$

$K = \|\mathbf{T}'(s)\| = \sqrt{\cos^2 s + \sin^2 s} = 1$

23. $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}, \ t = 1$

$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$

$\mathbf{T}(t) = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$

$\mathbf{T}'(t) = \mathbf{0}$

$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$

(The curve is a line.)

24. $\mathbf{r}(t) = t^2\mathbf{i} + \mathbf{j}, \ t = 2$

$\mathbf{v}(t) = 2t\mathbf{i}$

$\mathbf{T}(t) = \mathbf{i}$

$\mathbf{T}'(t) = \mathbf{0}$

$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$

25. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \ t = 1$

$\mathbf{v}(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$

$\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$

$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{j}$

$\mathbf{a}(1) = 2\mathbf{j}$

$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{N}(t) = \frac{1}{(t^4 + 1)^{1/2}}(\mathbf{i} + t^2\mathbf{j})$

$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$

$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{\sqrt{2}}{2}$

26. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{9}t^3\mathbf{j},\ t = 2$

$\mathbf{v}(t) = \mathbf{i} + \frac{1}{3}t^2\mathbf{j}$

$\mathbf{v}(2) = \mathbf{i} + \frac{4}{2}\mathbf{j},\ \|\mathbf{v}(2)\| = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$

$\mathbf{a}(t) = \frac{2}{3}t\mathbf{j}$

$\mathbf{a}(2) = \frac{4}{3}\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{i} + \frac{1}{3}t^2\mathbf{j}}{\sqrt{1 + \frac{t^4}{9}}} = \frac{3\mathbf{i} + t^2\mathbf{j}}{\sqrt{9 + t^4}}$$

$\mathbf{T}(2) = \frac{3\mathbf{i} + 4\mathbf{j}}{5}$

$\mathbf{N}(2) = \frac{-4\mathbf{i} + 3\mathbf{j}}{5}$

$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{4/5}{25/9} = \frac{36}{125}$$

27. $\mathbf{r}(t) = \langle t, \sin t\rangle,\ t = \frac{\pi}{2}$

$\mathbf{r}'(t) = \langle 1, \cos t\rangle, \|\mathbf{r}'(t)\| = \sqrt{1 + \cos^2 t}$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = \langle 1, 0\rangle,\ \left\|\mathbf{r}'\left(\frac{\pi}{2}\right)\right\| = 1$

$\mathbf{a}(t) = \langle 0, -\sin t\rangle,\ \mathbf{a}\left(\frac{\pi}{2}\right) = \langle 0, -1\rangle$

$\mathbf{T}(t) = \frac{1}{\sqrt{1 + \cos^2 t}}\langle 1, \cos t\rangle$

$\mathbf{T}\left(\frac{\pi}{2}\right) = \langle 1, 0\rangle$

$\mathbf{N}\left(\frac{\pi}{2}\right) = \langle 0, -1\rangle$

$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{1}{1} = 1$$

28. $\mathbf{r}(t) = \langle 5\cos t, 4\sin t\rangle,\ t = \frac{\pi}{3}$

$x(t) = 5\cos t,\ y(t) = 4\sin t$

$$K = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}} = \frac{|(-5\sin t)(-4\sin t) - (4\cos t)(-5\cos t)|}{\left[25\sin^2 t + 16\cos^2 t\right]^{3/2}} = \frac{20}{\left[25\sin^2 t + 16\cos^2 t\right]^{3/2}}$$

$$K\left(\frac{\pi}{3}\right) = \frac{20}{\left[25(3/4) + 16(1/4)^{3/2}\right]} = \frac{160\sqrt{91}}{8281}$$

29. $\mathbf{r}(t) = 4\cos 2\pi t\mathbf{i} + 4\sin 2\pi t\mathbf{j}$

$\mathbf{r}'(t) = -8\pi\sin 2\pi t\mathbf{i} + 8\pi\cos 2\pi t\mathbf{j}$

$\mathbf{T}(t) = -\sin 2\pi t\mathbf{i} + \cos 2t\mathbf{j}$

$\mathbf{T}'(t) = -2\pi\cos 2\pi t\mathbf{i} - 2\pi\sin 2\pi t\mathbf{j}$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2\pi}{8\pi} = \frac{1}{4}$$

30. $\mathbf{r}(t) = 2\cos \pi t\mathbf{i} + \sin \pi t\mathbf{j}$

$\mathbf{r}'(t) = -2\pi\sin \pi t\mathbf{i} + \pi\cos \pi t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \pi\sqrt{4\sin^2 \pi t + \cos^2 \pi t}$

$$\mathbf{T}(t) = \frac{-2\sin \pi t\mathbf{i} + \cos \pi t\mathbf{j}}{\sqrt{4\sin^2 \pi t + \cos^2 \pi t}}$$

$$\mathbf{T}'(t) = \frac{-2\pi\cos \pi t\mathbf{i} - 4\pi\sin \pi t\mathbf{j}}{\left(4\sin^2 \pi t + \cos^2 \pi t\right)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\frac{2\pi}{4\sin^2 \pi t + \cos^2 \pi t}}{\pi\sqrt{4\sin^2 \pi t + \cos^2 \pi t}} = \frac{2}{\left(4\sin^2 \pi t + \cos^2 \pi t\right)^{3/2}}$$

31. $\mathbf{r}(t) = a\cos \omega t\mathbf{i} + a\sin \omega t\mathbf{j}$

$\mathbf{r}'(t) = -a\omega\sin \omega t\mathbf{i} + a\omega\cos \omega t\mathbf{j}$

$\mathbf{T}(t) = -\sin \omega t\mathbf{i} + \cos \omega t\mathbf{j}$

$\mathbf{T}'(t) = -\omega\cos \omega t\mathbf{i} - \omega\sin \omega t\mathbf{j}$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\omega}{a\omega} = \frac{1}{a}$$

32. $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$

$$\mathbf{r}'(t) = -a\omega\sin(\omega t)\mathbf{i} + b\omega\cos(\omega t)\mathbf{j}$$

$$\mathbf{T}(t) = \frac{-a\sin(\omega t)\mathbf{i} + b\cos(\omega t)\mathbf{j}}{\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}$$

$$\mathbf{T}'(t) = \frac{-ab^2\omega\cos(\omega t)\mathbf{i} - a^2b\omega\sin(\omega t)\mathbf{j}}{\left[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)\right]^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{ab\omega}{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}{\omega\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}$$

$$= \frac{ab}{\left[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)\right]^{3/2}}$$

33. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$$

$$\mathbf{T}'(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\left(1 + 5t^2\right)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{\sqrt{5}}{\left(1 + 5t^2\right)}}{\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{\left(1 + 5t^2\right)^{3/2}}$$

34. $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \dfrac{1}{2}t^2\mathbf{k}$

$$\mathbf{r}'(t) = 4t\mathbf{i} + \mathbf{j} + t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{4t\mathbf{i} + \mathbf{j} + t\mathbf{k}}{\sqrt{1 + 17t^2}}$$

$$\mathbf{T}'(t) = \frac{4\mathbf{i} - 17t\mathbf{j} + \mathbf{k}}{\left(1 + 17t^2\right)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\mathbf{r}'(t)} = \frac{\sqrt{289t^2 + 17}}{\left(1 + 17t^2\right)^{3/2}} \Big/ \left(1 + 17t^2\right)^{1/2}$$

$$= \frac{\sqrt{17}}{\left(1 + 17t^2\right)^{3/2}}$$

35. $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k}$

$$\mathbf{r}'(t) = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{1}{5}[4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}]$$

$$\mathbf{T}'(t) = \frac{1}{5}[-3\cos t\mathbf{j} - 3\sin t\mathbf{k}]$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\mathbf{r}'(t)} = \frac{3/5}{5} = \frac{3}{25}$$

36. $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{2t}\cos t\mathbf{j} + e^{2t}\sin t\mathbf{k}$

$$\mathbf{r}'(t) = 2e^{2t}\mathbf{i} + \left(2e^{2t}\cos t - e^{2t}\sin t\right)\mathbf{j} + \left(2e^{2t}\sin t + e^{2t}\cos t\right)\mathbf{k} = e^{2t}\left[2\mathbf{i} + (2\cos t - \sin t)\mathbf{j} + (2\sin t + \cos t)\mathbf{k}\right]$$

$$\|\mathbf{r}'(t)\| = e^{2t}\left[4 + \left(4\cos^2 t - 4\cos t\sin t + \sin^2 t\right) + \left(4\sin^2 t + 4\sin t\cos t + \cos^2 t\right)\right]^{1/2} = e^{2t}[9]^{1/2} = 3e^{2t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2}{3}\mathbf{i} + \left(\frac{2}{3}\cos t - \frac{1}{3}\sin t\right)\mathbf{j} + \left(\frac{2}{3}\sin t + \frac{1}{3}\cos t\right)\mathbf{k}$$

$$\mathbf{T}'(t) = \left(-\frac{2}{3}\sin t - \frac{1}{3}\cos t\right)\mathbf{j} + \left(\frac{2}{3}\cos t - \frac{1}{3}\sin t\right)\mathbf{k}$$

$$\|\mathbf{T}'(t)\| = \left[\left(\frac{4}{9}\sin^2 t + \frac{1}{9}\cos^2 t + \frac{4}{9}\sin t\cos t\right) + \left(\frac{4}{9}\cos^2 t + \frac{1}{9}\sin^2 t - \frac{4}{9}\cos t\sin t\right)\right]^{1/2} = \frac{\sqrt{5}}{3}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{\sqrt{5}}{3}}{3e^{2t}} = \frac{\sqrt{5}}{9e^{2t}}$$

37. $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j},\ P(-3, 2) \Rightarrow t = -1$

$x = 3t,\ x' = 3,\ x'' = 0$

$y = 2t^2,\ y' = 4t,\ y'' = 4$

$$K = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}} = \frac{|3(4) - 0|}{\left[9 + (4t)^2\right]^{3/2}}$$

At $t = -1$, $K = \dfrac{12}{(9 + 16)^{3/2}} = \dfrac{12}{125}$

38. $\mathbf{r}(t) = e^t\mathbf{i} + 4t\mathbf{j},\ P(1, 0) \Rightarrow t = 0$

$x = e^t,\ x' = e^t,\ x'' = e^t$

$y = 4t,\ y' = 4,\ y'' = 0$

$$K = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}} = \frac{|0 - 4|}{(1 + 16)^{3/2}} = \frac{4}{17^{3/2}}$$

39. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^3}{4}\mathbf{k},\ P(2, 4, 2) \Rightarrow t = 2$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \dfrac{3}{4}t^2\mathbf{k}$

$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k},\ \|\mathbf{r}'(2)\| = \sqrt{26}$

$\mathbf{r}''(t) = 2\mathbf{j} + \dfrac{3}{2}t\mathbf{k}$

$\mathbf{r}''(2) = 2\mathbf{j} + 3\mathbf{k}$

$$\mathbf{r}'(2) \times \mathbf{r}'(2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ 0 & 2 & 3 \end{vmatrix} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

$\|\mathbf{r}'(2) \times \mathbf{r}''(2)\| = \sqrt{49} = 7$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{7}{26^{3/2}} = \frac{7\sqrt{26}}{676}$$

40. $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j} + e^t\mathbf{k},\ P(1, 0, 1) \Rightarrow t = 0$

$\mathbf{r}'(t) = \left(e^t \cos t - e^t \sin t\right)\mathbf{i} + \left(e^t \sin t + e^t \cos t\right)\mathbf{j} + e^t\mathbf{k}$

$$\|\mathbf{r}'(t)\| = e^t\sqrt{\left(\cos^2 t - 2\cos t \sin t + \sin^2 t\right) + \left(\sin^2 t + 2\sin t \cos t + \cos^2 t\right) + 1} = \sqrt{3}e^t$$

$\mathbf{T}(t) = \dfrac{1}{\sqrt{3}}\left[(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + \mathbf{k}\right]$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{3}}\left[(-\sin t - \cos t)\mathbf{i} + (\cos t - \sin t)\mathbf{j}\right]$

$\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \|\mathbf{r}'(0)\| = \sqrt{3}$

$\mathbf{T}'(0) = \dfrac{1}{\sqrt{3}}(-\mathbf{i} + \mathbf{j}) \Rightarrow \|\mathbf{T}'(0)\| = \dfrac{\sqrt{2}}{\sqrt{3}}$

$$K = \frac{\|\mathbf{T}'(0)\|}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}/\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{2}}{3}$$

41. $y = 6x,\ x = 3$

This is a line, so the curvature is 0.

$K = 0,\ \dfrac{1}{K}$ is undefined.

42. $y = x - \dfrac{4}{x}, x = 2$

$y' = 1 + \dfrac{4}{x^2}, y'' = -\dfrac{8}{x^3}$

At $x = 2, y' = 2$ and $y'' = -1$.

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{1}{\left[1 + 2^2\right]^{3/2}} = \frac{1}{5^{3/2}} \approx 0.0894$$

$$\frac{1}{K} = 5^{3/2}$$

43. $y = 5x^2 + 7, x = -1$

$y' = 10x, y'' = 10$

At $x = -1, y' = -10$ and $y'' = 10$.

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{10}{\left[1 + 10^2\right]^{3/2}} = \frac{10}{101^{3/2}}$$

$$\frac{1}{K} = \frac{101^{3/2}}{10}$$

44. $y = 2\sqrt{9 - x^2}, x = 0$

$y' = \dfrac{-2x}{\sqrt{9 - x^2}}, y'' = \dfrac{-18}{\left(9 - x^2\right)^{3/2}}$

At $x = 0, y' = 0$ and $y'' = -\dfrac{2}{3}$.

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\frac{2}{3}}{1} = \frac{2}{3}$$

$$\frac{1}{K} = \frac{3}{2}$$

45. $y = \sin 2x, x = \dfrac{\pi}{4}$

$y' = 2\cos 2x, y'' = -4\sin 2x$

At $x = \dfrac{\pi}{4}, y' = 0$ and $y'' = -4$.

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{4}{1} = 4$$

$$\frac{1}{K} = \frac{1}{4}$$

46. $y = e^{-x/4}, x = 8$

$y' = -\dfrac{1}{4}e^{-x/4}, y'' = \dfrac{1}{16}e^{-x/4}$

At $x = 8, y' = -\dfrac{1}{4}e^{-2}$ and $y'' = \dfrac{1}{16}e^{-2}$.

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\frac{1}{16}e^{-2}}{\left[1 + \frac{1}{16}e^{-4}\right]^{3/2}} = \frac{1}{16e^2\left[1 + \frac{1}{16e^4}\right]^{3/2}}$$

$$\frac{1}{K} = 16e^2\left[1 + \frac{1}{16e^4}\right]^{3/2}$$

47. $y = x^3, x = 2$

$y' = 3x^2, y'' = 6x$

At $x = 2, y = 8, y' = 12, y'' = 12$

$$K = \frac{12}{\left[1 + (12)^2\right]^{3/2}} = \frac{12}{(145)^{3/2}}$$

$$\frac{1}{K} = \frac{145\sqrt{145}}{12}$$

48. $y = x^n, x = 1, n \geq 2$

$y' = nx^{n-1}$

$y'' = n(n - 1)x^{n-2}$

At $x = 1, y = 1, y' = n, y'' = n(n - 1)$

$$K = \frac{n(n - 1)}{\left[1 + n^2\right]^{3/2}}$$

49. $y = (x-1)^2 + 3,\ y' = 2(x-1),\ y'' = 2$

$$K = \frac{2}{\left(1 + \left[2(x-1)\right]^2\right)^{3/2}} = \frac{2}{\left[1 + 4(x-1)^2\right]^{3/2}}$$

$$\frac{dK}{dx} = \frac{-24(x-1)}{\left[1 + 4(x-1)^2\right]^{5/2}}$$

(a) K is maximum when $x = 1$ or at the vertex $(1, 3)$.

(b) $\lim_{x\to\infty} K = 0$

50. $y = x^3,\ y' = 3x^2,\ y'' = 6x$

$$K = \left|\frac{6x}{\left(1 + 9x^4\right)^{3/2}}\right|$$

$$\frac{dK}{dx} = \frac{-6\left(45x^4 - 1\right)}{\left(1 + 9x^4\right)^{5/2}}$$

(a) K is maximum at $\left(\frac{1}{\sqrt[4]{45}}, \frac{1}{\sqrt[4]{45^3}}\right), \left(\frac{-1}{\sqrt[4]{45}}, \frac{-1}{\sqrt[4]{45^3}}\right)$.

(b) $\lim_{x\to\infty} K = 0$

51. $y = x^{2/3},\ y' = \frac{2}{3}x^{-1/3},\ y'' = -\frac{2}{9}x^{-4/3}$

$$K = \left|\frac{(-2/9)x^{-4/3}}{\left[1 + (4/9)x^{-2/3}\right]^{3/2}}\right| = \left|\frac{6}{x^{1/3}\left(9x^{2/3} + 4\right)^{3/2}}\right|$$

$$\frac{dK}{dx} = \frac{-8\left(9x^{2/3} + 1\right)}{x^{4/3}\left(9x^{2/3} + 4\right)^{5/2}}$$

(a) $K \to \infty$ as $x \to 0$. No maximum

(b) $\lim_{x\to\infty} K = 0$

52. $y = \frac{1}{x},\ y' = -\frac{1}{x^2},\ y'' = \frac{2}{x^3}$. Assume $x > 0$.

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\left|2/x^3\right|}{\left(1 + 1/x^4\right)^{3/2}} = \frac{2x^3}{\left(x^4 + 1\right)^{3/2}}$$

$$\frac{dK}{dx} = \frac{6x^2\left(1 - x^4\right)}{\left(x^4 + 1\right)^{5/2}}$$

(a) K has a maximum at $x = 1$
(and $x = -1$ by symmetry).

(b) $\lim_{x\to\infty} K = 0$

53. $y = \ln x,\ y' = \frac{1}{x},\ y'' = -\frac{1}{x^2}$

$$K = \left|\frac{-1/x^2}{\left[1 + (1/x)^2\right]^{3/2}}\right| = \frac{x}{\left(x^2 + 1\right)^{3/2}}$$

$$\frac{dK}{dx} = \frac{-2x^2 + 1}{\left(x^2 + 1\right)^{5/2}}$$

(a) K has a maximum when $x = \frac{1}{\sqrt{2}}$.

(b) $\lim_{x\to\infty} K = 0$

54. $y = e^x,\ y' = y'' = e^x$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{e^x}{\left(1 + e^{2x}\right)^{3/2}}$$

$$\frac{dK}{dx} = \frac{e^x\left(1 - 2e^{2x}\right)}{\left(1 + e^{2x}\right)^{5/2}}$$

(a) $1 - 2e^{2x} = 0 \Rightarrow e^{2x} = \frac{1}{2} \Rightarrow x = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}\ln 2$

K has maximum curvature at $x = -\frac{1}{2}\ln 2$.

(b) $\lim_{x\to\infty} K = 0$

55. $y = 1 - x^4,\ y' = -4x^3,\ y'' = -12x^2$

$$K = \frac{12x^2}{\left[1 + \left(-4x^3\right)^2\right]^{3/2}} = 0 \Rightarrow x = 0$$

Curvature is 0 at $(0, 1)$.

56. $y = (x-2)^6 + 3x$

$y' = 6(x-2)^5 + 3$

$y'' = 30(x-2)^4$

$$K = \frac{30(x-2)^4}{\left[1 + (y')^2\right]^{3/2}} = 0 \text{ at } x = 2$$

Curvature is 0 at $(2, 6)$.

57. $y = \cos\frac{x}{2}, y' = -\frac{1}{2}\sin\frac{x}{2}, y'' = -\frac{1}{4}\cos\frac{x}{2}$

$$K = \frac{\left|-\frac{1}{4}\cos\frac{x}{2}\right|}{\left[1 + (y')^2\right]^{3/2}} = 0 \Rightarrow \cos\frac{x}{2} = 0$$

So, $\frac{x}{2} = \frac{\pi}{2} + n\pi \Rightarrow x = \pi + 2n\pi$

Curvature is 0 at $(\pi + 2n\pi, 0)$.

58. $y = \sin x,\ y' = \cos x,\ y'' = -\sin x$

$$K = \frac{|\sin x|}{\left(1 + \cos^2 x\right)^{3/2}} = 0 \text{ for } x = n\pi.$$

Curvature is 0 for $x = n\pi$: $(n\pi, 0)$

59. $y = e^{cx}, y' = ce^{cx}, y'' = c^2e^{cx}$

$$K = \frac{c^2e^{cx}}{\left[1 + \left(ce^{cx}\right)^2\right]^{3/2}}$$

At $x = 0$, $K = \frac{c^2}{\left[1 + c^2\right]^{3/2}}$.

$$\frac{dK}{dc} = \frac{c\left(2 - c^2\right)}{\left(c^2 + 1\right)^{5/2}}$$

K is a maximum at $c = \pm\sqrt{2}$.

60. $K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}}$

At the smooth relative extremum $y' = 0$, so $K = |y''|$.

Yes, for example, $y = x^4$ has a curvature of 0 at its relative minimum $(0, 0)$. The curvature is positive at any other point on the curve.

61. $f(x) = x^4 - x^2$

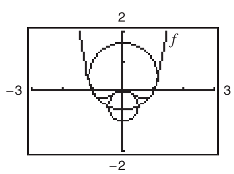
(a) $K = \frac{2\left|6x^2 - 1\right|}{\left|16x^6 - 16x^4 + 4x^2 + 1\right|^{3/2}}$

(b) For $x = 0$, $K = 2$. $f(0) = 0$. At $(0, 0)$, the circle of curvature has radius $\frac{1}{2}$. Using the symmetry of the graph of f, you obtain $x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$.

For $x = 1$, $K = \left(2\sqrt{5}\right)/5$. $f(1) = 0$. At $(1, 0)$, the circle of curvature has radius $\frac{\sqrt{5}}{2} = \frac{1}{K}$.

Using the graph of f, you see that the center of curvature is $\left(0, \frac{1}{2}\right)$. So,

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}.$$



To graph these circles, use

$$y = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2} \text{ and } y = \frac{1}{2} \pm \sqrt{\frac{5}{4} - x^2}.$$

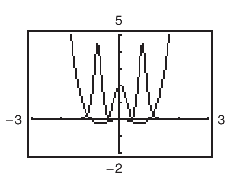
(c) The curvature tends to be greatest near the extrema of f, and K decreases as $x \to \pm\infty$. f and K, however, do not have the same critical numbers.

Critical numbers of f:

$x = 0, \pm\frac{\sqrt{2}}{2} \approx \pm 0.7071$

Critical numbers of K:

$x = 0, \pm 0.7647, \pm 0.4082$



62. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

(a) $\dfrac{dx}{dt} = 1, \dfrac{dy}{dt} = 2t$

$$s = \int_0^2 \sqrt{1 + 4t^2}\, dt = \frac{1}{2}\int_0^2 \sqrt{1 + 4t^2}(2)\, dt(u = 2t) = \frac{1}{2} \cdot \frac{1}{2}\left[2t\sqrt{1 + 4t^2} + \ln\left|2t + \sqrt{1 + 4t^2}\right|\right]_0^2 \text{ (Theorem 8.2)}$$

$$= \frac{1}{4}\left[4\sqrt{17} + \ln\left|4 + \sqrt{17}\right|\right] \approx 4.647$$

(b) Let $y = x^2,\ y' = 2x,\ y'' = 2$

At $t = 0,\ x = 0,\ y = 0,\ y' = 0,\ y'' = 2$

$$K = \frac{2}{[1 + 0]^{3/2}} = 2$$

At $t = 1,\ x = 1,\ y = 1,\ y' = 2,\ y'' = 2$

$$K = \frac{2}{\left[1 + (2)^2\right]^{3/2}} = \frac{2}{5^{3/2}} \approx 0.179$$

At $t = 2,\ x = 2,\ y = 4,\ y' = 4\ y'' = 2$

$$K = \frac{2}{[1 + 16]^{3/2}} = \frac{2}{17^{3/2}} \approx 0.0285$$

(c) As t changes from 0 to 2, the curvature decreases.

63. $y_1 = ax(b - x),\ y_2 = \dfrac{x}{x + 2}$

You observe that $(0, 0)$ is a solution point to both equations. So, the point P is origin.

$$y_1 = ax(b - x),\ y_1' = a(b - 2x),\ y_1'' = -2a$$

$$y_2 = \frac{x}{x + 2},\ y_2' = \frac{2}{(x + 2)^2},\ y_2'' = \frac{-4}{(x + 2)^3}$$

At P, $y_1'(0) = ab$ and $y_2'(0) = \dfrac{2}{(0 + 2)^2} = \dfrac{1}{2}$.

Because the curves have a common tangent at P, $y_1'(0) = y_2'(0)$ or $ab = \dfrac{1}{2}$. So, $y_1'(0) = \dfrac{1}{2}$.

Because the curves have the same curvature at P, $K_1(0) = K_2(0)$.

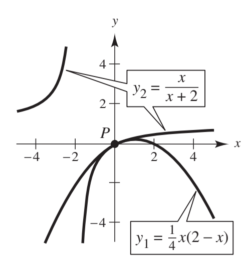
$$K_1(0) = \left|\frac{y_1''(0)}{\left[1 + (y_1(0))^2\right]^{3/2}}\right| = \left|\frac{-2a}{\left[1 + (1/2)^2\right]^{3/2}}\right|$$

$$K_2(0) = \left|\frac{y_2''(0)}{\left[1 + (y_2(0))^2\right]^{3/2}}\right| = \left|\frac{-1/2}{\left[1 + (1/2)^2\right]^{3/2}}\right|$$

So, $2a = \pm\dfrac{1}{2}$ or $a = \pm\dfrac{1}{4}$. In order that the curves intersect at only one point, the parabola must be concave downward. So,

$$a = \frac{1}{4} \text{ and } b = \frac{1}{2a} = 2.$$

$$y_1 = \frac{1}{4}x(2 - x) \text{ and } y_2 = \frac{x}{x + 2}$$



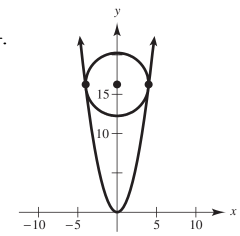
64. From the shape of the ellipse, you see that the curvature is greatest at the endpoints of the major axis, $(\pm 2, 0)$, and least at the endpoints of the minor axis, $(0, \pm 1)$.

65. (a) Imagine dropping the circle $x^2 + (y - k)^2 = 16$

into the parabola $y = x^2$. The circle will drop to the point where the tangents to the circle and parabola are equal.

$y = x^2$ and $x^2 + (y - k)^2 = 16 \Rightarrow x^2 + (x^2 - k)^2 = 16$

Taking derivatives, $2x + 2(y - k)y' = 0$ and $y' = 2x$. So, $(y - k)y' = -x \Rightarrow y' = \dfrac{-x}{y - k}$.



So, $\dfrac{-x}{y - k} = 2x \Rightarrow -x = 2x(y - k) \Rightarrow -1 = 2(x^2 - k) \Rightarrow x^2 - k = -\dfrac{1}{2}$.

So, $x^2 + (x^2 - k)^2 = x^2 + \left(-\dfrac{1}{2}\right)^2 = 16 \Rightarrow x^2 = 15.75$.

Finally, $k = x^2 + \dfrac{1}{2} = 16.25$, and the center of the circle is 16.25 units from the vertex of the parabola. Because the radius of the circle is 4, the circle is 12.25 units from the vertex.

(b) In 2-space, the parabola $z = y^2$ $\left(\text{or } z = x^2\right)$ has a curvature of $K = 2$ at $(0, 0)$. The radius of the largest sphere that will touch the vertex has radius $= 1/K = \dfrac{1}{2}$.

66. $s = \dfrac{c}{\sqrt{K}}$

$$y = \frac{1}{3}x^3$$

$$y' = x^2$$

$$y'' = 2x$$

$$K = \left|\frac{2x}{\left(1 + x^4\right)^{3/2}}\right|$$

When $x = 1$: $K = \dfrac{1}{\sqrt{2}}$

$$s = \frac{c}{\sqrt{1/\sqrt{2}}} = \sqrt[4]{2}c$$

$$30 = \sqrt[4]{2}c \Rightarrow c = \frac{30}{\sqrt[4]{2}}$$

At $x = \dfrac{3}{2}$, $K = \dfrac{3}{\left[1 + (81/16)\right]^{3/2}} \approx 0.201$

$$s\left(\frac{3}{2}\right) = \frac{c}{\sqrt{K}} = \frac{30/\sqrt[4]{2}}{\sqrt{K}} \approx 56.27 \text{ mi/h}$$

67. $P(x_0, y_0)$ point on curve $y = f(x)$. Let (α, β) be the center of curvature. The radius of curvature is $\dfrac{1}{K}$.

$y' = f'(x)$. Slope of normal line at (x_0, y_0) is $\dfrac{-1}{f'(x_0)}$.

Equation of normal line: $y - y_0 = \dfrac{-1}{f'(x_0)}(x - x_0)$

(α, β) is on the normal line: $-f'(x_0)(\beta - y_0) = \alpha - x_0$ Equation 1

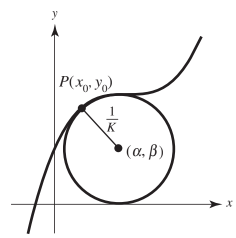
(x_0, y_0) lies on the circle: $(x_0 - \alpha)^2 + (y_0 - \beta)^2 = \left(\dfrac{1}{K}\right)^2 = \left[\dfrac{\left(1 + f'(x_0)^2\right)^{3/2}}{\left|f''(x_0)\right|}\right]^2$ Equation 2

Substituting Equation 1 into Equation 2:

$$\left[f'(x_0)(\beta - y_0)\right]^2 + (y_0 - \beta)^2 = \left(\frac{1}{K}\right)^2$$

$$(\beta - y_0)^2 + \left[1 + f'(x_0)^2\right] = \frac{\left(1 + f'(x_0)^2\right)^3}{\left(f''(x_0)\right)^2}$$

$$(\beta - y_0)^2 = \frac{\left[1 + f'(x_0)^2\right]^2}{f''(x_0)^2}$$



When $f''(x_0) > 0$, $\beta - y_0 > 0$, and if $f''(x_0) < 0$, then $\beta - y_0 < 0$.

So $\beta - y_0 = \dfrac{1 + f'(x_0)^2}{f''(x_0)}$

$$\beta = y_0 + \frac{1 + f'(x_0)^2}{f''(x_0)} = y_0 + z$$

Similarly, $\alpha = x_0 - f'(x_0)z$.

68. (a) $y = f(x) = e^x$, $f'(x) = f''(x) = e^x$, $(0, 1)$

$$z = \frac{1 + f'(0)^2}{f''(0)} = 2$$

$$(\alpha, \beta) = (0 - 2, 1 + 2) = (-2, 3)$$

(b) $y = \dfrac{x^2}{2}$, $y' = x$, $y'' = 1$, $\left(1, \dfrac{1}{2}\right)$

$$z = \frac{1 + f'(1)^2}{f''(1)} = 2$$

$$(\alpha, \beta) = \left(1 - 2, \frac{1}{2} + 2\right) = \left(-1, \frac{5}{2}\right)$$

(c) $y = x^2$, $y' = 2x$, $y'' = 2$, $(0, 0)$

$$z = \frac{1 + f'(0)^2}{f''(0)} = \frac{1}{2}$$

$$(\alpha, \beta) = \left(0, 0 + \frac{1}{2}\right) = \left(0, \frac{1}{2}\right)$$

69. $r(\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} = f(\theta)\cos\theta\mathbf{i} + f(\theta)\sin\theta\mathbf{j}$

$x(\theta) = f(\theta)\cos\theta$

$y(\theta) = f(\theta)\sin\theta$

$x'(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta$

$y'(\theta) = f(\theta)\cos\theta + f'(\theta)\sin\theta$

$x''(\theta) = -f(\theta)\cos\theta - f'(\theta)\sin\theta - f'(\theta)\sin\theta + f''(\theta)\cos\theta = -f(\theta)\cos\theta - 2f'(\theta)\sin\theta + f''(\theta)\cos\theta$

$y''(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta + f'(\theta)\cos\theta + f''(\theta)\sin\theta = -f(\theta)\sin\theta + 2f'(\theta)\cos\theta + f''(\theta)\sin\theta$

$$K = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}} = \frac{\left|f^2(\theta) - f(\theta)f''(\theta) + 2(f'(\theta))^2\right|}{\left[f^2(\theta) + (f'(\theta))^2\right]^{3/2}} = \frac{\left|r^2 - rr'' + 2(r')^2\right|}{\left[r^2 + (r')^2\right]^{3/2}}$$

70. (a) $r = 1 + \sin\theta$

$r' = \cos\theta$

$r'' = -\sin\theta$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{\left|2\cos^2\theta - (1 + \sin\theta)(-\sin\theta) + (1 + \sin\theta)^2\right|}{\sqrt{\left[\cos^2\theta + (1 + \sin\theta)^2\right]^3}} = \frac{3(1 + \sin\theta)}{\sqrt{8(1 + \sin\theta)^3}} = \frac{3}{2\sqrt{2(1 + \sin\theta)}}$$

(b) $r = \theta$

$r' = 1$

$r'' = 0$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{2 + \theta^2}{(1 + \theta^2)^{3/2}}$$

(c) $r = a\sin\theta$

$r' = a\cos\theta$

$r = -a\sin\theta$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{\left|2a^2\cos^2\theta + a^2\sin^2\theta + a^2\sin^2\theta\right|}{\sqrt{\left[a^2\cos^2\theta + a^2\sin^2\theta\right]^3}} = \frac{2a^2}{a^3} = \frac{2}{a}, a > 0$$

(d) $r = e^\theta$

$r' = e^\theta$

$r'' = e^\theta$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{2e^{2\theta}}{(2e^{2\theta})^{3/2}} = \frac{1}{\sqrt{2}e^\theta}$$

71. $r = e^{a\theta}, a > 0$

$r' = ae^{a\theta}$

$r'' = a^2e^{a\theta}$

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{\left|2a^2e^{2a\theta} - a^2e^{2a\theta} + e^{2a\theta}\right|}{\left[a^2e^{2a\theta} + e^{2a\theta}\right]^{3/2}} = \frac{1}{e^{a\theta}\sqrt{a^2 + 1}}$$

(a) As $\theta \to \infty, K \to 0$.

(b) As $a \to \infty$, $K \to 0$.

72. At the pole, $r = 0$.

$$K = \frac{\left|2(r')^2 - rr'' + r^2\right|}{\left[(r')^2 + r^2\right]^{3/2}} = \frac{\left|2(r')^2\right|}{|r'|^3} = \frac{2}{|r'|}$$

73. $r = 4\sin 2\theta$

$r' = 8\cos 2\theta$

At the pole: $K = \dfrac{2}{|r'(0)|} = \dfrac{2}{8} = \dfrac{1}{4}$

74. $r = \cos 3\theta$

$r' = -3\sin 3\theta$

At the pole,

$$\theta = \frac{\pi}{6},\ r'\left(\frac{\pi}{6}\right) = -3,$$

and

$$K = \frac{2}{|r'(\pi/6)|} = \frac{2}{3}.$$

75. $x = f(t), y = g(t)$

$$y' = \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$y'' = \frac{\dfrac{d}{dt}\left[\dfrac{g'(t)}{f'(t)}\right]}{\dfrac{dx}{dt}}$$

$$= \frac{\dfrac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^2}}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^3}$$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\left|\dfrac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^3}\right|}{\left[1 + \left(\dfrac{g'(t)}{f'(t)}\right)^2\right]^{3/2}}$$

$$= \frac{\left|\dfrac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^3}\right|}{\sqrt{\left\{\dfrac{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}{\left[f'(t)\right]^2}\right\}^3}} = \frac{\left|f'(t)g''(t) - g'(t)f''(t)\right|}{\left(\left[f'(t)\right]^2 + \left[g'(t)\right]^2\right)^{3/2}}$$

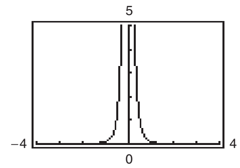
76. $x(t) = t^3,\ x'(t) = 3t^2,\ x''(t) = 6t$

$$y(t) = \frac{1}{2}t^2,\ y'(t) = t,\ y''(t) = 1$$

$$K = \frac{\left|(3t^2)(1) - (t)(6t)\right|}{\left[(3t^2)^2 + (t)^2\right]^{3/2}}$$

$$= \frac{3t^2}{|t^3|(9t^2 + 1)^{3/2}} = \frac{3}{|t|(9t^2 + 1)^{3/2}}$$

$K \to 0$ as $t \to \pm\infty$



77. $x(\theta) = a(\theta - \sin\theta)$ $\quad y(\theta) = a(1 - \cos\theta)$

$x'(\theta) = a(1 - \cos\theta)$ $\quad y'(\theta) = a\sin\theta$

$x''(\theta) = a\sin\theta$ $\quad y''(\theta) = a\cos\theta$

$$K = \frac{\left|x'(\theta)y''(\theta) - y'(\theta)x''(\theta)\right|}{\left[x'(\theta)^2 + y'(\theta)^2\right]^{3/2}}$$

$$= \frac{\left|a^2(1 - \cos\theta)\cos\theta - a^2\sin^2\theta\right|}{\left[a^2(1 - \cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}$$

$$= \frac{1}{a}\frac{|\cos\theta - 1|}{[2 - 2\cos\theta]^{3/2}}$$

$$= \frac{1}{a}\frac{1 - \cos\theta}{2\sqrt{2}[1 - \cos\theta]^{3/2}} \quad (1 - \cos \geq 0)$$

$$= \frac{1}{2a\sqrt{2 - 2\cos\theta}} = \frac{1}{4a}\csc\left(\frac{\theta}{2}\right)$$

Minimum: $\dfrac{1}{4a}$ $\quad (\theta = \pi)$

Maximum: none $\quad (K \to \infty$ as $\theta \to 0)$

78. (a) $\mathbf{r}(t) = 3t^2\mathbf{i} + (3t - t^3)\mathbf{j}$

$\mathbf{v}(t) = 6t\mathbf{i} + (3 - 3t^2)\mathbf{j}$

$$\frac{ds}{dt} = \|\mathbf{v}(t)\| = 3(1 + t^2),\ \frac{d^2s}{dt^2} = 6t$$

$$K = \frac{2}{3(1 + t^2)^2}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = 6t$$

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{2}{3(1 + t^2)^2} \cdot 9(1 + t^2)^2 = 6$$

(b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\frac{ds}{dt} = \|\mathbf{v}(t)\| = \sqrt{5t^2 + 1}$

$\frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & t \\ 0 & 2 & 1 \end{vmatrix} = -\mathbf{j} + 2\mathbf{k}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{5}}{(5t^2 + 1)^{3/2}}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{\sqrt{5}}{(5t^2 + 1)^{3/2}}(5t^2 + 1) = \frac{\sqrt{5}}{\sqrt{5t^2 + 1}}$$

79. $F = ma_{\mathbf{N}} = mK\left(\frac{ds}{dt}\right)^2$

$$= \left(\frac{5500 \text{ lb}}{32 \text{ ft/sec}^2}\right)\left(\frac{1}{100 \text{ ft}}\right)\left(\frac{30(5280) \text{ ft}}{3600 \text{ sec}}\right)^2 = 3327.5 \text{ lb}$$

80. $F = ma_{\mathbf{N}} = mK\left(\frac{ds}{dt}\right)^2$

$$= \left(\frac{6400 \text{ lb}}{32 \text{ ft/sec}^2}\right)\left(\frac{1}{250 \text{ ft}}\right)\left(\frac{35(5280) \text{ ft}}{3600 \text{ sec}}\right)^2 = \frac{94864}{45} \approx 2108.1 \text{ lb}$$

81. $y = \cosh x = \frac{e^x + e^{-x}}{2}$

$y' = \frac{e^x - e^{-x}}{2} = \sinh x$

$y'' = \frac{e^x + e^{-x}}{2} = \cosh x$

$$K = \frac{|\cosh x|}{\left[1 + (\sinh x)^2\right]^{3/2}} = \frac{\cosh x}{(\cosh^2 x)^{3/2}} = \frac{1}{\cosh^2 x} = \frac{1}{y^2}$$

82. (a) $K = \|\mathbf{T}'(s)\| = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds}\right\|$, by the Chain Rule

$$= \left\|\frac{d\mathbf{T}/dt}{ds/dt}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

(b) $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{r}'(t)}{ds/dt}$

$\mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$

$\mathbf{r}''(t) = \left(\frac{d^2s}{dt^2}\right)\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\left[\mathbf{T}(t) \times \mathbf{T}(t)\right] + \left(\frac{ds}{dt}\right)^2\left[\mathbf{T}(t) \times \mathbf{T}'(t)\right]$$

Because $\mathbf{T}(t) \times \mathbf{T}(t) = \mathbf{0}$ and $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$, you have:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \|\mathbf{r}'(t)\|^2\left[\mathbf{T}(t) \times \mathbf{T}'(t)\right]$$

$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t) \times \mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t)\|\|\mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2(1)K\|\mathbf{r}'(t)\|$ from (a)

So, $\frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = K.$

(c) $K = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)^3\|} = \dfrac{\dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}}{\|\mathbf{r}'(t)\|^2} = \dfrac{\dfrac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|}}{\|\mathbf{r}'(t)\|^2} = \dfrac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{r}'(t)\|^2}$

83. False. See Exploration on page 855.

84. True

85. True

86. True

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2$$

87. Let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $r = \|\mathbf{r}\| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$ and $r' = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. Then,

$$r\left(\frac{dr}{dt}\right) = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}\left[\frac{1}{2}\left\{[x(t)]^2 + [y(t)]^2 + [z(t)]^2\right\}^{-1/2} \cdot \left(2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)\right)\right]$$
$$= x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r} \cdot \mathbf{r}'.$$

88. $\mathbf{F} = m\mathbf{a} \Rightarrow m\mathbf{a} = \dfrac{-GmM}{r^3}\mathbf{r}$

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

Because $\mathbf{r}$ is a constant multiple of $\mathbf{a}$, they are parallel. Because $\mathbf{a} = \mathbf{r}''$ is parallel to $\mathbf{r}$, $\mathbf{r} \times \mathbf{r}'' = \mathbf{0}$. Also,

$\left(\dfrac{d}{dt}\right)(\mathbf{r} \times \mathbf{r}') = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{0} + \mathbf{0} = \mathbf{0}$. So, $\mathbf{r} \times \mathbf{r}'$ is a constant vector which we will denote by $\mathbf{L}$.

89. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where x, y, and z are function of t, and $r = \|\mathbf{r}\|$.

$$\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right] = \frac{r\mathbf{r}' - \mathbf{r}(dr/dt)}{r^2} = \frac{r\mathbf{r}' - \mathbf{r}\left[(\mathbf{r} \cdot \mathbf{r}')/r\right]}{r^2}$$
$$= \frac{r^2\mathbf{r}' - (\mathbf{r} \cdot \mathbf{r}')\mathbf{r}}{r^3} \text{ (using Exercise 105)}$$
$$= \frac{\left(x^2 + y^2 + z^2\right)\left(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}\right) - \left(xx' + yy' + zz'\right)\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right)}{r^3}$$
$$= \frac{1}{r^3}\left[\left(x'y^2 + x'z^2 - xyy' - xzz'\right)\mathbf{i} + \left(x^2y' + z^2y' - xx'y - zz'y\right)\mathbf{j} + \left(x^2z' + y^2z' - xx'z - yy'z\right)\mathbf{k}\right]$$
$$= \frac{1}{r^3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz' - y'z & -(xz' - x'z) & xy' - x'y \\ x & y & z \end{vmatrix} = \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}$$

90. $\dfrac{d}{dt}\left[\dfrac{\mathbf{r}'}{GM} \times \mathbf{L} - \dfrac{\mathbf{r}}{r}\right] = \dfrac{1}{GM}[\mathbf{r}' \times \mathbf{0} + \mathbf{r}'' \times \mathbf{L}] - \dfrac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}$

$$= \frac{1}{GM}\left[\mathbf{0} + \left(\frac{-GM\mathbf{r}}{r^3}\right) \times [\mathbf{r} \times \mathbf{r}']\right] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}$$
$$= -\frac{\mathbf{r}}{r^3} \times [\mathbf{r} \times \mathbf{r}'] - \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}$$
$$= \frac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} - [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\} = \mathbf{0}$$

So, $\left(\dfrac{\mathbf{r}'}{GM}\right) \times \mathbf{L} - \left(\dfrac{\mathbf{r}}{r}\right)$ is a constant vector which we will denote by $\mathbf{e}$.

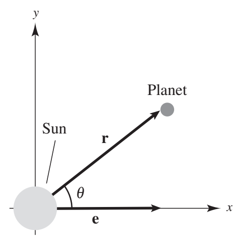
91. From Exercise 88, you have concluded that planetary motion is planar. Assume that the planet moves in the xy-plane with the sum at the origin. From Exercise 90, you have

$$\mathbf{r}' \times \mathbf{L} = GM\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right).$$

Because $\mathbf{r}' \times \mathbf{L}$ and $\mathbf{r}$ are both perpendicular to $\mathbf{L}$, so is $\mathbf{e}$.

So, $\mathbf{e}$ lies in the xy-plane. Situate the coordinate system so that $\mathbf{e}$ lies along the positive x-axis and θ is the angle between $\mathbf{e}$ and $\mathbf{r}$. Let $e = \|\mathbf{e}\|$. Then $\mathbf{r} \cdot \mathbf{e} = \|\mathbf{r}\|\|\mathbf{e}\| \cos\theta = re\cos\theta$. Also,

$$\begin{aligned}\|\mathbf{L}\|^2 &= \mathbf{L} \cdot \mathbf{L}\\ &= (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{L}\\ &= \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{L})\\ &= \mathbf{r} \cdot \left[GM\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right)\right]\\ &= GM\left[\mathbf{r} \cdot \mathbf{e} + \frac{\mathbf{r} \cdot \mathbf{r}}{r}\right]\\ &= GM\left[re\cos\theta + r\right].\end{aligned}$$



So, $$\frac{\|\mathbf{L}\|^2/GM}{1 + e\cos\theta} = r$$

and the planetary motion is a conic section. Because the planet returns to its initial position periodically, the conic is an ellipse.

92. $\|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\|$

Let: $\mathbf{r} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$

$$\mathbf{r}' = r(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\frac{d\theta}{dt}\left(\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta} \cdot \frac{d\theta}{dt}\right)$$

Then: $$\mathbf{r} \times \mathbf{r}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ r\cos\theta & r\sin\theta & 0\\ -r\sin\theta\frac{d\theta}{dt} & r\cos\theta\frac{d\theta}{dt} & 0\end{vmatrix} = r^2\frac{d\theta}{dt}\mathbf{k} \text{ and } \|\mathbf{L}\| = \|\mathbf{r} \times \mathbf{r}'\| = r^2\frac{d\theta}{dt}.$$

93. $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$

So,

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}\|\mathbf{L}\|$$

and $\mathbf{r}$ sweeps out area at a constant rate.

94. Let P denote the period. Then

$$A = \int_0^P \frac{dA}{dt}\,dt = \frac{1}{2}\|\mathbf{L}\|P.$$

Also, the area of an ellipse is πab where $2a$ and $2b$ are the lengths of the major and minor axes.

$$\begin{aligned}\pi ab &= \frac{1}{2}\|\mathbf{L}\|P\\ P &= \frac{2\pi ab}{\|\mathbf{L}\|}\end{aligned}$$

$$P^2 = \frac{4\pi^2a^2}{\|\mathbf{L}\|^2}\left(a^2 - c^2\right) = \frac{4\pi^2a^2}{\|\mathbf{L}\|^2}a^2\left(1 - e^2\right) = \frac{4\pi^2a^4}{\|\mathbf{L}\|^2}\left(\frac{ed}{a}\right) = \frac{4\pi^2ed}{\|\mathbf{L}\|^2}a^3 = \frac{4\pi^2\left(\|\mathbf{L}\|^2/GM\right)}{\|\mathbf{L}\|^2}a^3 = \frac{4\pi^2}{GM}a^3 = Ka^3$$

Review Exercises for Chapter 12

1. $\mathbf{r}(t) = \tan t\mathbf{i} + \mathbf{j} + t\mathbf{k}$

(a) Domain: $t \neq \dfrac{\pi}{2} + n\pi$, n an integer

(b) Continuous for all $t \neq \dfrac{\pi}{2} + n\pi$, n an integer

2. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \dfrac{1}{t-4}\mathbf{j} + \mathbf{k}$

(a) Domain: $[0, 4)$ and $(4, \infty)$

(b) Continuous except at $t = 4$

3. $\mathbf{r}(t) = \sqrt{t^2 - 9}\,\mathbf{i} - \mathbf{j} + \ln(t-1)\mathbf{k}$

(a) $t^2 - 9 \geq 0 \Rightarrow t^2 \geq 9 \Rightarrow t \leq -3, t \geq 3$

$t - 1 > 0 \Rightarrow t > 1$

Domain: $[3, \infty)$

(b) Continuous for all $t \geq 3$

4. $\mathbf{r}(t) = (2t+1)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$

(a) Domain: $(-\infty, \infty)$

(b) Continuous for all t

5. $\mathbf{r}(t) = (2t+1)\mathbf{i} + t^2\mathbf{j} - \sqrt{t+2}\,\mathbf{k}$

(a) $\mathbf{r}(0) = \mathbf{i} - \sqrt{2}\,\mathbf{k}$

(b) $\mathbf{r}(-2) = -3\mathbf{i} + 4\mathbf{j}$

(c) $\mathbf{r}(c-1) = (2c-1)\mathbf{i} + (c-1)^2\mathbf{j} - \sqrt{c+1}\,\mathbf{k}$

(d) $$\begin{aligned}\mathbf{r}(1+\Delta t) - \mathbf{r}(1) &= \left[2(1+\Delta t)+1\right]\mathbf{i} + (1+\Delta t)^2\mathbf{j} - \sqrt{3+\Delta t}\,\mathbf{k} - \left(3\mathbf{i} + \mathbf{j} - \sqrt{3}\,\mathbf{k}\right)\\ &= 2\Delta t\,\mathbf{i} + \left(\Delta t^2 + 2\Delta t\right)\mathbf{j} - \left(\sqrt{3+\Delta t} - \sqrt{3}\right)\mathbf{k}\end{aligned}$$

6. (a) $\mathbf{r}(0) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}\left(\dfrac{\pi}{2}\right) = -\dfrac{\pi}{2}\mathbf{k}$

(c) $\mathbf{r}(s-\pi) = 3\cos(s-\pi)\mathbf{i} + \left(1 - \sin(s-\pi)\right)\mathbf{j} - (s-\pi)\mathbf{k}$

(d) $$\begin{aligned}\mathbf{r}(\pi+\Delta t) - \mathbf{r}(\pi) &= \left(3\cos(\pi+\Delta t)\mathbf{i} + \left(1 - \sin(\pi+\Delta t)\right)\mathbf{j} - (\pi+\Delta t)\mathbf{k}\right) - (-3\mathbf{i} + \mathbf{j} - \pi\mathbf{k})\\ &= (-3\cos\Delta t + 3)\mathbf{i} + \sin\Delta t - \Delta t\,\mathbf{k}\end{aligned}$$

7. $P(3, 0, 5)$, $Q(2, -2, 3)$

$\mathbf{v} = \overrightarrow{PQ} = \langle -1, -2, -2\rangle$

$\mathbf{r}(t) = (3-t)\mathbf{i} - 2t\mathbf{j} + (5-2t)\mathbf{k}$, $0 \leq t \leq 1$

$x = 3 - t$, $y = -2t$, $z = 5 - 2t$, $0 \leq t \leq 1$

(Answers may vary)

8. $P(-2, -3, 8)$, $Q(5, 1, -2)$

$\mathbf{v} = \overrightarrow{PQ} = \langle 7, 4, -10\rangle$

$\mathbf{r}(t) = (-2+7t)\mathbf{i} + (-3+4t)\mathbf{j} + (8-10t)\mathbf{k}$, $0 \leq t \leq 1$

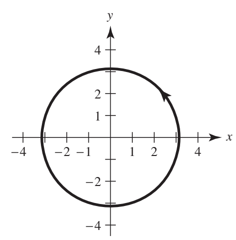
$x = -2 + 7t$, $y = -3 + 4t$, $z = 8 - 10t$, $0 \leq t \leq 1$

(Answers may vary)

9. $\mathbf{r}(t) = \langle \pi\cos t, \pi\sin t\rangle$

$x = \pi\cos t$, $y = \pi\sin t$

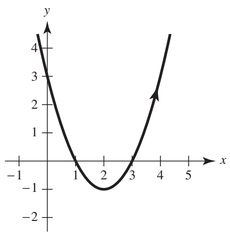
$x^2 + y^2 = \pi^2$, circle



10. $\mathbf{r}(t) = \langle t + 2, t^2 - 1\rangle$

$x = t + 2 \Rightarrow t = x - 2$

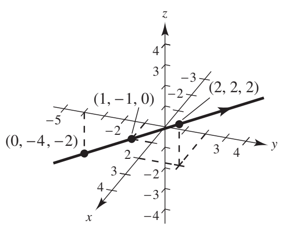
$y = t^2 - 1 = (x - 2)^2 - 1$, parabola



11. $\mathbf{r}(t) = (t + 1)\mathbf{i} + (3t - 1)\mathbf{j} + 2t\mathbf{k}$

$x = t + 1, y = 3t - 1, z = 2t$

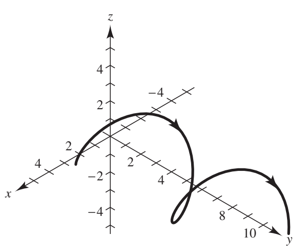
This is a line passing through the points $(1, -1, 0)$ and $(2, 2, 2)$.



12. $\mathbf{r}(t) = 2\cos t\mathbf{i} + t\mathbf{j} + 2\sin t\mathbf{k}$

$x = 2\cos t, y = t, z = 2\sin t$

$x^2 + z^2 = 4$, Circular helix



13. $3x + 4y - 12 = 0$

Let $x = t$, then $y = \dfrac{12 - 3t}{4}$.

$r(t) = t\mathbf{i} + \dfrac{12 - 3t}{4}\mathbf{j}$

Alternate solution: $x = 4t, y = 3 - 3t$

$r(t) = 4t\mathbf{i} + (3 - 3t)\mathbf{j}$

14. $y = 9 - x^2$

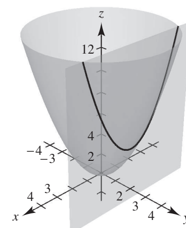
Let $x = t$, then $y = 9 - t^2$.

$r(t) = t\mathbf{i} + (9 - t^2)\mathbf{j}$

15. $z = x^2 + y^2, y = 2$

$z = x^2 + 4$

$x = t, y = 2, z = t^2 + 4$

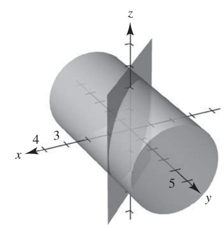


16. $x^2 + z^2 = 4, x - y = 0, t = x$

$x = t, y = t, z = \pm\sqrt{4 - t^2}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\,\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} - \sqrt{4 - t^2}\,\mathbf{k}$



17. $\displaystyle\lim_{t\to 3}\left(\sqrt{3 - t}\,\mathbf{i} + \ln t\mathbf{j} - \frac{1}{t}\mathbf{k}\right) = \ln 3\mathbf{j} - \frac{1}{3}\mathbf{k}$

18. $\displaystyle\lim_{t\to 0}\left(\frac{\sin 2t}{t}\mathbf{i} + e^{-t}\mathbf{j} + 4\mathbf{k}\right) = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

$\left(\textbf{Note: } \displaystyle\lim_{t\to 0}\frac{\sin 2t}{t} = \lim_{t\to 0}\frac{2\cos 2t}{1} = 2\right)$

19. $\mathbf{r}(t) = (t^2 + 4t)\mathbf{i} - 3t^2\mathbf{j}$

(a) $\mathbf{r}'(t) = (2t + 4)\mathbf{i} - 6t\mathbf{j}$

(b) $\mathbf{r}''(t) = 2\mathbf{i} - 6\mathbf{j}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (2t + 4)(2) + (-6t)(-6)$
$= 40t + 8$

20. $\mathbf{r}(t) = 5\cos t\mathbf{i} + 2\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -5\sin t\mathbf{i} + 2\cos t\mathbf{j}$

(b) $\mathbf{r}''(t) = -5\cos t\mathbf{i} - 2\sin t\mathbf{j}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (-5\sin t)(-5\cos t) + (2\cos t)(-2\sin t)$
$= 21\sin t\cos t$

21. $\mathbf{r}(t) = 2t^3\mathbf{i} + 4t\mathbf{j} - t^2\mathbf{k}$

(a) $\mathbf{r}'(t) = 6t^2\mathbf{i} + 4\mathbf{j} - 2t\mathbf{k}$

(b) $\mathbf{r}''(t) = 12t\mathbf{i} - 2\mathbf{k}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 6t^2(12t) + (-2t)(-2)$

$= 72t^3 + 4t$

(d) $\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6t^2 & 4 & -2t \\ 12t & 0 & -2 \end{bmatrix}$

$= -8\mathbf{i} - (-12t^2 + 24t^2)\mathbf{j} + (-48t)\mathbf{k}$

$= -8\mathbf{i} - 12t^2\mathbf{j} - 48t\mathbf{k}$

22. $\mathbf{r}(t) = (4t + 3)\mathbf{i} + t^2\mathbf{j} + (2t^2 + 4)\mathbf{k}$

(a) $\mathbf{r}'(t) = 4\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}$

(b) $\mathbf{r}''(t) = 2\mathbf{j} + 4\mathbf{k}$

(c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (2t)(2) + (4t)(4)$

$= 4t + 16t = 20t$

(d) $\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2t & 4t \\ 0 & 2 & 4 \end{bmatrix}$

$= (8t - 8t)\mathbf{i} - (16 - 0)\mathbf{j} + (8 - 0)\mathbf{k}$

$= -16\mathbf{j} + 8\mathbf{k}$

25. $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}, \mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$

$\mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}, \mathbf{u}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$

(a) $\mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

(b) $\frac{d}{dt}[\mathbf{u}(t) - 2\mathbf{r}(t)] = \mathbf{u}'(t) - 2\mathbf{r}'(t)$

$= (\mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}) - 2(3\mathbf{i} + \mathbf{j})$

$= (1 - 6)\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$

$= -5\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$

(c) $\frac{d}{dt}(3t)\mathbf{r}(t) = (3t)\mathbf{r}'(t) + 3\mathbf{r}(t)$

$= (3t)(3\mathbf{i} + \mathbf{j}) + 3[3t\mathbf{i} + (t - 1)\mathbf{j}]$

$= 9t\mathbf{i} + 3t\mathbf{j} + 9t\mathbf{i} + (3t - 3)\mathbf{j}$

$= 18t\mathbf{i} + (6t - 3)\mathbf{j}$

(d) $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

$= [(3t)(1) + (t - 1)(2t) + (0)(2t^2)] + [(3)(t) + (1)(t^2) + (0)(\frac{2}{3}t^3)]$

$= (3t + 2t^2 - 2t) + (3t + t^2)$

$= 4t + 3t^2$

23. $\mathbf{r}(t) = (t - 1)^3\mathbf{i} + (t - 1)^4\mathbf{j}$

$\mathbf{r}'(t) = 3(t - 1)^2\mathbf{i} + 4(t - 1)^3\mathbf{j}$

Not smooth at $t = 1$.

Smooth on $(-\infty, 1)$ and $(1, \infty)$

24. $\mathbf{r}(t) = \frac{t}{t - 2}\mathbf{i} + t\mathbf{j} + \sqrt{1 + t}\mathbf{k}, t \geq -1, t \neq 2$

$\mathbf{r}'(t) = \frac{-2}{(t - 2)^2}\mathbf{i} + \mathbf{j} + \frac{1}{2\sqrt{1 + t}}\mathbf{k}$

Smooth on $(-1, 2)$ and $(2, \infty)$

(e) $\dfrac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

$$= \left[\left(2t^3 - 2t^2\right)\mathbf{i} - 6t^3\mathbf{j} + \left(6t^2 - t + 1\right)\mathbf{k}\right] + \left[\tfrac{2}{3}t^3\mathbf{i} - 2t^3\mathbf{j} + \left(3t^2 - t\right)\mathbf{k}\right]$$

$$= \left(\tfrac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + \left(9t^2 - 2t + 1\right)\mathbf{k}$$

(f) $\dfrac{d}{dt}\left[\mathbf{u}(2t)\right] = 2\mathbf{u}'(2t)$

$$= 2\left[\mathbf{i} + 2(2t)\mathbf{j} + 2(2t)^2\mathbf{k}\right]$$

$$= 2\mathbf{i} + 8t\mathbf{j} + 16t^2\mathbf{k}$$

26. $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k},\ \mathbf{u}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \dfrac{1}{t}\mathbf{k}$

$\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{k},\ \mathbf{u}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} - \dfrac{1}{t^2}\mathbf{k}$

(a) $\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{k}$

(b) $\dfrac{d}{dt}\left[\mathbf{u}(t) - 2\mathbf{r}(t)\right] = \mathbf{u}'(t) - 2\mathbf{r}'(t)$

$$= \left(\cos t\mathbf{i} - \sin t\mathbf{j} - \frac{1}{t^2}\mathbf{k}\right) - 2\left(\cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{k}\right)$$

$$= -\cos t\mathbf{i} + \sin t\mathbf{j} - \left(2 + \frac{1}{t^2}\right)\mathbf{j}$$

(c) $\dfrac{d}{dt}\left[(3t)\mathbf{r}(t)\right] = (3t)\mathbf{r}'(t) + 3\mathbf{r}(t)$

$$= (3t)\left(\cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{k}\right) + 3\left(\sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k}\right)$$

$$= (3t\cos t + 3\sin t)\mathbf{i} + (3\cos t - 3t\sin t)\mathbf{j} + 6t\,\mathbf{k}$$

(d) $\dfrac{d}{dt}\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

$$= \left[(\sin t)(\cos t) + (\cos t)(-\sin t) + (t)\left(-\frac{1}{t^2}\right)\right] + \left[(\cos t)(\sin t) + (-\sin t)(\cos t) + (1)\left(\frac{1}{t}\right)\right]$$

$$= \sin t\cos t - \sin t\cos t - \frac{1}{t} + \sin t\cos t - \sin t\cos t + \frac{1}{t}$$

$$= 0$$

(e) $\dfrac{d}{dt}\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

$$= \left[\left(-\frac{1}{t^2}\cos t + t\sin t\right)\mathbf{i} - \left(-\frac{1}{t^2}\sin t + t\cos t\right)\mathbf{j} + \left(-\sin^2 t - \cos^2 t\right)\mathbf{k}\right]$$

$$+ \left[\left(-\frac{1}{t}\sin t - \cos t\right)\mathbf{i} - \left(\frac{1}{t}\cos t - \sin t\right)\mathbf{j} + \left(\cos^2 t + \sin^2 t\right)\mathbf{k}\right]$$

$$= \left[\left(t - \frac{1}{t}\right)\sin t - \left(1 + \frac{1}{t^2}\right)\cos t\right]\mathbf{i} - \left[\left(\frac{1}{t} - t\right)\cos t - \left(1 + \frac{1}{t^2}\right)\sin t\right]\mathbf{j}$$

(f) $\dfrac{d}{dt}\left[\mathbf{u}(2t)\right] = 2\mathbf{u}'(2t)$

$$= 2\left[\cos(2t)\mathbf{i} - \sin(2t)\mathbf{j} - \frac{1}{(2t)^2}\mathbf{k}\right]$$

$$= 2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j} - \frac{1}{2t^2}\mathbf{k}$$

27. $\int\left(t^2\mathbf{i} + 5t\mathbf{j} + 8t^3\mathbf{k}\right)dt = \dfrac{t^3}{3}\mathbf{i} + \dfrac{5}{2}t^2\mathbf{j} + 2t^4\mathbf{j} + \mathbf{C}$

28. $\int(6\mathbf{i} - 2t\mathbf{j} + \ln t\,\mathbf{k})dr = 6t\mathbf{i} - t^2\mathbf{j} + (t\ln t - t)\mathbf{k} + \mathbf{C}$

29. $\int\left(3\sqrt{t}\,\mathbf{i} + \dfrac{2}{t}\mathbf{j} + \mathbf{k}\right)dt = 2t^{3/2}\,\mathbf{i} + 2\ln|t|\,\mathbf{j} + t\mathbf{k} + \mathbf{C}$

30. $\int\left(\sin t\mathbf{i} + \mathbf{j} + e^{2t}\mathbf{k}\right)dt = -\cos t\mathbf{i} + t\mathbf{j} + \dfrac{1}{2}e^{2t}\mathbf{k} + \mathbf{C}$

31. $\int_{-2}^{2}\left(3t\mathbf{i} + 2t^2\,\mathbf{j} - t^3\,\mathbf{k}\right)dt = \left[\dfrac{3t^2}{2}\mathbf{i} + \dfrac{2t^3}{3}\mathbf{j} - \dfrac{t^4}{4}\mathbf{k}\right]_{-2}^{2} = \dfrac{32}{3}\mathbf{j}$

32. $\int_0^3\left(t\mathbf{i} + \sqrt{t}\,\mathbf{j} + 4t\mathbf{k}\right)dt = \left[\dfrac{t^2}{2}\mathbf{i} + \dfrac{2}{3}t^{3/2}\mathbf{j} + 2t^2\mathbf{k}\right]_0^1 = \dfrac{9}{2}\mathbf{i} + 2\sqrt{3}\mathbf{j} + 18\mathbf{k}$

33. $\int_0^2\left(e^{t/2}\mathbf{i} - 3t^2\,\mathbf{j} - \mathbf{k}\right)dt = \left[2e^{t/2}\,\mathbf{i} - t^3\,\mathbf{j} - t\mathbf{k}\right]_0^2 = (2e - 2)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

34. $\int_0^{\pi/3}(2\cos t\mathbf{i} + \sin t\mathbf{j} + 3\mathbf{k})\,dt = [2\sin t\mathbf{i} - \cos t\mathbf{j} + 3t\mathbf{k}]_0^{\pi/3} = \left(\sqrt{3}\,\mathbf{i} - \dfrac{1}{2}\mathbf{j} + \pi\mathbf{k}\right) - (-\mathbf{j}) = \sqrt{3}\,\mathbf{i} + \dfrac{1}{2}\mathbf{j} + \pi\mathbf{k}$

35. $\mathbf{r}(t) = \int\left(2t\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}\right)dt = t^2\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

$\mathbf{r}(t) = \left(t^2 + 1\right)\mathbf{i} + \left(e^t + 2\right)\mathbf{j} - \left(e^{-t} + 4\right)\mathbf{k}$

36. $\mathbf{r}(t) = \int\left(\sec t\mathbf{i} + \tan t\mathbf{j} + t^2\,\mathbf{k}\right)dt$

$= \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \dfrac{t^3}{3}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 3\mathbf{k}$

$\mathbf{r}(t) = \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \left(\dfrac{t^3}{3} + 3\right)\mathbf{k}$

37. $\mathbf{r}(t) = 4t\mathbf{i} + t^3\mathbf{j} - t\mathbf{k},\ t = 1$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = 4\mathbf{i} + 3t^2\mathbf{j} - \mathbf{k}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{16 + 9t^4 + 1} = \sqrt{17 + 9t^4}$

$\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{j}$

(b) $\mathbf{v}(1) = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$\mathbf{a}(1) = 6\mathbf{j}$

38. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + 5t\mathbf{j} + 2t^2\mathbf{k},\ t = 4$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \dfrac{1}{2\sqrt{t}}\mathbf{i} + 5\mathbf{j} + 4t\mathbf{k}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{\dfrac{1}{4t} + 25 + 16t^2}$

$\mathbf{a}(t) = \mathbf{r}''(t) = -\dfrac{1}{4t^{3/2}}\mathbf{i} + 4\mathbf{k}$

(b) $\mathbf{v}(4) = \dfrac{1}{4}\mathbf{i} + 5\mathbf{j} + 16\mathbf{k}$

$\mathbf{a}(4) = -\dfrac{1}{32}\mathbf{i} + 4\mathbf{k}$

39. $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t, 3t \rangle, t = \pi$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t, 3 \rangle$

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 9}$$
$$= 3\sqrt{\cos^2 t \sin^2 t\left(\cos^2 t + \sin^2 t\right) + 1}$$
$$= 3\sqrt{\cos^2 t \sin^2 t + 1}$$

$$\mathbf{a}(t) = \left\langle -6\cos t\left(-\sin^2 t\right) + \left(-3\cos^2 t\right)\cos t, 6\sin t\cos^2 t + 3\sin^2 t(-\sin t), 0 \right\rangle$$
$$= \left\langle 3\cos t\left(2\sin^2 t - \cos^2 t\right), 3\sin t\left(2\cos^2 t - \sin^2 t\right), 0 \right\rangle$$

(b) $\mathbf{v}(\pi) = \langle 0, 0, 3 \rangle$

$\mathbf{a}(\pi) = \langle 3, 0, 0 \rangle$

40. $\mathbf{r}(t) = \langle t, -\tan t, e^t \rangle, t = 0$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, -\sec^2 t, e^t \rangle$

$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{1 + \sec^4 t + e^{2t}}$

$\mathbf{a}(t) = \langle 0, -2\sec^2 t \tan t, e^t \rangle$

(b) $\mathbf{v}(0) = \langle 1, -1, 1 \rangle$

$\mathbf{a}(0) = \langle 0, 0, 1 \rangle$

41. $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - 16t^2\right]\mathbf{j}$

$$= (120\cos 30°)t\mathbf{i} + \left(3.5 + (120\sin 30°)t - 16t^2\right)\mathbf{j}$$
$$= 60\sqrt{3}t\mathbf{i} + \left(3.5 + 60t - 16t^2\right)\mathbf{j}$$

$\mathbf{r}'(t) = \mathbf{v}(t) = 60\sqrt{3}\mathbf{i} + (60 - 32t)\mathbf{j}$

Find the maximum height

$$\mathbf{y}'(t) = 60 - 32t = 0 \Rightarrow t = \frac{60}{32} = \frac{15}{8} = 1.875$$

$$y\left(\frac{15}{8}\right) = 3.5 + 60\left(\frac{15}{8}\right) - 16\left(\frac{15}{8}\right)^2 = 59.75 \text{ ft}, \text{ Maximum height}$$

$$375 = 60\sqrt{3}t \Rightarrow t = \frac{375}{60\sqrt{3}} = \frac{25}{4\sqrt{3}} = \frac{25\sqrt{3}}{12} \approx 3.608$$

$$y\left(\frac{25\sqrt{3}}{12}\right) \approx 11.67 \text{ ft}$$

The baseball clears the 8-foot fence.

42. $\mathbf{r}(t) = (\mathbf{v}_0 \cos\theta)t\mathbf{i} + \left[h + (\mathbf{v}_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$$= (400\cos 60°)t\mathbf{i} + \left[6 + (400\sin 60°)t - 16t^2\right]\mathbf{j}$$

$\mathbf{r}'(t) = 200\mathbf{i} + \left[200\sqrt{3} - 32t\right]\mathbf{k}$

Range: $6 + 400\sin 60° t - 16t^2 = 0 \Rightarrow t \approx 21.668$ sec

$(400\cos 60°)(21.668) \approx 4333.6$ ft

Maximum height: $200\sqrt{3} - 32t = 0 \Rightarrow t = \dfrac{200\sqrt{3}}{32} \approx 10.825$ sec

$$6 + (400\sin 60°)\left(\frac{200\sqrt{3}}{32}\right) - 16\left(\frac{200\sqrt{3}}{32}\right)^2 \approx 1881 \text{ ft}$$

43. $\mathbf{r}(t) = 6t\mathbf{i} - t^2\mathbf{j},\ t = 2$

$\mathbf{r}'(t) = 6\mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(2) = 6\mathbf{i} - 4\mathbf{j},$

$\|\mathbf{r}'(2)\| = \sqrt{36 + 16} = 2\sqrt{13}$

$\mathbf{T}(2) = \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$

44. $\mathbf{r}(t) = 2\sin t\mathbf{i} + 4\cos t\mathbf{j},\ t = \frac{\pi}{6}$

$\mathbf{r}'(t) = 2\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$\mathbf{r}'\left(\frac{\pi}{6}\right) = \sqrt{3}\mathbf{i} - 2\mathbf{j}$

$\left\|\mathbf{r}'\left(\frac{\pi}{6}\right)\right\| = \sqrt{3 + 4} = \sqrt{7}$

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{6}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{6}\right)\right\|} = \frac{1}{\sqrt{7}}\left(\sqrt{3}\mathbf{i} - 2\mathbf{j}\right)$$

$$= \frac{\sqrt{21}}{7}\mathbf{i} - \frac{2\sqrt{7}}{7}\mathbf{j}$$

45. $\mathbf{r}(t) = e^{2t}\mathbf{i} + \cos t\mathbf{j} - \sin 3t\mathbf{k},\ P(1, 1, 0)$

$\mathbf{r}'(t) = 2e^{2t}\mathbf{i} - \sin t\mathbf{j} - 3\cos 3t\mathbf{k}$

At $P(1, 1, 0),\ t = 0.$

$\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$

$\mathbf{r}'(0) = 2\mathbf{i} - 3\mathbf{k}$

$\|\mathbf{r}'(0)\| = \sqrt{4 + 9} = \sqrt{13}$

$\mathbf{T}(0) = \frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{k})$

Direction numbers: $a = 2, b = 0, c = -3$

Parametric equations: $x = 1 + 2t,\ y = 1,\ z = -3t$

46. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k},\ P\left(2, 4, \frac{16}{3}\right)$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$

$t = 2$ at $P\left(2, 4, \frac{16}{3}\right)$

$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}}{9} = \frac{1}{9}\mathbf{i} + \frac{4}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}$$

Direction numbers: $a = 1, b = 4, c = 8$

Parametric equations:

$x = t + 2,\ y = 4t + 4,\ z = 8t + \frac{16}{3}$

47. $\mathbf{r}(t) = 2t\mathbf{i} + 3t^2\mathbf{j},\ t = 1$

$\mathbf{r}'(t) = 2\mathbf{i} + 6t\mathbf{j},\ \mathbf{r}'(1) = 2\mathbf{i} + 6\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{4 + 36t^2},\ \|\mathbf{r}'(1)\| = \sqrt{40} = 2\sqrt{10}$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{2\mathbf{i} + 6\mathbf{j}}{2\sqrt{10}} = \frac{1}{\sqrt{10}}(\mathbf{i} + 3\mathbf{j})$$

$\mathbf{N}(1)$ is orthogonal to $\mathbf{T}(1)$ and points towards the concave side. Hence,

$\mathbf{N}(1) = \frac{1}{\sqrt{10}}(-3\mathbf{i} + \mathbf{j}).$

48. $\mathbf{r}(t) = t\mathbf{i} + \ln t\,\mathbf{j},\ t = 2$

$\mathbf{r}'(t) = \mathbf{i} + \frac{1}{t}\mathbf{j},\ \mathbf{r}'(2) = \mathbf{i} + \frac{1}{2}\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + \frac{1}{t^2}} = \frac{1}{t}\sqrt{t^2 + 1},\ \|\mathbf{r}'(2)\| = \frac{1}{2}\sqrt{5}$

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{2}{\sqrt{5}}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

$\mathbf{N}(2)$ is orthogonal to $\mathbf{T}(2)$ and points towards the concave side. Hence,

$\mathbf{N}(t) = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j} = \frac{\sqrt{5}}{5}\mathbf{i} - \frac{2\sqrt{5}}{5}\mathbf{j}.$

49. $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\,\mathbf{j} + 3\mathbf{k},\ t = \frac{\pi}{4}$

$\mathbf{r}'(t) = -6\sin 2t\mathbf{i} + 6\cos 2t\,\mathbf{j},\ \|\mathbf{r}'(t)\| = 6$

$\mathbf{T}(t) = -\sin 2t\mathbf{i} + \cos 2t\,\mathbf{j},\ \mathbf{T}(\pi/4) = -\mathbf{i}$

$\mathbf{T}'(t) = -2\cos 2t\mathbf{i} - 2\sin 2t\,\mathbf{j},\ \|\mathbf{T}'(t)\| = 2$

$\mathbf{N}(t) = -\cos 2t\mathbf{i} - \sin 2t\,\mathbf{j}$

$\mathbf{N}(\pi/4) = -\mathbf{j}$

50. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\,\mathbf{j} + \mathbf{k},\ t = \frac{2\pi}{3}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\,\mathbf{j},\ \|\mathbf{r}'(t)\| = 4$

$\mathbf{T}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j},\ \mathbf{T}\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$

$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j},\ \|\mathbf{T}'(t)\| = 1$

$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\,\mathbf{j},\ \mathbf{N}\left(\frac{2\pi}{3}\right) = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$

51. $\mathbf{r}(t) = \frac{3}{t}\mathbf{i} - 6t\,\mathbf{j},\ t = 3$

$\mathbf{v}(t) = -\frac{3}{t^2}\mathbf{i} - 6\mathbf{j},\ \mathbf{v}(3) = -\frac{1}{3}\mathbf{i} - 6\mathbf{j}$

$\mathbf{a}(t) = \frac{6}{t^3}\mathbf{i},\ \mathbf{a}(3) = \frac{2}{9}\mathbf{i}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\left(-\frac{3}{t^2}\right)\mathbf{i} - 6\mathbf{j}}{\sqrt{\frac{9}{t^4} + 36}} = \frac{-3\mathbf{i} - 6t^2\mathbf{j}}{3\sqrt{1 + 4t^4}}$$

$$\mathbf{T}(3) = \frac{-3\mathbf{i} - 54\mathbf{j}}{3\sqrt{1 + 324}} = \frac{-\mathbf{i} - 18\mathbf{j}}{\sqrt{325}} = -\frac{\sqrt{13}}{65}\mathbf{i} - \frac{18\sqrt{13}}{65}\mathbf{j}$$

$\mathbf{N}(3)$ is orthogonal to $\mathbf{T}(3)$, and points in the direction the curve is bending. Hence,

$$\mathbf{N}(3) = \frac{18\mathbf{i} - \mathbf{j}}{\sqrt{325}} = \frac{18\sqrt{13}}{65}\mathbf{i} - \frac{\sqrt{13}}{65}\mathbf{j}.$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = -\frac{2}{9\sqrt{325}} = -\frac{2\sqrt{13}}{585}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{4}{\sqrt{325}} = \frac{4\sqrt{13}}{65}$$

52. $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\,\mathbf{j},\ t = \frac{\pi}{6}$

$\mathbf{v}(t) = -6\sin 2t\mathbf{i} + 6\cos 2t\,\mathbf{j},\ \mathbf{v}\left(\frac{\pi}{6}\right) = -3\sqrt{3}\mathbf{i} + 3\mathbf{j}$

$\mathbf{a}(t) = -12\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j},\ \mathbf{a}\left(\frac{\pi}{6}\right) = -6\mathbf{i} - 6\sqrt{3}\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{36\sin^2 2t + 36\cos^2 2t} = 6$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{6}(-6\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j})$$
$$= -\sin 2t\mathbf{i} + \cos 2t\,\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$\mathbf{T}'(t) = -2\cos 2t\mathbf{i} - 2\sin 2t\mathbf{j},\ \|\mathbf{T}'(t)\| = 2$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos 2t\mathbf{i} - \sin 2t\,\mathbf{j}$$

$$\mathbf{N}\left(\frac{\pi}{6}\right) = -\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \left(-6\mathbf{i} - 6\sqrt{3}\mathbf{j}\right)\cdot\left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 3\sqrt{3} - 3\sqrt{3} = 0$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \left(-6\mathbf{i} - 6\sqrt{3}\mathbf{j}\right)\cdot\left(-\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}\right) = 3 + 9 = 12$$

53. $\mathbf{r}(t) = \sin t\mathbf{i} - 3t\mathbf{j} + \cos t\mathbf{k},\ t = \dfrac{\pi}{6}$

$\mathbf{r}'(t) = \mathbf{v}(t) = \cos t\mathbf{i} - 3\mathbf{j} - \sin t\mathbf{k}$

$\mathbf{r}''(t) = \mathbf{a}(t) = -\sin t\mathbf{i} - \cos t\mathbf{k}$

$\mathbf{r}'\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}\mathbf{i} - 3\mathbf{j} - \dfrac{1}{2}\mathbf{k};\ \mathbf{r}''\left(\dfrac{\pi}{6}\right) = -\dfrac{1}{2}\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{k}$

$\left\|\mathbf{r}'\left(\dfrac{\pi}{6}\right)\right\| = \sqrt{\dfrac{3}{4} + 9 + \dfrac{1}{4}} = \sqrt{10}$

$\mathbf{T}\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{10}}\left(\dfrac{\sqrt{3}}{2}\mathbf{i} - 3\mathbf{j} - \dfrac{1}{2}\mathbf{k}\right)$

$a_{\mathbf{T}} = a \cdot \mathbf{T} = \dfrac{1}{\sqrt{10}}\left(-\dfrac{1}{2}\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{k}\right) \cdot \left(\dfrac{\sqrt{3}}{2}\mathbf{i} - 3\mathbf{j} - \dfrac{1}{2}\mathbf{k}\right) = 0$

$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \|\mathbf{a}\| = 1$

54. $\mathbf{r}(t) = \dfrac{t^3}{3}\mathbf{i} - 6t\mathbf{j} + t^2\mathbf{k},\ t = 2$

$\mathbf{r}'(t) = t^2\mathbf{i} - 6\mathbf{j} + 2t\mathbf{k}$

$\mathbf{r}''(t) = 2t\mathbf{i} + 2\mathbf{k}$

$\mathbf{r}'(2) = 4\mathbf{i} - 6\mathbf{j} + 4\mathbf{k},\ \mathbf{r}''(2) = 4\mathbf{i} + 2\mathbf{k}$

$\|\mathbf{r}'(2)\| = \sqrt{16 + 36 + 16} = \sqrt{68} = 2\sqrt{17}$

$\mathbf{T}(2) = \dfrac{1}{\sqrt{17}}(2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$

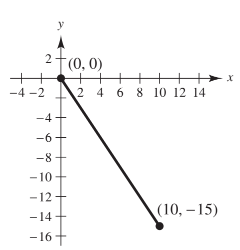
$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{17}}(4\mathbf{i} + 2\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$$
$$= \frac{1}{\sqrt{17}}(8 + 4) = \frac{12}{\sqrt{17}}$$

$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - \mathbf{a}_{\mathbf{T}}^2} = \sqrt{20 - \dfrac{144}{17}} = \dfrac{14\sqrt{17}}{17}$

55. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j},\ [0, 5]$

$\mathbf{r}'(t) = 2\mathbf{i} - 3\mathbf{j}$

$s = \displaystyle\int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^5 \sqrt{4 + 9}\, dt = \left[\sqrt{13}\, t\right]_0^5 = 5\sqrt{13}$



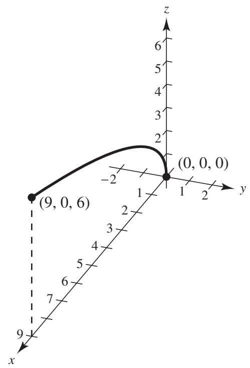
56. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{k}, [0, 3]$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{k}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^3 \sqrt{4t^2 + 4}\, dt$$

$$= \left[\ln\left|\sqrt{t^2 + 1} + t\right| + t\sqrt{t^2 + 1}\right]_0^3$$

$$= \ln\left(\sqrt{10} + 3\right) + 3\sqrt{10} \approx 11.3053$$



57. $\mathbf{r}(t) = 2\sin t\mathbf{i} + \mathbf{j}, \left[\dfrac{\pi}{2}, \pi\right]$

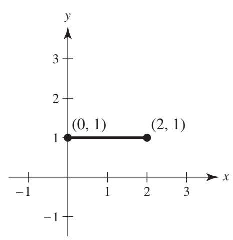
$\mathbf{r}'(t) = 2\cos t\mathbf{i}$

$\|\mathbf{r}'(t)\| = \sqrt{4\cos^2 t} = 2\cos t$

$$s = \int_{\pi/2}^{\pi} 2\cos t\, dt$$

$$= [2\sin t]_{\pi/2}^{\pi}$$

$$= [0 - (-2)] = 2$$

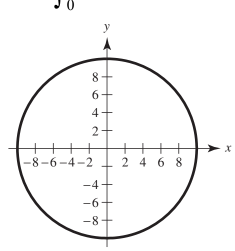


58. $\mathbf{r}(t) = 10\cos t\mathbf{i} + 10\sin t\mathbf{j}, [0, 2\pi]$

$\mathbf{r}'(t) = -10\sin t\mathbf{i} + 10\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 10$

$$s = \int_0^{2\pi} 10\, dt = 20\pi$$

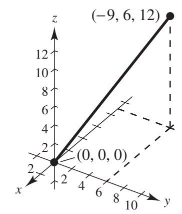


59. $\mathbf{r}(t) = -3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}, [0, 3]$

$\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^3 \sqrt{9 + 4 + 16}\, dt$$

$$= \int_0^3 \sqrt{29}\, dt = 3\sqrt{29}$$

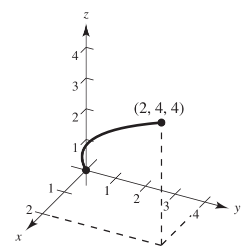


60. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}, [0, 2]$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{5 + 4t^2}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^2 \sqrt{5 + 4t^2}\, dt$$

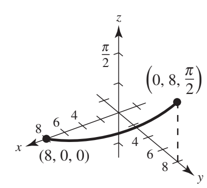
$$= \sqrt{21} + \tfrac{5}{4}\ln 5 - \tfrac{5}{4}\ln\left(\sqrt{105} - 4\sqrt{5}\right) \approx 6.2638$$



61. $\mathbf{r}(t) = \langle 8\cos t, 8\sin t, t\rangle, \left[0, \dfrac{\pi}{2}\right]$

$\mathbf{r}'(t) = \langle -8\sin t, 8\cos t, 1\rangle, \|\mathbf{r}'(t)\| = \sqrt{65}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^{\pi/2} \sqrt{65}\, dt = \frac{\pi\sqrt{65}}{2}$$



62. $\mathbf{r}(t) = \langle 2(\sin t - t\cos t), 2(\cos t + t\sin t), t\rangle, \left(0, \dfrac{\pi}{2}\right)$

$\mathbf{r}'(t) = \langle 2t\sin t, 2t\cos t, 1\rangle, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^{\pi/2} \sqrt{4t^2 + 1}\, dt$$

$$= \frac{1}{4}\ln\left(\sqrt{\pi^2 + 1} + \pi\right) + \frac{\pi}{4}\sqrt{\pi^2 + 1} \approx 3.055$$

63. $\mathbf{r}(t) = 3t\,\mathbf{i} + 2t\,\mathbf{j}$

Line

$K = 0$

64. $\mathbf{r}(t) = 2\sqrt{t}\,\mathbf{i} + 3t\,\mathbf{j}$

$\mathbf{r}'(t) = \dfrac{1}{\sqrt{t}}\mathbf{i} + 3\mathbf{j},\ \|\mathbf{r}'(t)\| = \sqrt{\dfrac{1}{t} + 9} = \sqrt{\dfrac{1+9t}{t}}$

$\mathbf{r}''(t) = -\dfrac{1}{2}t^{-3/2}\,\mathbf{i}$

$$\mathbf{r}'\times\mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{1}{\sqrt{t}} & 3 & 0 \\ -\dfrac{1}{2}t^{-3/2} & 0 & 0 \end{vmatrix} = \frac{3}{2}t^{-3/2}\mathbf{k};\ \|\mathbf{r}'\times\mathbf{r}''\| = \frac{3}{2t^{3/2}}$$

$$K = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{3/2t^{3/2}}{(1+9t)^{3/2}/t^{3/2}} = \frac{3}{2(1+9t)^{3/2}}$$

65. $\mathbf{r}(t) = 2t\,\mathbf{i} + \dfrac{1}{2}t^2\,\mathbf{j} + t^2\,\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k},\ \|\mathbf{r}'\| = \sqrt{5t^2 + 4}$

$\mathbf{r}''(t) = \mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}'\times\mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = -4\mathbf{j} + 2\mathbf{k},\ \|\mathbf{r}'\times\mathbf{r}''\| = \sqrt{20}$$

$$K = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{20}}{(5t^2+4)^{3/2}} = \frac{2\sqrt{5}}{(4+5t^2)^{3/2}}$$

66. $\mathbf{r}(t) = 2t\mathbf{i} + 5\cos t\mathbf{j} + 5\sin t\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} - 5\sin t\mathbf{j} + 5\cos t\mathbf{k},\ \|\mathbf{r}'(t)\| = \sqrt{29}$

$\mathbf{r}''(t) = 5\cos t\mathbf{j} - 5\sin t\mathbf{k}$

$$\mathbf{r}'\times\mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5\sin t & 5\cos t \\ 0 & -5\cos t & -5\sin t \end{vmatrix} = 25\mathbf{i} + 10\sin t\,\mathbf{j} - 10\cos t\,\mathbf{k}$$

$\|\mathbf{r}'\times\mathbf{r}''\| = \sqrt{725}$

$$K = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{725}}{(29)^{3/2}} = \frac{\sqrt{25\cdot 29}}{29\sqrt{29}} = \frac{5}{29}$$

67. $\mathbf{r}(t) = \dfrac{1}{2}t^2\,\mathbf{i} + t\,\mathbf{j} + \dfrac{1}{3}t^3\,\mathbf{k},\ P\left(\dfrac{1}{2}, 1, \dfrac{1}{3}\right) \Rightarrow t = 1$

$\mathbf{r}'(t) = t\mathbf{i} + \mathbf{j} + t^2\mathbf{k},\ \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{r}''(t) = \mathbf{i} + 2t\mathbf{k},\ \mathbf{r}''(1) = \mathbf{i} + 2\mathbf{k}$

$$\mathbf{r}'\times\mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$K = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{4+1+1}}{(\sqrt{3})^3} = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}$$

68. $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + t\,\mathbf{k},\ P(-4, 0, \pi) \Rightarrow t = \pi$

$\mathbf{r}'(t) = -4\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + \mathbf{k},\ \mathbf{r}'(\pi) = -3\mathbf{j} + \mathbf{k}$

$\mathbf{r}''(t) = -4\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j},\ \mathbf{r}''(\pi) = 4\mathbf{i}$

$$\mathbf{r}'\times\mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & 1 \\ 4 & 0 & 0 \end{vmatrix} = 4\mathbf{j} + 12\mathbf{k}$$

$$K = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{16+144}}{(9+1)^{3/2}} = \frac{2}{5}$$

69. $y = \dfrac{1}{2}x^2 + x,\ x = 4$

$y' = x + 1$

$y'' = 1$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{1}{\left(1 + x^2\right)^{3/2}}$$

At $x = 4$, $K = \dfrac{1}{26^{3/2}}$ and $r = 26^{3/2} = 26\sqrt{26}$.

70. $y = e^{-x/2},\ x = 0$

$y' = -\dfrac{1}{2}e^{-x/2},\ y'' = \dfrac{1}{4}e^{-x/2}$

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \frac{\frac{1}{4}e^{-x/2}}{\left[1 + \frac{1}{4}e^{-x}\right]^{3/2}}$$

At $x = 0$, $K = \dfrac{1/4}{(5/4)^{3/2}} = \dfrac{2}{5^{3/2}} = \dfrac{2}{5\sqrt{5}} = \dfrac{2\sqrt{5}}{25}$,

$r = \dfrac{5\sqrt{5}}{2}$.

71. $y = \ln x, x = 1$

$y' = \dfrac{1}{x}$

$y'' = -\dfrac{1}{x^2}$

$K = \dfrac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \dfrac{1/x^2}{\left[1 + (1/x)^2\right]^{3/2}}$

At $x = 1$, $K = \dfrac{1}{2^{3/2}} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$ and $r = 2\sqrt{2}$.

72. $y = \tan x, x = \dfrac{\pi}{4}$

$y' = \sec^2 x$

$y'' = 2\sec^2 x \tan x$

$K = \dfrac{|y''|}{\left[1 + (y')^2\right]^{3/2}} = \dfrac{\left|2\sec^2 x \tan x\right|}{\left[1 + \sec^4 x\right]^{3/2}}$

At $x = \dfrac{\pi}{4}$, $K = \dfrac{4}{5^{3/2}} = \dfrac{4}{5\sqrt{5}} = \dfrac{4\sqrt{5}}{25}$ and $r = \dfrac{5\sqrt{5}}{4}$.

73. $\mathbf{F} = ma_{\mathbf{N}} = mk\left(\dfrac{ds}{dt}\right)^2$

$= \left(\dfrac{7200 \text{ lb}}{32 \text{ ft/sec}^2}\right)\left(\dfrac{1}{150 \text{ ft}}\right)\left(\dfrac{25(5280)\text{ft}}{3600 \text{ sec}}\right)^2$

≈ 2016.67 lb

Problem Solving for Chapter 12

1. $x(t) = \int_0^t \cos\left(\dfrac{\pi u^2}{2}\right) du, y(t) = \int_0^t \sin\left(\dfrac{\pi u^2}{2}\right) du$

$x'(t) = \cos\left(\dfrac{\pi t^2}{2}\right), y'(t) = \sin\left(\dfrac{\pi t^2}{2}\right)$

(a) $s = \int_0^a \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^a dt = a$

(b) $x''(t) = -\pi t \sin\left(\dfrac{\pi t^2}{2}\right), y''(t) = \pi t \cos\left(\dfrac{\pi t^2}{2}\right)$

$K = \dfrac{\left|\pi t \cos^2\left(\dfrac{\pi t^2}{2}\right) + \pi t \sin^2\left(\dfrac{\pi t^2}{2}\right)\right|}{1} = \pi t$

At $t = a$, $K = \pi a$.

(c) $K = \pi a = \pi$ (length)

2. $x^{2/3} + y^{2/3} = a^{2/3}$

$\dfrac{2}{3}x^{-1/3} + \dfrac{2}{3}y^{-1/3}y' = 0$

$y' = \dfrac{-y^{1/3}}{x^{1/3}}$ Slope at $P(x, y)$.

$\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$

$\mathbf{r}'(t) = -3\cos^2 t \sin t\,\mathbf{i} + 3\sin^2 t \cos t\,\mathbf{j}$

$\|\mathbf{r}'(t)\|\mathbf{i} = |3\cos t \sin t|$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

$\mathbf{T}'(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

$Q(0, 0, 0)$ origin

$P = (\cos^3 t, \sin^3 t, 0)$ on curve.

$\overrightarrow{PQ} \times \mathbf{T} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos^3 t & \sin^3 t & 0 \\ -\cos t & \sin t & 0 \end{vmatrix} = (\cos^3 t \sin t - \sin^3 t \cos t)\mathbf{k}$

$D = \dfrac{\|\overrightarrow{PQ} \times \mathbf{T}\|}{\|\mathbf{T}\|} = |\cos t \sin t|$

$K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{1}{|3\cos t \sin t|}$

So, the radius of curvature, $\dfrac{1}{K}$, is three times the distance from the origin to the tangent line.

3. Bomb: $\mathbf{r}_1(t) = \langle 5000 - 400t, 3200 - 16t^2 \rangle$

Projectile: $\mathbf{r}_2(t) = \langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - 16t^2 \rangle$

At 1600 ft: Bomb:

$3200 - 16t^2 = 1600 \Rightarrow t = 10$ seconds.

Projectile will travel 5 seconds:

$$5(v_0 \sin\theta) - 16(25) = 1600$$
$$v_0 \sin\theta = 400.$$

Horizontal position:

At $t = 10$, bomb is at $5000 - 400(10) = 1000$.

At $t = 5$, projectile is at $5v_0 \cos\theta$.

So, $v_0 \cos\theta = 200$.

Combining,

$$\frac{v_0 \sin\theta}{v_0 \cos\theta} = \frac{400}{200} \Rightarrow \tan\theta = 2 \Rightarrow \theta \approx 63.43°.$$

$$v_0 = \frac{200}{\cos\theta} \approx 447.2 \text{ ft/sec}$$

4. Bomb: $\mathbf{r}_1(t) = \langle 5000 + 400t, 3200 - 16t^2 \rangle$

Projectile: $\mathbf{r}_2(t) = \langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - 16t^2 \rangle$

At 1600 ft: Bomb:

$3200 - 16t^2 = 1600 \Rightarrow t = 10$

Projectile will travel 5 seconds:

$$5(v_0 \sin\theta) - 16(25) = 1600$$
$$v_0 \sin\theta = 400.$$

Horizontal position:

At $t = 10$, bomb is at $5000 + 400(10) = 9000$.

At $t = 5$, projectile is at $(v_0 \cos\theta)5$.

So,

$$5v_0 \cos\theta = 9000$$
$$v_0 \cos\theta = 1800.$$

Combining,

$$\frac{v_0 \sin\theta}{v_0 \cos\theta} = \frac{400}{1800} \Rightarrow \tan\theta = \frac{2}{9} \Rightarrow \theta \approx 12.5°.$$

$$v_0 = \frac{1800}{\cos\theta} \approx 1843.9 \text{ ft/sec}$$

5. $x'(\theta) = 1 - \cos\theta, y'(\theta) = \sin\theta, 0 \le \theta \le 2\pi$

$$\sqrt{x'(\theta)^2 y'(\theta)^2} = \sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2 - 2\cos\theta} = \sqrt{4\sin^2\frac{\theta}{2}}$$

$$s(t) = \int_\pi^t 2\sin\frac{\theta}{2}\,d\theta = \left[-4\cos\frac{\theta}{2}\right]_\pi^t = -4\cos\frac{t}{2}$$

$x''(\theta) = \sin\theta, y''(\theta) = \cos\theta$

$$K = \frac{|(1-\cos\theta)\cos\theta - \sin\theta\sin\theta|}{\left(2\sin\frac{\theta}{2}\right)^3} = \frac{|\cos\theta - 1|}{8\sin^3\frac{\theta}{2}} = \frac{1}{4\sin\frac{\theta}{2}}$$

So, $\rho = \dfrac{1}{K} = 4\sin\dfrac{t}{2}$ and

$$s^2 + \rho^2 = 16\cos^2\left(\frac{t}{2}\right) + 16\sin^2\left(\frac{t}{2}\right) = 16.$$

6. $r = 1 - \cos\theta$

$r' = \sin\theta$

$$s(t) = \int_\pi^t \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\,d\theta = \int_\pi^t \sqrt{2 - 2\cos\theta}\,d\theta$$
$$= \int_\pi^t 2\sin\frac{\theta}{2}\,d\theta = \left[-4\cos\frac{\theta}{2}\right]_\pi^t = -4\cos\frac{t}{2}$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{\left[(r')^2 + r^2\right]^{3/2}}$$
$$= \frac{|2\sin^2\theta - (1-\cos\theta)(\cos\theta) + (1-\cos\theta)^2|}{8\sin^3\frac{\theta}{2}}$$
$$= \frac{|3 - 3\cos\theta|}{8\sin^3\frac{\theta}{2}} = \frac{3}{4}\frac{\sin^2\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} = \frac{3}{4\sin\frac{\theta}{2}}$$

$$\rho = \frac{1}{K} = \frac{4\sin\frac{\theta}{2}}{3}$$

$$s^2 + 9\rho^2 = 16\cos^2\frac{\theta}{2} + 16\sin^2\frac{\theta}{2} = 16$$

7. $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t)\cdot\mathbf{r}(t)$

$$\frac{d}{dt}\left(\|\mathbf{r}(t)\|\right)^2 = 2\|\mathbf{r}(t)\|\frac{d}{dt}\|\mathbf{r}(t)\| = \mathbf{r}(t)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\mathbf{r}(t) \Rightarrow \frac{d}{dt}\|\mathbf{r}(t)\| = \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{\|\mathbf{r}(t)\|}$$

8. (a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ position vector

$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$

$$\frac{d\mathbf{r}}{dt} = \left[\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right]\mathbf{i} + \left[\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right]\mathbf{j}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2}\cos\theta - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}\right]\mathbf{i}$$

$$+ \left[\frac{d^2r}{dt^2}\sin\theta + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}\right]$$

$$a_r = \mathbf{a}\cdot\mathbf{u}_r = \mathbf{a}\cdot(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$= \left[\frac{d^2r}{dt^2}\cos^2\theta - 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 - r\cos\theta\sin\theta\frac{d^2\theta}{dt^2}\right]$$

$$+ \left[\frac{d^2r}{dt^2}\sin^2\theta + 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\sin\theta\frac{d^2\theta}{dt^2}\right] = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$a_\theta = \mathbf{a}\cdot u_\theta = \mathbf{a}\cdot(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = 2\frac{dr}{dr}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{u}_r)\mathbf{u}_r + (\mathbf{a}\cdot\mathbf{u}_\theta)u_\theta = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\mathbf{u}_\theta$$

(b) $\mathbf{r} = 42{,}000\cos\left(\frac{\pi t}{12}\right)\mathbf{i} + 42{,}000\sin\left(\frac{\pi t}{12}\right)\mathbf{j}$

$\mathbf{r} = 42{,}000, \dfrac{dr}{dt} = 0, \dfrac{d^2r}{dt^2} = 0$

$\dfrac{d\theta}{dt} = \dfrac{\pi}{12}, \dfrac{d^2\theta}{dt^2} = 0$

So, $\mathbf{a} = -42{,}000\left(\dfrac{\pi}{12}\right)^2\mathbf{u}_r = -\dfrac{875}{3}\pi^2\mathbf{u}_r.$

Radial component: $-\dfrac{875}{3}\pi^2$

Angular component: 0

9. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3t\mathbf{k}, t = \dfrac{\pi}{2}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 3\mathbf{k}, \|\mathbf{r}'(t)\| = 5$

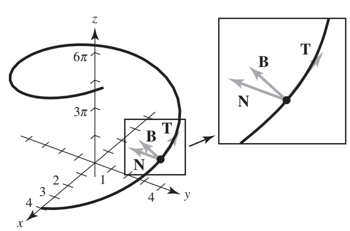
$\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$\mathbf{T} = -\dfrac{4}{5}\sin t\mathbf{i} + \dfrac{4}{5}\cos t\mathbf{j} + \dfrac{3}{5}\mathbf{k}$

$\mathbf{T}' = -\dfrac{4}{5}\cos t\mathbf{i} - \dfrac{4}{5}\sin t\mathbf{j}$

$\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \dfrac{3}{5}\sin t\mathbf{i} - \dfrac{3}{5}\cos t\mathbf{j} + \dfrac{4}{5}\mathbf{k}$

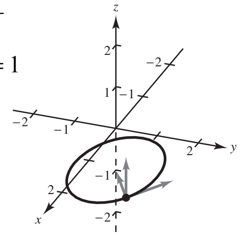


At $t = \dfrac{\pi}{2}$, $\mathbf{T}\left(\dfrac{\pi}{2}\right) = -\dfrac{4}{5}\mathbf{i} + \dfrac{3}{5}\mathbf{k}$

$\mathbf{N}\left(\dfrac{\pi}{2}\right) = -\mathbf{j}$

$\mathbf{B}\left(\dfrac{\pi}{2}\right) = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{k}$

10. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} - \mathbf{k},\ t = \dfrac{\pi}{4}$



$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j},\ \|\mathbf{r}'(t)\| = 1$

$\mathbf{T} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

$\mathbf{T}' = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$

$\mathbf{N} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$

$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \mathbf{k}$

At $t = \dfrac{\pi}{4}$, $\mathbf{T}\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{N}\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{B}\left(\dfrac{\pi}{4}\right) = \mathbf{k}$

11. (a) $\|\mathbf{B}\| = \|\mathbf{T} \times \mathbf{N}\| = 1$ constant length $\Rightarrow \dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = (\mathbf{T} \times \mathbf{N}') + (\mathbf{T}' \times \mathbf{N})$$

$$\mathbf{T} \cdot \frac{d\mathbf{B}}{ds} = \mathbf{T} \cdot (\mathbf{T} \times \mathbf{N}') + \mathbf{T} \cdot (\mathbf{T}' \times \mathbf{N})$$

$$= (\mathbf{T} \times \mathbf{T}) \cdot \mathbf{N}' + \mathbf{T} \cdot \left(\mathbf{T}' \times \frac{\mathbf{T}'}{\|\mathbf{T}'\|}\right) = 0$$

So, $\dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$ and $\dfrac{d\mathbf{B}}{ds} \perp \mathbf{T} \Rightarrow \dfrac{d\mathbf{B}}{ds} = \tau\mathbf{N}$

for some scalar τ.

(b) $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Using Section 11.4, exercise 58,

$$\mathbf{B} \times \mathbf{N} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{N} = -\mathbf{N} \times (\mathbf{T} \times \mathbf{N})$$
$$= -[(\mathbf{N} \cdot \mathbf{N})\mathbf{T} - (\mathbf{N} \cdot \mathbf{T})\mathbf{N}]$$
$$= -\mathbf{T}$$

$$\mathbf{B} \times \mathbf{T} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{T} = -\mathbf{T} \times (\mathbf{T} \times \mathbf{N})$$
$$= -[(\mathbf{T} \cdot \mathbf{N})\mathbf{T} - (\mathbf{T} \cdot \mathbf{T})\mathbf{N}]$$
$$= \mathbf{N}.$$

Now, $K\mathbf{N} = \left\|\dfrac{d\mathbf{T}}{ds}\right\| \dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \mathbf{T}'(s) = \dfrac{d\mathbf{T}}{ds}$

Finally,

$$\mathbf{N}'(s) = \frac{d}{ds}(\mathbf{B} \times \mathbf{T}) = (\mathbf{B} \times \mathbf{T}') + (\mathbf{B}' \times \mathbf{T})$$
$$= (\mathbf{B} \times K\mathbf{N}) + (-\tau\mathbf{N} \times \mathbf{T})$$
$$= -K\mathbf{T} + \tau\mathbf{B}.$$

12. $y = \dfrac{1}{32}x^{5/2}$

$y' = \dfrac{5}{64}x^{3/2}$

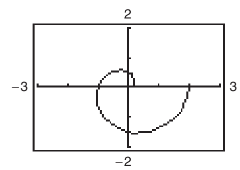
$y'' = \dfrac{15}{128}x^{1/2}$

$$K = \left|\frac{\frac{15}{128}x^{1/2}}{\left(1 + \frac{25}{4096}x^3\right)^{3/2}}\right|$$

At the point $(4, 1)$, $K = \dfrac{120}{(89)^{3/2}} \Rightarrow r = \dfrac{1}{K} = \dfrac{(89)^{3/2}}{120} \approx 7.$

13. $\mathbf{r}(t) = \langle t\cos \pi t, t\sin \pi t\rangle,\ 0 \le t \le 2$

(a)



(b) Length $= \displaystyle\int_0^2 \|\mathbf{r}'(t)\|\,dt$

$$= \int_0^2 \sqrt{\pi^2 t^2 + 1}\,dt$$

≈ 6.766 (graphing utility)

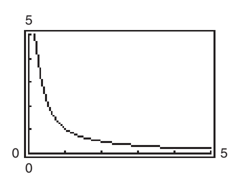
(c) $K = \dfrac{\pi(\pi^2 t^2 + 2)}{[\pi^2 t^2 + 1]^{3/2}}$

$K(0) = 2\pi$

$K(1) = \dfrac{\pi(\pi^2 + 2)}{(\pi^2 + 1)^{3/2}} \approx 1.04$

$K(2) \approx 0.51$

(d)



(e) $\displaystyle\lim_{t \to \infty} K = 0$

(f) As $t \to \infty$, the graph spirals outward and the curvature decreases.

14. (a) Eliminate the parameter to see that the Ferris wheel has a radius of 15 meters and is centered at $16\mathbf{j}$. At $t = 0$, the friend is located at $\mathbf{r}_1(0) = \mathbf{j}$, which is the low point on the Ferris wheel.

(b) If a revolution takes Δt seconds, then

$$\frac{\pi(t + \Delta t)}{10} = \frac{\pi t}{10} + 2\pi$$

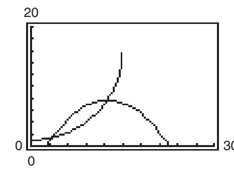
and so $\Delta t = 20$ seconds. The Ferris wheel makes three revolutions per minute.

(c) The initial velocity is $r_2'(t_0) = -8.03\mathbf{i} + 11.47\mathbf{j}$. The speed is $\sqrt{8.03^2 + 11.47^2} \approx 14$ meters per second. The angle of inclination is $\arctan\left(\dfrac{11.47}{8.03}\right) \approx 0.96$ radians or $55°$.

(d) Although you may start with other values, $t_0 = 0$ is a fine choice. The graph at the right shows two points of intersection. At $t = 3.15$ sec the friend is near the vertex of the parabola, which the object reaches when

$$t - t_0 = -\frac{11.47}{2(-4.9)} \approx 1.17 \text{ seconds.}$$

So, after the friend reaches the low point on the Ferris wheel, wait $t_0 = 2$ seconds before throwing the object in order to allow it to be within reach.



(e) The approximate time is 3.15 seconds after starting to rise from the low point on the Ferris wheel. The friend has a constant speed of $\|\mathbf{r}_1'(t)\| = 15$ meters per second. The speed of the object at that time is

$$\|\mathbf{r}_2'(3.15)\| = \sqrt{8.03^2 + [11.47 - 9.8(3.15 - 2)]^2} \approx 8.03 \text{ meters per second.}$$

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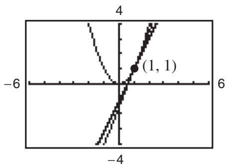
$P(1, 1)$ and $Q_1(1.5, f(1.5))$: $m = \dfrac{2.25 - 1}{1.5 - 1} = \dfrac{1.25}{0.5} = 2.5$

$P(1, 1)$ and $Q_2(1.1, f(1.1))$: $m = \dfrac{1.21 - 1}{1.1 - 1} = \dfrac{0.21}{0.1} = 2.1$

$P(1, 1)$ and $Q_3(1.01, f(1.01))$: $m = \dfrac{1.0201 - 1}{1.01 - 1} = \dfrac{0.0201}{0.01} = 2.01$

$P(1, 1)$ and $Q_4(1.001, f(1.001))$: $m = \dfrac{1.002001}{1.001 - 1} = \dfrac{0.002001}{0.001} = 2.001$

$P(1, 1)$ and $Q_5(1.0001, f(1.0001))$: $m = \dfrac{1.00020001}{1.0001 - 1} = \dfrac{0.00020001}{0.0001} = 2.0001$



$m = 2$

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The width of each rectangle is $\dfrac{1}{5}$.

The area of the inscribed set of rectangles is:

$$\begin{aligned} A &= \frac{1}{5}\cdot f(0) + \frac{1}{5}\cdot f\left(\frac{1}{5}\right) + \frac{1}{5}\cdot f\left(\frac{2}{5}\right) + \frac{1}{5}\cdot f\left(\frac{3}{5}\right) + \frac{1}{5}\cdot f\left(\frac{4}{5}\right) \\ &= \frac{1}{5}\left[0 + \frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25}\right] \\ &= \frac{1}{5}\left[\frac{30}{25}\right] \\ &= \frac{6}{25} = 0.24. \end{aligned}$$

The area of the circumscribed set of rectangles is:

$$\begin{aligned} A &= \frac{1}{5}\cdot f\left(\frac{1}{5}\right) + \frac{1}{5}\cdot f\left(\frac{2}{5}\right) + \frac{1}{5}\cdot f\left(\frac{3}{5}\right) + \frac{1}{5}\cdot f\left(\frac{4}{5}\right) + \frac{1}{5}\cdot f(1) \\ &= \frac{1}{5}\left[\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25} + 1\right] \\ &= \frac{1}{5}\left[\frac{55}{25}\right] \\ &= \frac{11}{25} = 0.44. \end{aligned}$$

The area of the region is approximately $\dfrac{1}{2}\left(\dfrac{6}{25} + \dfrac{11}{25}\right) = \dfrac{17}{50} = 0.34.$

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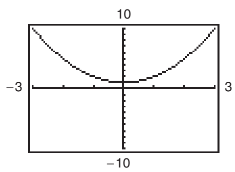
x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25

The graph of $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ agrees with the graph of $g(x) = x - 1$ at all points but one, where $x = 2$. If you trace along f getting closer and closer to $x = 2$, the value of y will get closer and closer to 1.

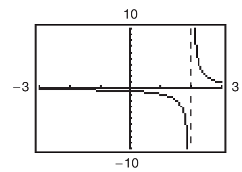
Chapter 2, Section 4, page 94

a.



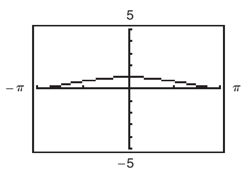
$y = x^2 + 1$ looks continuous on $(-3, 3)$.

b.



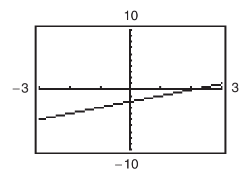
$y = \dfrac{1}{x - 2}$ does not look continuous on $(-3, 3)$.

c.



$y = \dfrac{\sin x}{x}$ looks continuous on $(-\pi, \pi)$.

d.



$y = \dfrac{x^2 - 4}{x + 2}$ looks continuous on $(-3, 3)$.

You cannot trust the results you obtain graphically. In these examples, only part **a.** is continuous. Part **b.** is discontinuous at $x = 2$; part **c.** is discontinuous at $x = 0$; part **d.** is discontinuous at $x = -2$.

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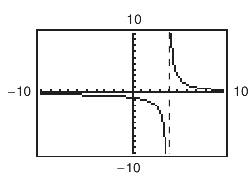
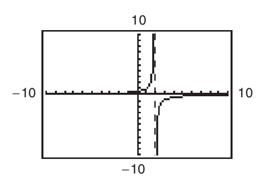
a.



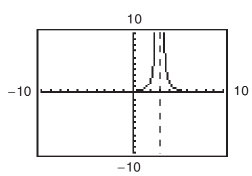
$c = 4;\ \lim\limits_{x\to 4^-} f(x) = -\infty;\ \lim\limits_{x\to 4^+} f(x) = +\infty$

b.



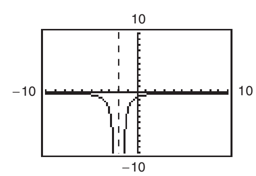
$c = 2;\ \lim\limits_{x\to 2^-} f(x) = +\infty;\ \lim\limits_{x\to 2^+} f(x) = -\infty$

c.



$c = 3;\ \lim\limits_{x\to 3^-} f(x) = +\infty;\ \lim\limits_{x\to 3^+} f(x) = +\infty$

d.

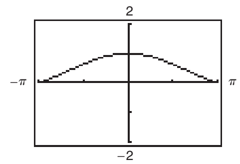


$c = -2;\ \lim\limits_{x\to 2^-} f(x) = -\infty;\ \lim\limits_{x\to -2^+} f(x) = -\infty$

S E C T I O N   P R O J E C T S

Chapter 2, Section 5, page 114   Graphs and Limits of Trigonometric Functions

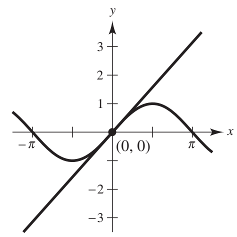
(a) On the graph of f, it appears that the y-coordinates of points lie as close to 1 as desired as long as you consider only those points with an x-coordinate near to but not equal to 0.



(b) Use a table of values of x and $f(x)$ that includes several values of x near 0. Check to see if the corresponding values of $f(x)$ are close to 1. In this case, because f is an even function, only positive values of x are needed.

x	0.5	0.1	0.01	0.001
$f(x)$	0.9589	0.9983	1.0000	1.0000

(c) The slope of the sine function at the origin appears to be 1. (It is necessary to use radian measure and have the same unit of length on both axes.)



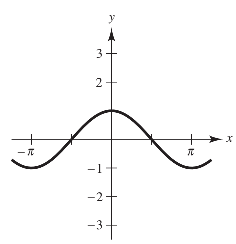
(d) In the notation of Section 2.1, $c = 0$ and $c + \Delta x = x$. Thus, $m_{\text{sec}} = \dfrac{\sin x - 0}{x - 0}$.

This formula has a value of 0.998334 if $x = 0.1$; $m_{\text{sec}} = 0.999983$ if $x = 0.01$.

The exact slope of the tangent line to g at $(0, 0)$ is $\displaystyle\lim_{x\to 0} m_{\text{sec}} = \lim_{x\to 0}\frac{\sin x}{x} = 1$.

(e) The slope of the tangent line to the cosine function at the point $(0, 1)$ is 0. The analytical proof is as follows:

$$\lim_{\Delta x\to 0} m_{\text{sec}} = \lim_{\Delta x\to 0}\frac{\cos(0 + \Delta x) - 1}{\Delta x} = -\lim_{\Delta x\to 0}\frac{1 - \cos(\Delta x)}{\Delta x} = 0.$$



(f) The slope of the tangent line to the graph of the tangent function at $(0, 0)$ is:

$$\lim_{\Delta x\to \infty} m_{\text{sec}} = \lim_{\Delta x\to 0}\frac{\tan(0 + \Delta x) - 0}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sin(\Delta x)}{\Delta x}\cdot\frac{1}{\cos\Delta x} = 1\cdot\frac{1}{1} = 1.$$

Chapter 3, Section 5, page 177   Optical Illusions

(a)
$$x^2 + y^2 = C^2$$
$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$

at the point $(3, 4)$, $y' = -\dfrac{3}{4}$

(b)
$$xy = C$$
$$xy' + y = 0$$
$$y' = -\frac{y}{x}$$

at the point $(1, 4)$, $y' = -4$

