Introduction to the Micromechanics of Composite Materials 1st edition Yin Solutions Manual

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Chapter 2

Vectors and tensors

2.1 Exercise

1. Prove the following identity equations using the index notation:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]$$

Proof:

(1)$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$=$a_i(\varepsilon_{ijk}b_jc_k) = (\varepsilon_{ijk}a_ib_j)c_k$= $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

(2)$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$=$\varepsilon_{ijk}a_j(\varepsilon_{klm}b_lc_m) = (\varepsilon_{ijk}\varepsilon_{lmk})a_jb_lc_m$= $(\delta_{ij}\delta_{jm} - \delta_{im}\delta_{jl})a_jb_lc_m$

=$a_jb_ic_j$−$a_jb_jc_i$=$(a_jc_j)b_j - (a_jb_j)c_i$=$(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

(3)$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$=$(\varepsilon_{ijk}a_ib_j)(\varepsilon_{klm}c_ld_m)$=$a_i(\varepsilon_{ijk}b_j(\varepsilon_{lmk}c_ld_m)$=$\mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]$

2. For a rotational matrix $\mathbf{Q}$ between two Cartesian coordinates $\mathbf{x}$ and $\mathbf{x}$' with a common origin, say $Q_{ij} = cos(x_i, x_j^{'})$. The angular relationship between any two Cartesian reference frames $\mathbf{x}$ and $\mathbf{x}$' can be represented in terms of three Euler angles(ϕ,θ,ψ) as follows: a) the first rotation is by an angle ϕ about the x_3 axis, b) the second rotation is by an angle $\theta\in(0,\pi)$about the $x_1$axis, and c) the third rotation is by an angle ψ about the the x_3 axis again. Then the transformation matrix can be written as:

$$\mathbf{Q} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & \cos\psi\sin\phi + \cos\theta\sin\psi\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}.$$

Prove

$$Q_{ik}Q_{jk} = \delta_{ij}.$$

Proof:

We calculate this one by one

$Q_{1k}Q_{1k}$

$=(\cos\psi\cos\phi-\cos\theta\sin\psi\sin\phi)^2+(\cos\psi\sin\phi+\cos\theta\sin\psi\cos\phi)^2+(\sin\psi\sin\theta)^2$

$=\cos^2\psi\cos^2\phi-2\cos\psi\cos\phi\cos\theta\sin\psi\sin\phi+\cos^2\theta\sin^2\psi\cos^2\phi$

$+\cos^2\psi\sin^2\phi+2\cos\theta\sin\psi\cos\phi+\cos^2\theta\sin^2\psi\cos^2\phi+\sin^2\psi\sin^2\theta$

$=\left(\cos^2\psi\cos^2\phi+\cos^2\psi\sin^2\phi\right)$

$+\left(\cos^2\theta\sin^2\psi\cos^2\phi+\cos^2\theta\sin^2\psi\cos^2\phi\right)+\sin^2\psi\sin^2\theta$

$=\cos^2\psi+\cos^2\theta\sin^2\psi+\sin^2\psi\sin^2\theta$

$=1$

$Q_{1k}Q_{2k}$

$=(\cos\psi\cos\phi-\cos\theta\sin\psi\sin\phi)(-\sin\psi\cos\phi-\cos\theta\cos\psi\sin\phi)$

$+(\cos\psi\sin\phi+\cos\theta\sin\psi\cos\phi)(-\sin\psi\sin\phi+\cos\theta\cos\psi\cos\phi)$

$+(\sin\psi\sin\theta)(\cos\psi\sin\theta)$

$=-\cos\psi\cos^2\phi\sin\psi-\cos^2\psi\cos\phi\cos\theta\sin\phi+\cos\theta\sin^2\psi\sin\phi\cos\phi+\cos^2\theta\sin\psi\sin^2\phi\cos\psi$

$-\cos\psi\sin^2\phi\sin\psi+\cos^2\psi\cos\phi\cos\theta\sin\phi-\cos\theta\sin^2\psi\sin\phi\cos\phi+\cos^2\theta\sin\psi\cos^2\phi\cos\psi$

$+\sin\psi\cos\psi\sin^2\theta$

$=-\left(\cos\psi\cos^2\phi\sin\psi+\cos\psi\sin^2\phi\sin\psi\right)$

$+\left(\cos^2\theta\sin\psi\sin^2\phi\cos\psi+\cos^2\theta\sin\psi\cos^2\phi\cos\psi\right)$

$+\sin\psi\cos\psi\sin^2\theta$

$=-\cos\psi\sin\psi+\cos^2\theta\sin\psi\cos\psi+\sin\psi\cos\psi\sin^2\theta$

$=0$

$Q_{1k}Q_{3k}$

$=(\cos\psi\cos\phi-\cos\theta\sin\psi\sin\phi)(\sin\theta\sin\phi)$

$+(\cos\psi\sin\phi+\cos\theta\sin\psi\cos\phi)(-\sin\theta\cos\phi)$

$+(\sin\psi\sin\theta)\cos\theta$

$=\cos\psi\cos\phi\sin\theta\sin\phi-\cos\theta\sin\psi\sin^2\phi\sin\theta$

-$\cos\psi\sin\phi\sin\theta\cos\phi-\cos\theta\sin\psi\cos^2\phi\sin\theta$

$+\sin\psi\sin\theta\cos\theta$

$=-\cos\theta\sin\psi\left(\sin^2\phi+\cos^2\phi\right)\sin\theta+\sin\psi\sin\theta\cos\theta$

$=0$

$Q_{2k}Q_{2k}$

$=(-\sin\psi\cos\phi-\cos\theta\cos\psi\sin\phi)(-\sin\psi\sin\phi-\cos\theta\cos\psi\sin\phi)$

$+(\sin\psi\sin\phi+\cos\theta\cos\psi\cos\phi)\,(-\sin\psi\sin\phi+\cos\theta\cos\psi\cos\phi)$

$+(\cos\psi\sin\theta)\,(\cos\psi\sin\theta)$

$=\sin^2\psi\cos^2\phi+2\sin\psi\cos\phi\cos\theta\cos\psi\sin\phi+\cos^2\theta\cos^2\psi\sin^2\phi$

$+\sin^2\psi\sin^2\phi-2\sin\psi\sin\phi\cos\theta\cos\psi\cos\phi+\cos^2\theta\cos^2\psi\cos^2\phi$

$+\cos^2\psi\sin^2\theta$

$=\left(\sin^2\psi\cos^2\phi+\sin^2\psi\sin^2\phi\right)$

$+\left(\cos^2\theta\cos^2\psi\sin^2\phi+\cos^2\theta\cos^2\psi\cos^2\phi\right)$

$+\cos^2\psi\sin^2\theta$

$=\sin^2\psi+\cos^2\theta\cos^2\psi+\cos^2\psi\sin^2\theta$

$=1$

$Q_{2k}Q_{3k}$

$=(-\sin\psi\cos\phi-\cos\theta\cos\psi\sin\phi)\,(\sin\theta\sin\phi)$

$+(-\sin\psi\sin\phi+\cos\theta\cos\psi\cos\phi)\,(-\sin\theta\cos\phi)$

$+(\cos\psi\sin\theta)\,(\cos\theta)$

$=-\sin\psi\cos\phi\sin\theta\sin\phi-\cos\theta\cos\psi\sin\phi\sin\theta\sin\phi$

$+\sin\psi\sin\phi\sin\theta\cos\phi-\cos\theta\cos\psi\cos\phi\sin\theta\cos\phi$

$+\cos\psi\sin\theta\cos\theta$

$=-\cos\theta\cos\psi\left(\sin^2\phi+\cos^2\phi\right)\sin\theta+\cos\psi\sin\theta\cos\theta$

$=0$

$Q_{3k}Q_{3k}$

$=(\sin\theta\sin\phi)^2+(-\sin\theta\cos\phi)^2+\cos^2\theta$

$=\sin^2\theta\left(\sin^2\phi+\cos^2\phi\right)+\cos^2\theta$

$=1$

And for symmetry

$Q_{2k}Q_{1k}=Q_{1k}Q_{2k}=0$

$Q_{3k}Q_{1k}=Q_{1k}Q_{3k}=0$

$Q_{3k}Q_{2k}=Q_{2k}Q_{3k}=0$

So

$Q_{ik}Q_{jk}=\delta_{ij}$

3. Prove the following tensors are isotropic for the corresponding tensor ranks:

$$A_{ij}=k\delta_{ij};\;A_{ijk}=ke_{ijk};\;A_{ijkl}=\alpha\delta_{ij}\delta_{kl}+\beta\delta_{ik}\delta_{jl}+\gamma\delta_{il}\delta_{jk};$$

Proof:

1. We need to point out that the transformation matrix $\mathbf{Q}$, it satisfies

$Q_{mi}Q_{ni} = \delta_{mn}$

For $\lambda\delta_{ij}$,under the transformation matrix Q,it becomes

$Q_{mi}Q_{nj}\lambda\delta_{ij} = \lambda Q_{mi}Q_{ni} = \lambda\delta_{mn} = \lambda\delta_{ij}$

Since $\mathbf{Q}$ is transformation matrix, it satisfies $Q_{mi}Q_{ni} = \delta_{mn}$

2. For $A_{ijk} = \lambda e_{ijk}$ under a transformation, we can write

$A'_{lmn} = \lambda Q_{li}Q_{mj}Q_{nk}e_{ijk}$

Considering $e_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k$, we can write

$A'_{lmn} = \lambda Q_{li}Q_{mj}Q_{nk}\left(\mathbf{e}_i \times \mathbf{e}_j\right) \cdot \mathbf{e}_k$

$= \lambda\left(Q_{li}\mathbf{e}_i \times Q_{mj}\mathbf{e}_j\right) \cdot Q_{nk}\mathbf{e}_k$

$= \lambda\left(\mathbf{e}'_l \times \mathbf{e}'_m\right) \cdot \mathbf{e}'_n$

Considering the propoerties of the base vector, $\left(\mathbf{e}'_l \times \mathbf{e}'_m\right) \cdot \mathbf{e}'_n = \left(\mathbf{e}_l \times \mathbf{e}_m\right) \cdot \mathbf{e}_n$ as the volume of the cubes of the vectors, we can write

$A'_{lmn} = \lambda\left(\mathbf{e}'_l \times \mathbf{e}'_m\right) \cdot \mathbf{e}'_n = \lambda\left(\mathbf{e}_l \times \mathbf{e}_m\right) \cdot \mathbf{e}_n = A_{lmn}$

So it is isotropic

3. For forth rank tensor $A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$

$A'_{mnst} = Q_{mi}Q_{ni}Q_{sk}Q_{tk}\left(\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}\right)$

$=\alpha Q_{mi}Q_{ni}Q_{sk}Q_{tk} + \beta Q_{mi}Q_{nj}Q_{si}Q_{tj} + \gamma Q_{mi}Q_{nj}Q_{sj}Q_{ti}$

$=\alpha\left(Q_{mi}Q_{ni}\right)\left(Q_{sk}Q_{tk}\right) + \beta\left(Q_{mi}Q_{si}\right)\left(Q_{nj}Q_{tj}\right) + \gamma\left(Q_{mi}Q_{ti}\right)\left(Q_{bj}Q_{sj}\right)$

$=\alpha\delta_{mn}\delta_{st} + \beta\delta_{mst}\delta_{nt} + \gamma\delta_{mt}\delta_{ns}$

$=A_{mnst}$

So it is isotropic

4. Consider vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (3, 2, 1)$. Determine the following quantities:

$$\mathbf{a} \cdot \mathbf{b}, \mathbf{a} \times \mathbf{b}, \mathbf{a} \otimes \mathbf{b}.$$

Now if the coordinate is rotated along the axis x_3 at 30° counterclockwise, i.e. the $x_1 - x_2$ plane is rotated 30° counterclockwise and the x_3 axis remains the same, recalculate the above quantities in the new coordinate system.

Answer:

$\mathbf{a} \cdot \mathbf{b} = a_i b_i = 1 \times 3 + 2 \times 2 + 3 \times 1 = 10$

$\mathbf{a} \times \mathbf{b} = \varepsilon_{kij}a_i b_j = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = (2 \times 1 - 3 \times 2)\, i + (3 \times 3 - 1 \times 1)\, j + (1 \times 2 - 2 \times 3)\, k = (-4, 8, -4)$

$$\mathbf{a}\otimes\mathbf{b}=a_ib_j=\begin{bmatrix}1\times3 & 1\times2 & 1\times1\\ 2\times3 & 2\times2 & 2\times1\\ 3\times3 & 3\times2 & 3\times1\end{bmatrix}=\begin{bmatrix}3 & 2 & 1\\ 6 & 4 & 2\\ 9 & 6 & 3\end{bmatrix}$$

Then we consider the condition of rotation,

The rotation matrix is

$$Q=\begin{bmatrix}\cos a & -\sin a & \\ \sin a & \cos a & \\ & & 1\end{bmatrix}=\begin{bmatrix}\cos 30 & -\sin 30 & \\ \sin 30 & \cos 30 & \\ & & 1\end{bmatrix}$$

After the rotation of the coordinate, the new quantities of the vectors would be

$$\mathbf{a}=(1,2,3)\begin{bmatrix}\cos 30 & -\sin 30 & \\ \sin 30 & \cos 30 & \\ & & 1\end{bmatrix}=\left(\frac{\sqrt{3}}{2}+1,-\frac{1}{2}+\sqrt{3},3\right)=(1.8660\ 1.2321\ 3.0000)$$

$$\mathbf{b}=(3,2,1)\begin{bmatrix}\cos 30 & -\sin 30 & \\ \sin 30 & \cos 30 & \\ & & 1\end{bmatrix}=\left(\frac{3\sqrt{3}}{2}+1,-\frac{3}{2}+\sqrt{3},1\right)=(3.5981\ 0.2321\ 1.0000)$$

Then

$$\mathbf{a}\cdot\mathbf{b}=a_ib_i=\left(\frac{\sqrt{3}}{2}+1\right)\times\left(\frac{3\sqrt{3}}{2}+1\right)+\left(-\frac{1}{2}+\sqrt{3}\right)\times\left(-\frac{3}{2}+\sqrt{3}\right)+3\times1$$

$$=\left(\frac{13}{4}+2\sqrt{3}\right)+\left(\frac{15}{4}-2\sqrt{3}\right)+3=10$$

$$\mathbf{a}\times\mathbf{b}=\varepsilon_{kij}a_ib_j=\begin{vmatrix} i & j & k\\ \frac{\sqrt{3}}{2}+1 & -\frac{1}{2}+\sqrt{3} & 3\\ \frac{3\sqrt{3}}{2}+1 & -\frac{3}{2}+\sqrt{3} & 1\end{vmatrix}$$

$$=\left[-\frac{1}{2}+\sqrt{3}-\left(-\frac{9}{2}+3\sqrt{3}\right)\right]i+\left[\frac{9\sqrt{3}}{2}+3-\left(\frac{\sqrt{3}}{2}+1\right)\right]j+\left[\frac{\sqrt{3}}{4}-\left(4+\frac{\sqrt{3}}{4}\right)\right]k$$

$$=\left(4-2\sqrt{3},4\sqrt{3}+2,-4\right)=(0.5359\ 8.9282\ -4.0000)$$

Note that this quantity of the vector is in the new coordinate system

$$\mathbf{a}\otimes\mathbf{b}=a_ib_j=\begin{bmatrix}\left(\frac{\sqrt{3}}{2}+1\right)\times\left(\frac{3\sqrt{3}}{2}+1\right) & \left(\frac{3\sqrt{3}}{2}+1\right)\times\left(-\frac{3}{2}+\sqrt{3}\right) & \left(\frac{\sqrt{3}}{2}+1\right)\times1\\ \left(-\frac{1}{2}+\sqrt{3}\right)\times\left(\frac{3\sqrt{3}}{2}+1\right) & \left(-\frac{1}{2}+\sqrt{3}\right)\times\left(-\frac{3}{2}+\sqrt{3}\right) & \left(-\frac{1}{2}+\sqrt{3}\right)\times1\\ 3\times\left(\frac{3\sqrt{3}}{2}+1\right) & 3\times\left(-\frac{3}{2}+\sqrt{3}\right) & 3\times1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{13}{4}+2\sqrt{3} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{2}+1\\ 4+\frac{\sqrt{3}}{4} & \frac{15}{4}-2\sqrt{3} & -\frac{1}{2}+\sqrt{3}\\ \frac{9\sqrt{3}}{2}+3 & -\frac{9}{2}+3\sqrt{3} & 3\end{bmatrix}=\begin{bmatrix}6.7141 & 0.4330 & 1.8660\\ 4.4330 & 0.2859 & 1.2321\\ 10.7942 & 0.6962 & 3.0000\end{bmatrix}$$

Still in the new coordinate system.

5. An orthogonal material has the stiffness tensor given in form of $C_{ijkl}=C^1_{IK}\delta_{ij}\delta_{kl}+C^2_{IJ}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)$ with

$$\mathbf{C^1}=\begin{bmatrix}1 & 0.5 & 0.5\\ 0.5 & 1.5 & 0.75\\ 0.5 & 0.75 & 2\end{bmatrix};\ \mathbf{C^2}=\begin{bmatrix}0.5 & 0.25 & 0.25\\ 0.25 & 0.75 & 0.5\\ 0.25 & 0.5 & 1\end{bmatrix}.$$

(a) Determine the components of the stiffness in form of a 6 by 6 matrix.

(b) Calculate the components of the compliance.

(c) Write the compliance tensor in the form of $D_{ijkl} = D^1_{IK}\delta_{ij}\delta_{kl} + D^2_{IJ}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Determine D^1_{IK} and D^2_{IJ} in terms of C^1_{IK} and C^2_{IJ}. Check whether the calculation in (b) is consistent with the formulation.

Answer:

(a) The stiffness matrix transfers a strain tensor to a stress tensor as $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$. Due to the symmetry of stress and strain, we can use a list of 6 components to represent the 3x3 tensor to save space in numerical programming as follows:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{pmatrix}$$

Notice that the strain list is the engineering strain instead of the tensorial strain, whose difference is in the shear strain components, i.e. $\gamma_{23} = 2\epsilon_{23}$, etc. but the normal strain s keep the same. If tensorial strain is used, the above relation becomes

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1123} & 2C_{1131} & 2C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2223} & 2C_{2231} & 2C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & 2C_{3323} & 2C_{3331} & 2C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & 2C_{2323} & 2C_{2331} & 2C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & 2C_{3123} & 2C_{3131} & 2C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1223} & 2C_{1231} & 2C_{1212} \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{12} \end{pmatrix}$$

where the symmetry of the stiffness tensor is lost. Therefore, we generally use the engineering strain in the 6-component notation. However, for orthotropic materials, because the components without symmetry are zero, this issue is not real. In the following, let us use engineering strain

$$C_{ijkl} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{bmatrix}$$

$$= \begin{bmatrix} C^1_{11} + C^2_{11} \times 2 & C^1_{12} & C^1_{13} & 0 & 0 & 0 \\ C^1_{21} & C^1_{22} + C^2_{22} \times 2 & C^1_{32} & 0 & 0 & 0 \\ C^1_{31} & C^1_{32} & C^1_{33} + C^2_{33} \times 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & C^2_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & C^2_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & C^2_{12} \end{bmatrix}$$

$$= \begin{bmatrix} 1+0.5\times 2 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 1.5+0.75\times 2 & 0.75 & 0 & 0 & 0 \\ 0.5 & 0.75 & 1+0.5\times 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 3 & 0.75 & 0 & 0 & 0 \\ 0.5 & 0.75 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 \end{bmatrix}$$

(b)

Using Matlab, we calculate the transvers if the stiffness matrix, compliance matrix D

$$D = C^{-1} = \begin{bmatrix} 0.5320 & -0.0756 & -0.0523 & 0 & 0 & 0 \\ -0.0756 & 0.3605 & -0.0581 & 0 & 0 & 0 \\ -0.0523 & -0.0581 & 0.2674 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Notice that this is for engineering strain. If tensorial strain is used, the bottom-right quardrant shall be applied a multiplyer of 1/2.

(c)

There are not definite relationship between $D^1_{IK}\, D^2_{IJ}$and$C^1_{IK}\, C^2_{IJ}$

From the calculation above, we see that the ralationship between $C^1_{IK}\, C^2_{IJ}$and $C_{ijkl}$is not one to

one.

There are numerous sets of C^1_{IK} and C^2_{IJ} that can get the same C_{ijkl} as long as $C^1_{ii} + 2C^2_{ii}\,(i = 1, 2, 3)$

keeps the same value.

The same is for D^1_{IK} and D^2_{IJ}, their compositioncan be various and may not be determined uniquely.

However, we can still get the one way to obtain the general relation between $C^1_{IK}\, C^2_{IJ}$and$D^1_{IK}\, D^2_{IJ}$

Let

$$C^3 = \begin{bmatrix} C^1_{11} + C^2_{11}\times 2 & C^1_{12} & C^1_{13} \\ C^1_{21} & C^1_{22} + C^2_{22}\times 2 & C^1_{32} \\ C^1_{31} & C^1_{32} & C^1_{33} + C^2_{33}\times 2 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} D^1_{11} + D^2_{11}\times 2 & D^1_{12} & D^1_{13} \\ D^1_{21} & D^1_{22} + D^2_{22}\times 2 & D^1_{32} \\ D^1_{31} & D^1_{32} & D^1_{33} + D^2_{33}\times 2 \end{bmatrix}$$

$C^4 = \begin{bmatrix} C_{23}^2 & & \\ & C_{13}^2 & \\ & & C_{12}^2 \end{bmatrix}$

$D^4 = \begin{bmatrix} D_{23}^2 & & \\ & D_{13}^2 & \\ & & D_{12}^2 \end{bmatrix}$

Since CD = I

We have

$C^3 D^3 = I$

$C^4 D^4 = I$

So when $C_{IK}^1\, C_{IJ}^2$ is determined,we can calculate get $D^3\, D^4$,based on D^4 we get

$\begin{cases} D_{23}^2 = D_{32}^2 = (C_{23}^2)^{-1} \\ D_{13}^2 = D_{31}^2 = (C_{13}^2)^{-1} \\ D_{12}^2 = D_{21}^2 = (C_{12}^2)^{-1} \end{cases}$

Then $D_{11}^2\, D_{22}^2\, D_{33}^2$, could be chosen to be any value

From D^3we get

$D_{ij}^1 = D_{ij}^3\, (i \neq j)$

And

$\begin{cases} D_{11}^1 = D_{11}^3 - 2 \times D_{11}^2 \\ D_{22}^1 = D_{22}^3 - 2 \times D_{22}^2 \\ D_{33}^1 = D_{33}^3 - 2 \times D_{33}^2 \end{cases}$

So by now we solve the problem. If we have the value of $C_{IK}^1\, C_{IJ}^2$, we can get the according value for

$D_{IK}^1\, D_{IJ}^2$,but they are not uniquely determined. From the above derivation, we can also provide an analytical form for the case of tensorial strains.

6. Prove the following identities

$$\nabla \times (\nabla \phi) = \mathbf{0},$$
$$\nabla \cdot (\nabla \times \mathbf{a}) = 0,$$
$$\nabla \times (\nabla \times \mathbf{a}) = \nabla (\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

where ϕ is a scalar field and $\mathbf{a}$ a vector field.

Proof:

(1) $\nabla \times (\nabla \phi) = \varepsilon_{ijk}\phi_{,kj} = -\varepsilon_{ikj}\phi_{,kj} = -\varepsilon_{ijk}\phi_{,kj} = 0$(Switch j,k in the last two formulas)

(2) $\nabla \cdot (\nabla \times \mathbf{a}) = (\varepsilon_{ijk} a_{j,k})_{,i} = \varepsilon_{ijk} a_{j,ki} = -\varepsilon_{kji} a_{j,ik} = -\varepsilon_{ijk} a_{j,ki} = 0$(Switch i, k in the last two formulas)

(3)$\nabla \times (\nabla \times \mathbf{a})= \varepsilon_{ijk}(\varepsilon_{kmn}a_{n,m})_{,j} = (\varepsilon_{ijk}\varepsilon_{kmn})a_{n,mj}$

$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})a_{n,mj} = a_{j,ij} - a_{i,jj} = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$

7. Given a scalar field in a 3-dimensional infinite domain as $\psi = |\mathbf{x} - \mathbf{x}^0|$, which is a distance from $\mathbf{x}$ to a known point $\mathbf{x}^0$, derive $\nabla\nabla\psi$, which is a second rank tensor. Similarly, for $\phi = 1/|\mathbf{x} - \mathbf{x}^0|$, derive $\nabla\nabla\phi$.

Answer:

Let $r = \left|x - x^0\right|^2$

First we get

$\nabla\psi = \frac{d\psi}{dr}\frac{dr}{dx_i} = \frac{1}{2\left|x - x^0\right|} \cdot 2(x_i - x_i^0)$

$=\frac{(x_i - x_i^0)}{\left|x - x^0\right|}$

$\nabla\nabla\psi = -\frac{1}{2\left|x - x^0\right|^3}(x_i - x_i^0) \cdot 2(x_j - x_j^0) + \frac{\delta_{ij}}{\left|x - x^0\right|}$

$=\frac{-(x_i - x_i^0)(x_j - x_j^0) + \delta_{ij}\left|x - x^0\right|^2}{\left|x - x^0\right|^3}$

For $\phi = \frac{1}{\left|x - x^0\right|}$

$\nabla\psi = \frac{d\psi}{dr}\frac{dr}{dx_i} = -\frac{1}{2\left|x - x^0\right|} \cdot 2(x_i - x_i^0)$

$=\frac{-(x_i - x_i^0)}{\left|x - x^0\right|^3}$

$\nabla\nabla\psi = \frac{3}{2\left|x - x^0\right|^5}(x_i - x_i^0) \cdot 2(x_j - x_j^0) + \frac{\delta_{ij}}{\left|x - x^0\right|^3}$

$=\frac{3(x_i - x_i^0)(x_j - x_j^0) - \delta_{ij}\left|x - x^0\right|^2}{\left|x - x^0\right|^5}$

8. Given a vector field $\mathbf{a} = \mathbf{x}$ in a spherical domain Ω: $x_1^2 + x_2^2 + x_3^2 \leq 1$, use the Helmholtz's decomposition to write $\mathbf{a} = \nabla\varphi + \nabla \times \mathbf{b}$ in terms of a scalar potential and a vector potential $\mathbf{b}$. Find the specific form of φ and $\mathbf{b}$ in terms of x_i.

Answer:

From the textbook, we have

$a(x) = \nabla^2 u(x) = \nabla(\nabla \cdot u(x)) - \nabla \times (\nabla \times u(x))$

We choose

$\varphi = \nabla \cdot u(x)$

$b = -\nabla \times u(x)$

Then we can get

$a(x) = \nabla^2 u(x) = \nabla\varphi + \nabla \times b$

So now we caculate $u(x)$

$\nabla^2 u(x) = a(x)$=x

We can get

$u(x) = \left(\frac{x_1^3}{6} + x_1 f_1(x_2, x_3) + g_1(x_2, x_3), \frac{x_2^3}{6} + x_2 f_2(x_1, x_3) + g_2(x_1, x_3), \frac{x_3^3}{6} + x_3 f_3(x_1, x_2) + g_3(x_1, x_2)\right)$

Here $f_1(x_2, x_3)$and $g_1(x_2, x_3)$are any functions

$\varphi = \nabla \cdot u(x) = \frac{x_1^2}{2} + f_1(x_2, x_3) + \frac{x_2^2}{2} + f_2(x_1, x_3) + \frac{x_3^2}{2} + f_3(x_1, x_2)$

$b = -\nabla \times u(\mathbf{x}) = -\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1(\mathbf{x}) & u_2(\mathbf{x}) & u_3(\mathbf{x}) \\ i & j & k \end{vmatrix}$

$= \begin{bmatrix} -\left(x_3 \frac{\partial f_3(x_1,x_2)}{\partial x_2} + \frac{\partial g_3(x_1,x_2)}{\partial x_2} - x_2 \frac{\partial f_2(x_1,x_3)}{\partial x_3} - \frac{\partial g_2(x_1,x_3)}{\partial x_3}\right) \\ x_2 \frac{\partial f_1(x_2,x_3)}{\partial x_3} + \frac{\partial g_1(x_2,x_3)}{\partial x_3} - x_3 \frac{\partial f_3(x_1,x_2)}{\partial x_1} - \frac{\partial g_3(x_1,x_2)}{\partial x_1} \\ x_2 \frac{\partial f_2(x_1,x_3)}{\partial x_1} + \frac{\partial g_2(x_1,x_3)}{\partial x_1} - x_1 \frac{\partial f_1(x_2,x_3)}{\partial x_2} - \frac{\partial g_1(x_2,x_3)}{\partial x_2} \end{bmatrix}$

9. Consider a scalar field $\varphi = x_1^2 + x_2^2 + x_3^2$ in a spherical domain : $x_1^2 + x_2^2 + x_3^2 \leq 1$. Confirm the following equations:

$$\int_\Omega \nabla\varphi dV = \int_{\partial\Omega} \mathbf{n}\varphi dS$$

$$\int_\Omega \nabla^2\varphi dV = \int_{\partial\Omega} \mathbf{n.}\nabla\varphi dS$$

Answer:

(1)

For the left hand side

$\int \nabla\varphi dV = \int 2x_i dV$

$=\overset{[}{r}= 0]r = 1\int \overset{[}{\theta}= 0]\theta = \pi\int \overset{[}{\alpha}= 0]\alpha = 2\pi\int (2r\sin\theta\cos a) r^2 \sin\theta dr d\theta d\alpha \mathbf{i}$

$+\overset{[}{r}= 0]r = 1\int \overset{[}{\theta}= 0]\theta = \pi\int \overset{[}{\alpha}= 0]\alpha = 2\pi\int (2r\sin\theta\sin a) r^2 \sin\theta dr d\theta d\alpha \mathbf{j}$

$+\overset{[}{r}= 0]r = 1\int \overset{[}{\theta}= 0]\theta = \pi\int \overset{[}{\alpha}= 0]\alpha = 2\pi\int (2r\cos\theta) r^2 \sin\theta dr d\theta d\alpha \mathbf{k}$

=0+0+0=0

For the right hand side

$\int_\partial \mathbf{n}\varphi dV$

$=\overset{[}{\theta}= 0]\theta = \pi\int \overset{[}{\alpha}= 0]\alpha = 2\pi\int (\sin\theta\cos a)\sin\theta d\theta d\alpha \mathbf{i}$

$+\overset{[}{\theta}= 0]\theta = \pi\int \overset{[}{\alpha}= 0]\alpha = 2\pi\int (\sin\theta\sin a)\sin\theta d\theta d\alpha \mathbf{j}$

$+\overset{[}{\theta}= 0]\theta = \pi\int \overset{[}{\alpha}= 0]\alpha = 2\pi\int (\sin\theta\cos a) r^2 \sin\theta d\theta d\alpha \mathbf{k}$

=0+0+0=0

They are equal

(2)

For the left hand side

$\int_{\Omega} \nabla^2 \varphi dV = \int_{\Omega} 6dV = 8\pi$

while

$\int_{\partial\Omega} \mathbf{n.}\nabla\varphi dS = \overset{[}{\theta} = 0]\theta = \pi \int \overset{[}{\alpha} = 0]\alpha = 2\pi \int 2\left[\sin^2\theta\left(\cos^2\alpha + \sin^2\alpha\right) + \cos^2\theta\right]\sin\theta d\theta d\alpha$

$= \overset{[}{\theta} = 0]\theta = \pi \int \overset{[}{\alpha} = 0]\alpha = 2\pi \int 2\sin\theta d\theta d\alpha$

$=8\pi$

They are equal.

10. Prove $G\left(\mathbf{x}, \mathbf{x}'\right) = ln\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ is the Green's function for the 2-dimensional Laplace equation.

Proof:

$G\left(\mathbf{x}, \mathbf{x}'\right) = ln\frac{1}{|\mathbf{x}-\mathbf{x}'|}$

When x$\neq x$'

$\nabla^2 G\left(\mathbf{x}, \mathbf{x}'\right) = \frac{\partial}{\partial x_i}\left\{|x - x'| \frac{\partial}{\partial x_i}\left[(x - x')^2\right]^{-\frac{1}{2}}\right\}$

$= \frac{\partial}{\partial x_i}\left\{|x - x'|\left(-\frac{1}{2}\right)\left[(x - x')^2\right]^{-\frac{3}{2}} 2\left(x_i - x'_i\right)\right\}$

$=-\frac{\partial}{\partial x_i}\frac{\left(x_i - x'_i\right)^2}{\left|x - x'\right|^2}$

$=-\frac{2}{|x-x'|^2} - \left\{-1\left[(x - x')^2\right]^{-2} 2\left(x_i - x'_i\right)^2\right\}$

$= -\frac{2}{|x-x'|^2} + \frac{2\left(x_i - x'_i\right)^2}{|x-x'|^4}$

$= -\frac{2}{|x-x'|^2} + \frac{2|x-x'|^2}{|x-x'|^4}$

$=0$

Here since it's two dimensional, $\frac{\partial}{\partial x_i} x_i = 2$

When x=x' , this point is nondifferentiable

From above we know that

$\frac{\partial G(x,x')}{\partial x_i} = -\frac{\left(x_i - x'_i\right)}{|x-x'|^2}$

If we take derivative with x', then it is

$\frac{\partial G(x,x')}{\partial x'_i} = -\frac{\left(x_i - x'_i\right)}{|x-x'|^2}$

Using Guass equation

$\int_\Omega \nabla^2 G(x,x')\,dx' = \int_{\partial\Omega} \nabla G(x,x')\,\mathbf{n}dx' = \int_{\partial\Omega} \frac{(x_i - x_i')}{|x-x'|^2} n_i dx'$

Since $\nabla^2 G(x,x') = 0$ everywhere except x=x', we can choose any $\partial\Omega$. Just choose it to be a circle

radius of one and centered at x,

Then

$\int_\Omega \nabla^2 G(x,x')\,dx'$

$=\int_{\partial\Omega} \frac{(x_i - x_i')}{|x-x'|^2} n_i dx'$

$=\overset{[}{\theta}= 0]\theta = 2\pi\int \overset{[}{\theta}= 0]\theta = \pi\int - \left(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta\right)\sin\theta d\theta d\varphi$

$=\overset{[}{\theta}= 0]\theta = 2\pi\int \overset{[}{\theta}= 0]\theta = \pi\int - 1\sin\theta d\theta d\varphi$

= -2π

So it is Green function in two dimension

And it is better written in the form of

$G(\mathbf{x},\mathbf{x}') = -\frac{1}{2\pi} ln \frac{1}{|\mathbf{x}-\mathbf{x}'|}$

11. Derive the Green's function for a cantilever beam bending.

Answer:

For the cantilever, using the same procedure in class, we have

$EI \cdot G(x-x') = \frac{(x-x')^3}{6} H(x-x') + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$

There is no displacement and tilt angle at the starting point x=0,

And the moment caused by this point force is x' at x=0 and 0 at x=L

So boundary condition is

$$\begin{cases} G(x-x') = 0\,(x=0) \\ G'(x-x') = 0\,(x=0) \\ G''(x-x') = x'\,(x=0) \\ G''(x-x') = 0\,(x=L) \end{cases}$$

From these we can determine

$$\begin{cases} C_1 = -1 \\ C_2 = x' \\ C_3 = 0 \\ C_4 = 0 \end{cases}$$

So the Green function is

$G(x-x') = \frac{1}{EI}\left[\frac{(x-x')^3}{6} H(x-x') + \frac{1}{6}x^3 + \frac{x'}{2}x^2\right]$

The beam's deflection shape function can be written as

$w = \int_{\Omega} G(x, x') \, q(x') \, dx'$"

12. Write the 6 equations implied by compatibility condition for strains $\nabla \times \epsilon \times \nabla = \mathbf{0}$.

Answer:

$\nabla \times \epsilon \times \nabla = \left(\varepsilon_{kim} \frac{\partial \varepsilon_{ij}}{\partial x_k}\right) \varepsilon_{jln} \frac{\partial}{\partial x_1} = \left(\varepsilon_{kim} \varepsilon_{jnl}\right) \partial \varepsilon_{ij,kl} = \mathbf{0}$

when$(m = 1, n = 1)$

$(\varepsilon_{ki1} \varepsilon_{j1l}) \, \partial \varepsilon_{ij,kl} = (\varepsilon_{231} \varepsilon_{213}) \, \partial \varepsilon_{32,23} \, (k = 2, i = 3, j = 2, l = 3)$

$+(\varepsilon_{231} \varepsilon_{312}) \, \partial \varepsilon_{33,22} \, (k = 2, i = 3, j = 3, l = 2)$

$+(\varepsilon_{321} \varepsilon_{213}) \, \partial \varepsilon_{22.33} \, (k = 3, i = 2, j = 2, l = 3)$

$+(\varepsilon_{321} \varepsilon_{312}) \, \partial \varepsilon_{23,32} \, (k = 3, i = 2, j = 3, l = 2)$

$= -\varepsilon_{32,23} + \varepsilon_{33,22} + \varepsilon_{22,33} - \varepsilon_{23,32}$

$=0$

So similar when $(m = 2, n = 2)$

We get

$-\varepsilon_{21,12} + \varepsilon_{22,11} + \varepsilon_{11,22} - \varepsilon_{12,21} = 0$

When $(m = 3, n = 3)$

$-\varepsilon_{13,31} + \varepsilon_{11.33} + \varepsilon_{33,11} - \varepsilon_{31,13} = 0$

$(m = 1, n = 2)$

$(\varepsilon_{ki1} \varepsilon_{j2l}) \, \partial \varepsilon_{ij,kl} = (\varepsilon_{231} \varepsilon_{213}) \, \partial \varepsilon_{31,23} \, (k = 2, i = 3, j = 1, l = 3)$

$+(\varepsilon_{231} \varepsilon_{321}) \, \partial \varepsilon_{33,21} \, (k = 2, i = 3, j = 3, l = 1)$

$+(\varepsilon_{321} \varepsilon_{123}) \, \partial \varepsilon_{21.33} \, (k = 3, i = 2, j = 1, l = 3)$

$+(\varepsilon_{321} \varepsilon_{321}) \, \partial \varepsilon_{23,31} \, (k = 3, i = 2, j = 3, l = 1)$

When $(m = 3, n = 1)$

$\varepsilon_{23,12} + \varepsilon_{22.33} + \varepsilon_{13,22} - \varepsilon_{12.23} = 0$

In conclude the six equations we get are:

$$\begin{cases} \varepsilon_{33,22} + \varepsilon_{22.33} - 2\varepsilon_{23,32} = 0 \\ \varepsilon_{22,11} + \varepsilon_{11,22} - 2\varepsilon_{12,21} = 0 \\ \varepsilon_{11.33} + \varepsilon_{33,11} - 2\varepsilon_{31,13} = 0 \\ \varepsilon_{31,23} - \varepsilon_{33,21} - \varepsilon_{21,33} + \varepsilon_{23,31} = 0 \\ \varepsilon_{12,31} - \varepsilon_{11,32} - \varepsilon_{32,11} + \varepsilon_{23,12} = 0 \\ \varepsilon_{12,31} - \varepsilon_{11,32} - \varepsilon_{32,11} + \varepsilon_{31,12} = 0 \end{cases}$$

13. Solve the boundary value problem: For a solid elastic ball fully boned by an outer layer, the radius of the ball and outer surface radii are written as $R_i < R_o$, respectively. The

Young's moduli and Poisson's ratios of the ball and outer layer are respectively written as E_i and v_i and E_o and v_o. When a pressure P is applied on the outer surface, derive the displacement distribution along the radial direction in two materials.

Answer:

For strain condition

$\varepsilon_{rr} = u_{r,r}$

$\varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u_r}{r}$

Equilibrium equation

$\sigma_{rr,r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} = 0$

Constitutive equation

$\sigma_{rr} = \lambda\left(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi}\right) + 2\mu\varepsilon_{rr} = \left(2\mu + \lambda\right) u_{r,r} + 2\lambda\frac{u_r}{r}$

Here to be convenient ,we use λ μ to replace E v

The relationship is

$\lambda = \frac{Ev}{(1+v)(1-2v)}$

$\mu = \frac{E}{2(1+v)}$

According to the textbook,the displacement of this kind of problem could be written as

$u_r = C_1 r + \frac{C_2}{r^2}$ Out layer

$u_r^* = C_1^* r + \frac{C_2^*}{r^2}$ inside sphere

Assure the stress between the two different materials is $\sigma_{rr}|_{r=Ri} = P^*$

We have two separate problems

$u_r = C_1 r + \frac{C_2}{r^2} \quad Ri \le r \le Ro$

$\begin{cases} \sigma_{rr}|_{r=Ro} = -P \\ \sigma_{rr}|_{r=Ri} = -P^* \end{cases} \quad (1)$

$u_r^* = C_1^* r + \frac{C_2^*}{r^2} \quad r \le Ri$

$\begin{cases} u_r^*|_{r=0} = 0 \\ \sigma_{rr}^*|_{r=Ri} = P^* \end{cases} \quad (2)$

$u_r = u_r^*|_{r=Ri}$

For the first boundary value problem, we get

$C_1 = \frac{R_o^3 P^* - (2R_o^3 - R_i^3)P}{(2\mu_o + 3\lambda_o)(R_0^3 - R_i^3)}$

$C_2 = \frac{R_o^3 R_i^3 (P* - P)}{4\mu_o (R_0^3 - R_i^3)}$

$u_r = \frac{R_i^3 P^* - (2R_o^3 - R_i^3)P}{(2\mu_o + 3\lambda_o)(R_0^3 - R_i^3)} r + \frac{R_o^3 R_i^3 (P* - P)}{4\mu_o (R_0^3 - R_i^3)} \frac{1}{r^2}$

For the second boundary condition,

At r=0 there is no displacement, so we can get $C_2* = \frac{P*}{2\mu_i+3\lambda_i}$

$u_r* = \frac{P*}{2\mu+3\lambda}r$

Then the problem is to decide P*

The displacement of the two material at the interface must be equal

$u_r = u_r^*|_{r=Ri}$

So we have

$\frac{R_i^3P^*-(2R_o^3-R_i^3)P}{(2\mu_o+3\lambda_o)(R_0^3-R_i^3)}R_i + \frac{R_o^3R_i^3(P*-P)}{4\mu_o(R_0^3-R_i^3)}\frac{1}{R_I^2} = \frac{P*}{2\mu+3\lambda}R_i$

From this we get

$P^* = \left[\left(\frac{(2R_o^3-R_i^3)R_i}{(2\mu_o+3\lambda_o)(R_0^3-R_i^3)} + \frac{R_o^3R_i}{4\mu_o(R_0^3-R_i^3)}\right)/\left(\frac{R_i^4}{(2\mu_o+3\lambda_o)(R_0^3-R_i^3)} + \frac{R_o^3R_i}{4\mu_o(R_0^3-R_i^3)} - \frac{R_i}{2\mu_i+3\lambda_i}\right)\right]P$

In conclude ,the solution of this problem is

For the inner solid elastic ball,

$u_r* = \frac{P*}{2\mu+3\lambda}r$

For the outer layer

$u_r = \frac{R_i^3P^*-(2R_o^3-R_i^3)P}{(2\mu_o+3\lambda_o)(R_0^3-R_i^3)}r + \frac{R_o^3R_i^3(P*-P)}{4\mu_o(R_0^3-R_i^3)}\frac{1}{r^2}$

Here

$P^* = \left[\left(\frac{(2R_o^3-R_i^3)R_i}{(2\mu_o+3\lambda_o)(R_0^3-R_i^3)} + \frac{R_o^3R_i}{4\mu_o(R_0^3-R_i^3)}\right)/\left(\frac{R_i^4}{(2\mu_o+3\lambda_o)(R_0^3-R_i^3)} + \frac{R_o^3R_i}{4\mu_o(R_0^3-R_i^3)} - \frac{R_i}{2\mu_i+3\lambda_i}\right)\right]P$

$\lambda = \frac{Ev}{(1+v)(1-2v)}$

$\mu = \frac{E}{2(1+v)}$.

14. The stress tensor in a semi-infinite domain $x_3 > 0$ is given by $\sigma_{ij} = (\alpha x_i x_j x_3)/|x|^5$, where $|x| = (x_i x_i)^{1/2}$ and α is a constant. Determine whether there are any body forces in the domain using the equilibrium equation.

Answer:

The equilibrium equation is

$\sigma_{ij,j} + b_i = 0$

From the expression of stress we can get

$\sigma_{ij,j} = \frac{(\alpha\delta_{ij}x_jx_3)}{|x|^5} + \frac{(\alpha\delta_{jj}x_ix_3)}{|x|^5} + \frac{(\alpha x_ix_jx_3)}{|x|^7}\left(\frac{-5}{2}\right)(2x_j)$

$= \frac{(\alpha x_ix_3)}{|x|^5} + \frac{(3\alpha x_ix_3)}{|x|^5} + \frac{(\alpha x_ix_3)}{|x|^5} - \frac{5\left(\alpha x_i|x|^2x_3\right)}{|x|^7}$

$= \frac{5\alpha x_ix_3}{|x|^5} - \frac{5\alpha x_ix_3}{|x|^5}$

$=0$

So

$b_i = 0 - 0 = 0$

Body force is zero.

15. In an infinite solid with material elastic constants μ and v, a concentrated force P = (0, 0, 1) is applied at the origin. Calculate the surface force distribution on the sphere with radius 1 and the center at the origin. Integrate it over the surface for the resultant surface force.

Answer:

First we use the formula in textbook

$$u_i = \frac{P}{16\pi\mu(1-v)}\left[(3-4v)\frac{\delta_{ij}}{\left|x-x'\right|} + \frac{\left(x_i - x_i'\right)\left(x_j - x_j'\right)}{\left|x-x'\right|^3}\right]P_j$$

In this case , we get

$$u_i = \frac{P}{16\pi\mu(1-v)}\left(\frac{x_1x_3}{\left|x\right|^3}, \frac{x_2x_3}{\left|x\right|^3}, \frac{3-4v}{\left|x\right|} + \frac{x_3^2}{\left|x\right|^3}\right)$$

Then we can calculate strain

$$\varepsilon = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$=\frac{P}{16\pi\mu(1-v)}\begin{bmatrix} \frac{\partial\left(\frac{x_1x_3}{|x|^3}\right)}{\partial x_1} & \frac{1}{2}\left[\frac{\partial\left(\frac{x_2x_3}{|x|^3}\right)}{\partial x_1} + \frac{\partial\left(\frac{x_1x_3}{|x|^3}\right)}{\partial x_2}\right] & \frac{1}{2}\left[\frac{\partial\left(\frac{3-4v}{|x|} + \frac{x_3^2}{|x|^3}\right)}{\partial x_1} + \frac{\partial\left(\frac{x_1x_3}{|x|^3}\right)}{\partial x_3}\right] \\ \frac{1}{2}\left[\frac{\partial\left(\frac{x_2x_3}{|x|^3}\right)}{\partial x_1} + \frac{\partial\left(\frac{x_1x_3}{|x|^3}\right)}{\partial x_2}\right] & \frac{\partial\left(\frac{x_2x_3}{|x|^3}\right)}{\partial x_2} & \frac{1}{2}\left[\frac{\partial\left(\frac{3-4v}{|x|}\frac{x_3^2}{|x|^3}\right)}{\partial x_2} + \frac{\partial\left(\frac{x_2x_3}{|x|^3}\right)}{\partial x_3}\right] \\ \frac{1}{2}\left[\frac{\partial\left(\frac{3-4v}{|x|} + \frac{x_3^2}{|x|^3}\right)}{\partial x_1} + \frac{\partial\left(\frac{x_1x_3}{|x|^3}\right)}{\partial x_3}\right] & \frac{1}{2}\left[\frac{\partial\left(\frac{3-4v}{|x|}\frac{x_3^2}{|x|^3}\right)}{\partial x_2} + \frac{\partial\left(\frac{x_2x_3}{|x|^3}\right)}{\partial x_3}\right] & \frac{\partial\left(\frac{3-4v}{|x|}\frac{x_3^2}{|x|^3}\right)}{\partial x_3} \end{bmatrix}$$

$$=\frac{P}{16\pi\mu(1-v)}\begin{bmatrix} \frac{x_3}{|x|^3} - \frac{3x_1^2x_3}{|x|^5} & -\frac{3x_1^2x_2x_3}{|x|^5} & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_1 - \cdots\right. \\ -\frac{3x_1^2x_2x_3}{|x|^5} & \frac{x_3}{|x|^3} - \frac{3x_2^2x_3}{|x|^5} & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_2 - \cdots\right. \\ \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_1 - \frac{3x_3^2}{|x|^5}x_1 + \frac{x_1}{|x|^3} - \frac{3x_1x_3^2}{|x|^5}\right) & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_2 - \frac{3x_3^2}{|x|^5}x_2 + \frac{x_2}{|x|^3} - \frac{3x_2x_3^2}{|x|^5}\right) & -\frac{3-4v}{|x|^3}x_3 \cdots \end{bmatrix}$$

According to the elasticity constitutive equation using μ, v,the ralation between stress and strain is

$$\sigma_{ij} = \frac{2\mu v}{1-2v}\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$\varepsilon_{kk} = \frac{P}{16\pi\mu(1-v)}\left(\frac{4x_3}{|x|^3} - \frac{3(x_1^2+x_2^2+x_3^2)x_3}{|x|^5} - \frac{3-4v}{|x|^3}x_3\right)$$

$$= \frac{P}{16\pi\mu(1-v)}\left(\frac{x_3}{|x|^3} - \frac{3-4v}{|x|^3}x_3\right)$$

$$= -\frac{P}{16\pi\mu(1-v)}\left(\frac{2-4v}{|x|^3}x_3\right)$$

$$\sigma_{ij} = \frac{2\mu v}{1-2v}\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$= \frac{2\mu v}{1-2v}\frac{P}{16\pi\mu(1-v)}\begin{bmatrix} -\frac{2-4v}{|x|^3}x_3 & 0 & 0 \\ 0 & -\frac{2-4v}{|x|^3}x_3 & 0 \\ 0 & 0 & -\frac{2-4v}{|x|^3}x_3 \end{bmatrix}$$

$$+2\mu\frac{P}{16\pi\mu(1-v)}\begin{bmatrix} \frac{x_3}{|x|^3} - \frac{3x_1^2x_3}{|x|^5} & -\frac{3x_1x_2x_3}{|x|^5} & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_1 \cdots \right. \\ -\frac{3x_1x_2x_3}{|x|^5} & \frac{x_3}{|x|^3} - \frac{3x_2^2x_3}{|x|^5} & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_2 \cdots \right. \\ \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_1 - \frac{3x_3^2}{|x|^5}x_1 + \frac{x_1}{|x|^3} - \frac{3x_1x_3^2}{|x|^5}\right) & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_2 - \frac{3x_3^2}{|x|^5}x_2 + \frac{x_2}{|x|^3} - \frac{3x_2x_3^2}{|x|^5}\right) & -\frac{3-4v}{|x|^3}\cdots \end{bmatrix}$$

$$= \frac{vP}{4\pi(1-v)}\begin{bmatrix} -\frac{x_3}{|x|^3} & 0 & 0 \\ 0 & -\frac{x_3}{|x|^3} & 0 \\ 0 & 0 & -\frac{x_3}{|x|^3} \end{bmatrix}$$

$$+\frac{P}{8\pi(1-v)}\begin{bmatrix} \frac{x_3}{|x|^3} - \frac{3x_1^2x_3}{|x|^5} & -\frac{3x_1x_2x_3}{|x|^5} & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_1 - \cdots \right. \\ -\frac{3x_1x_2x_3}{|x|^5} & \frac{x_3}{|x|^3} - \frac{3x_2^2x_3}{|x|^5} & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_2 - \cdots \right. \\ \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_1 - \frac{3x_3^2}{|x|^5}x_1 + \frac{x_1}{|x|^3} - \frac{3x_1x_3^2}{|x|^5}\right) & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_2 - \frac{3x_3^2}{|x|^5}x_2 + \frac{x_2}{|x|^3} - \frac{3x_2x_3^2}{|x|^5}\right) & -\frac{3-4v}{|x|^3}x_3 - \cdots \end{bmatrix}$$

$$f_i = \sigma_{ij}n_j\,(n_j = (x_1, x_2, x_3))$$

$$= \frac{vP}{4\pi(1-v)}\begin{bmatrix} -\frac{x_3}{|x|^3} & 0 & 0 \\ 0 & -\frac{x_3}{|x|^3} & 0 \\ 0 & 0 & -\frac{x_3}{|x|^3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$+\frac{P}{8\pi(1-v)}\begin{bmatrix} \frac{x_3}{|x|^3}-\frac{3x_1^2x_3}{|x|^5} & -\frac{3x_1x_2x_3}{|x|^5} & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_1-\frac{3x \cdots}{|x| \cdots}\right. \\ -\frac{3x_1x_2x_3}{|x|^5} & \frac{x_3}{|x|^3}-\frac{3x_2^2x_3}{|x|^5} & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_2-\frac{3x \cdots}{|x| \cdots}\right. \\ \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_1-\frac{3x_3^2}{|x|^5}x_1+\frac{x_1}{|x|^3}-\frac{3x_1x_3^2}{|x|^5}\right) & \frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_2-\frac{3x_3^2}{|x|^5}x_2+\frac{x_2}{|x|^3}-\frac{3x_2x_3^2}{|x|^5}\right) & -\frac{3-4v}{|x|^3}x_3- \cdots \end{bmatrix}$$

$$=\frac{vP}{4\pi(1-v)}\begin{bmatrix} -\frac{x_1x_3}{|x|^3} \\ -\frac{x_2x_3}{|x|^3} \\ -\frac{x_3^2}{|x|^3} \end{bmatrix}+\frac{P}{8\pi(1-v)}\begin{bmatrix} \frac{x_1x_3}{|x|^3}-\frac{3x_1^3x_3}{|x|^5}-\frac{3x_1x_2^2x_3}{|x|^5}+\frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_1x_3-\frac{3x_3^3}{|x|^5}x_1+\frac{x_1x_3}{|x|^3}-\frac{3x_1x_3^4}{|x|^5}\right) \\ \frac{x_2x_3}{|x|^3}-\frac{3x_2^3x_3}{|x|^5}-\frac{3x_1^2x_2x_3}{|x|^5}+\frac{1}{2}\left(-\frac{3-4v}{|x|^3}x_2x_3-\frac{3x_3^3}{|x|^5}x_2+\frac{x_2x_3}{|x|^3}-\frac{3x_2x_3^4}{|x|^5}\right) \\ \frac{1}{2}\left(-\frac{3-4v}{|x|}-\frac{3x_3^2}{|x|^3}+\frac{1}{|x|}-\frac{3x_3^2}{|x|^3}\right)-\frac{1}{2}\frac{3-4v}{|x|^3}x_3^2+\frac{3x_3^2}{2|x|^3} \end{bmatrix}$$

$$=\frac{vP}{4\pi(1-v)}\begin{bmatrix} -\frac{x_1x_3}{|x|^3} \\ -\frac{x_2x_3}{|x|^3} \\ -\frac{x_3^2}{|x|^3} \end{bmatrix}+\frac{P}{8\pi(1-v)}\begin{bmatrix} -\frac{(3-2v)x_1x_3}{|x|^3} \\ -\frac{(3-2v)x_2x_3}{|x|^3} \\ -\frac{1-2v}{|x|}-\frac{3-2v}{|x|^3}x_3^2 \end{bmatrix}$$

$$=\frac{P}{8\pi(1-v)}\begin{bmatrix} -\frac{2vx_1x_3}{|x|^3}-\frac{(3-2v)x_1x_3}{|x|^3} \\ -\frac{2vx_2x_3}{|x|^3}-\frac{(3-2v)x_2x_3}{|x|^3} \\ -\frac{2vx_3^2}{|x|^3}-\frac{1-2v}{|x|}-\frac{3-2v}{|x|^3}x_3^2 \end{bmatrix}$$

$$=\frac{P}{8\pi(1-v)}\begin{bmatrix} -\frac{3x_1x_3}{|x|^3} \\ -\frac{3x_2x_3}{|x|^3} \\ -\frac{1-2v}{|x|}-\frac{3x_3^2}{|x|^3} \end{bmatrix}$$

For the resultan surface force , they are

$f=\int_{\partial\Omega} f_i$

$$=\int_{\partial\Omega}\frac{P}{8\pi(1-v)}\begin{bmatrix} \frac{3x_1x_3}{|x|^3} \\ \frac{3x_2x_3}{|x|^3} \\ -\frac{1-2v}{|x|}+\frac{3x_3^2}{|x|^3} \end{bmatrix}_i$$

$$=\frac{P}{8\pi(1-v)}\begin{bmatrix} 0 \\ 0 \\ -(1-2v)\ [a=0]2\pi\int\ [\theta=0]\theta=\pi\int\sin\theta d\theta da-3\ [a=0]2\pi\int\ [\theta=0]\theta=\pi\int\cos^2\theta\sin\theta d\theta da \end{bmatrix}$$

$$= \frac{P}{8\pi(1-v)} \begin{bmatrix} 0 \\ 0 \\ -(1-2v)\, 2\pi \left[-\cos\theta\right]_0^\pi - 3 \cdot 2\pi \left[\frac{-\cos^3\theta}{3}\right]_0^\pi \end{bmatrix}$$

$$= \frac{P}{8\pi(1-v)} \begin{bmatrix} 0 \\ 0 \\ -(1-2v)\, 4\pi - 4\pi \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -P \end{bmatrix}$$

The first two items are zero because symmetry.

The result is correct in which the force around the sphere balance the concentrated force.

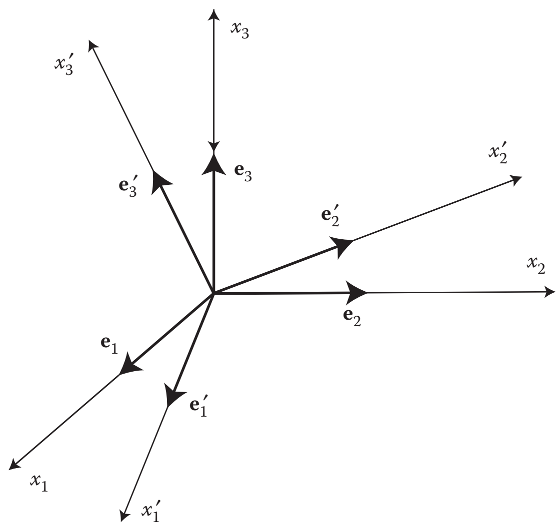

Figure 2.1 Three-dimensional Cartesian coordinate and coordinate rotation.

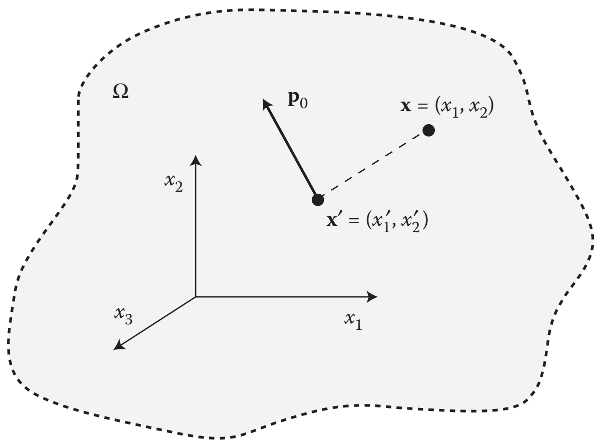

Figure 2.2 Green's function.

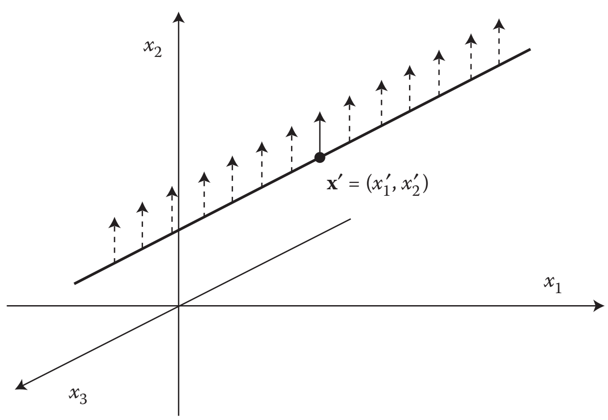

Figure 2.3 **Dimension reduction of Green's function.**

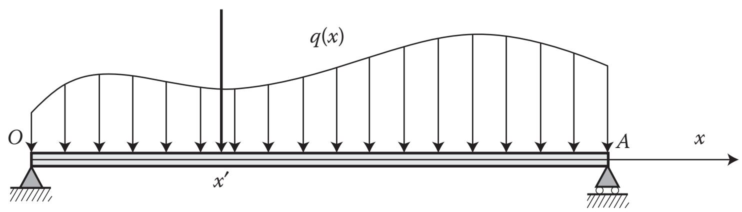

Figure 2.4 Straight beam under distribution loading.

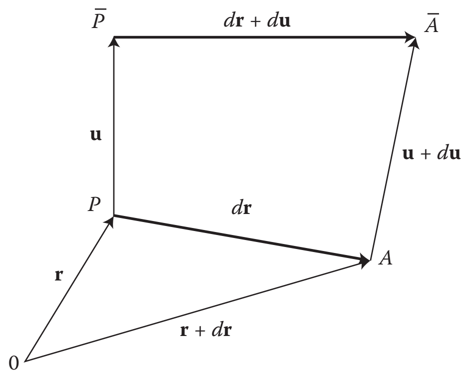

Figure 2.5 Deformation of an infinitesimal line element.

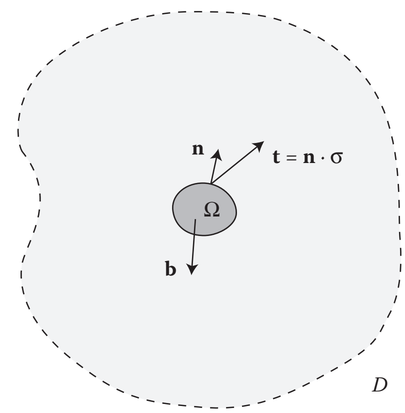

Figure 2.6 **The equilibrium of an arbitrary domain.**

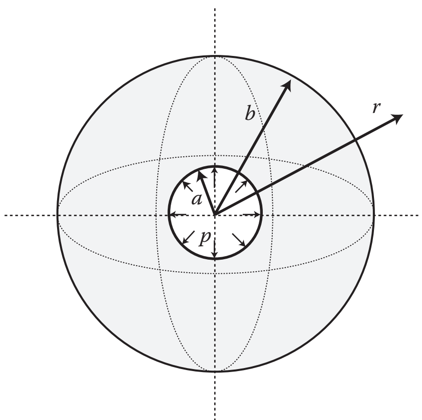

Figure 2.7 **A ball subjected to a uniform inner pressure** p**.**

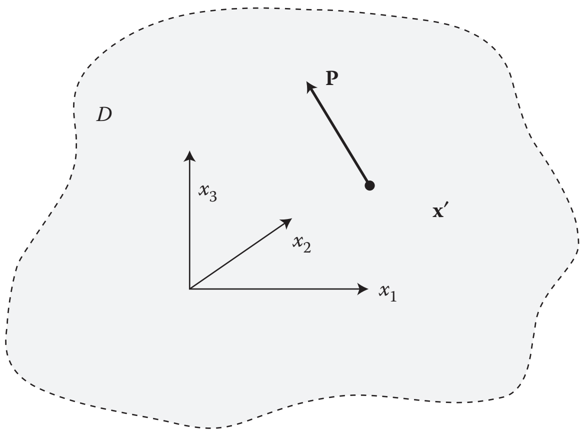

Figure 2.8 **Concentrated force acting on an infinite domain.**

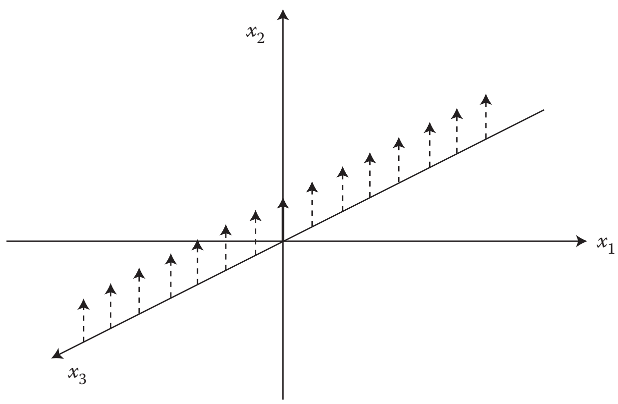

Figure 2.9 Dimension reduction of Kelvin's solution.