Engineering Mechanics Dynamics 1st edition Tongue Solutions Manual

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Answers to Selected Chapter Exercises

Chapter 2

2.1.1	$\Delta x_{max} = 0.311\,\text{mi},\ \Delta t_{max} = 14.5\,\text{s}$
2.1.3	$\overline{v} = 0.4\,\text{m/s},\ v_{max} = 1\,\text{m/s},\ v_{min} = 0.1\,\text{m/s}$
2.1.5	$v = 19.1\,\text{ft/s}$
2.1.7	$x = 270\,\text{ft}$
Do It 2.1.1	$\dot{x}(2) = -48\,\text{ft/s},\ \ddot{x}(10) = 20\,\text{ft/s}^2$
2.1.9	$a_0 = 17.6\,\text{ft/s}^2,\ a_0 = 17.6\,\text{ft/s}^2 \dfrac{(1\,g)}{(32.2\,\text{ft/s}^2)} = 0.55\,g$
Do It 2.1.2	$\Delta x_{max} = 1.72 \times 10^3\,\text{ft} = 0.326\,\text{mi},\ \Delta t = 0.26\,\text{s}$
2.1.11	$v_{max} = 1.29 \times 10^3\,\text{in/s},\ a_{max} = 9.48 \times 10^5\,\text{in/s}^2 = 2.45 \times 10^3 g$
2.1.13	21% increase in generated force
2.1.15	$v_{term} = 200\,\text{ft/s},\ t_{term} = 0.00367\,\text{s}$
2.1.17	$s(4\,\text{s}) - s(0) = 80\frac{2}{3}\,\text{m}$
Do It 2.1.3	$y_{max} = 114\,\text{m},\ y_{max} = 127\,\text{m},\ e = 12.3\%$
Do It 2.1.4	$0.854\,\text{s},\ 26.5\,\text{ft/s} = 18\,\text{mph}$
2.1.19	$t^* = 0.348\,\text{s},\ v_{coll} = 36.7\,\text{ft/s} = 25\,\text{mph}$
2.1.21	$\lvert \dot{s}_f \rvert = 7318\,\text{mph}$
2.1.23	$\dot{s}_{impact} = 52.8\ \text{ft/s}$
2.1.25	$0 < \dot{s} \leq 10,\ \Delta t = 1.842\,\text{s}$

2.2.1

$$\begin{array}{c|c|c|} & \vec{c}_1 & \vec{c}_2 \\ \hline \vec{b}_1 & -\cos\phi & -\sin\phi \\ \hline \vec{b}_2 & \sin\phi & -\cos\phi \\ \hline \end{array}$$

2.2.3

$$\begin{array}{c|c|c|} & \vec{b}_1 & \vec{b}_2 \\ \hline \vec{c}_1 & -\sin\phi & \cos\phi \\ \hline \vec{c}_2 & -\cos\phi & -\sin\phi \\ \hline \end{array}, \quad \begin{array}{c|c|c|} & \vec{\imath} & \vec{\jmath} \\ \hline \vec{c}_1 & \cos(\theta+\phi) & \sin(\theta+\phi) \\ \hline \vec{c}_2 & -\sin(\theta+\phi) & \cos(\theta+\phi) \\ \hline \end{array}$$

Do It 2.2.1	$\lVert\vec{r}(1)\rVert = 0.471\,\text{ft},\ \lVert\vec{v}(1)\rVert = 0.983\,\text{ft/s},\ \lVert\vec{a}(1)\rVert = 2.73\,\text{ft/s}^2$
Do It 2.2.2	$\vec{p} = -8.70\vec{b}_1 + 2.08\vec{b}_2$
2.2.5	$\vec{p} = -5.00\,\vec{b}_1 - 0.01\,\vec{b}_2$

Do It 2.2.3	$\theta_{\min} = 0.333\,\text{rad} = 19.06°,\ t = 2.116\,\text{s}$
2.2.7	$t = 1.791\,\text{s},\ x = 31.67\,\text{ft},\ y_{\max} = 24.85\,\text{ft}$
2.2.9	$v_0 = 8.21\,\text{ft/s},\ t = 0.485\,\text{s}$
2.2.11	$v_0 = 10.0\,\text{m/s},\ t = 1.37\,\text{s}$
2.2.13	59.9 degrees
2.2.15	$\eta = 0.7854\ \text{rad} = 45°$
2.2.17	$R = \dfrac{73.57\,\text{m}}{\cos\beta} = 84.95\ \text{m}$
2.2.19	55 mph
2.2.21	$\ddot{y} = \dfrac{-{v_0}^2 r^2}{\left\lvert (r^2 - x^2)^{3/2} \right\rvert},\ y^{(3)} = \dfrac{-3{v_0}^3 x r^2}{\left\lvert (r^2 - x^2)^{5/2} \right\rvert}$
2.2.23	$v_0 = \sqrt{\dfrac{\left(L + g\left(-\frac{3\pi}{\alpha}\right)^{\frac{2}{3}}\right)^2 + 4d^2}{4\left(-\frac{3\pi}{\alpha}\right)^{\frac{2}{3}}}}$
Do It 2.3.1	$\lVert\vec{v}_A\rVert = 8.72\,\text{m/s},\ \lVert\vec{a}_A\rVert = 6.35\,\text{m/s}^2$
2.3.1	$\vec{v}_{Bill} = (13.6\vec{e}_r + 6.80\vec{e}_\theta)\,\text{ft/s},$
	$\vec{a}_{Bill} = (0.845\vec{e}_r + 0.6225\vec{e}_\theta)\,\text{ft/s}^2$
2.3.3	$\vec{v}_A = r_A\dot{\theta}\vec{e}_\theta$
Do It 2.3.2	$\vec{v}_B = (0.4\vec{e}_r + 0.4\vec{e}_\theta)\,\text{m/s},$

$\vec{a}_B = (0.08\vec{e}_r + 0.36\vec{e}_\theta)\,\text{m/s}^2,\ \vec{v}_B = (0.146\vec{\imath} + 0.546\vec{\jmath})\,\text{m/s},$

$\vec{a}_B = (-0.111\vec{\imath} + 0.352\vec{\jmath})\,\text{m/s}^2$

2.3.5	$\dot{\theta} = 0.0\bar{5}\ \text{rad/s}$
2.3.7	$\dot{r} \neq \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$
2.3.9	No
2.3.11	$\dot{\theta} = 0.733\,\text{rad/s},\ \vec{a}_A = -0.762\vec{\jmath}\,\text{ft/s}^2$
2.3.13	$\theta = 60°,\ \ddot{\theta} = 0$
2.3.15	$\dot{r} = 4.830\,\text{ft/s},\ \dot{\theta} = 0.647\,\text{rad/s},\ \ddot{r} = 1.807\,\text{ft/s}^2,$
	$\ddot{\theta} = -2.996\,\text{rad/s}^2$

2.3.17	$r = 31.11\,\text{ft}$, $\dot{r} = -3.830\,\text{ft/s}$, $\dot{\theta} = 0.103\,\text{rad/s}$, $\ddot{r} = -0.472\,\text{ft/s}^2$, $\ddot{\theta} = -5.34 \times 10^{-3}\,\text{rad/s}^2$
Do It 2.3.3	$\|\vec{v}_A\| = 15.97\,\text{ft/s}$, $\|\vec{a}_A\| = 53.33\,\text{ft/s}^2$
2.3.19	$v_G = [0.173\,\text{m/s} + (2.2\,\text{m})(1.52\,\text{rad/s})] = 3.52\,\text{m/s}$
2.3.21	$\ddot{r} \neq 193.6\,\text{ft/s}^2$
2.3.23	$\vec{a} = \left(-\frac{\sqrt{2}}{2}\vec{\imath} + \frac{9\sqrt{2}}{2}\vec{\jmath}\right)\text{m/s}^2$, $\dot{\theta} = \pm 2\,\text{rad/s}$, $\dot{r} = \pm 1.44\,\text{m/s}$
Do It 2.3.4	$\vec{v}_A = (100\text{ ft/s})\vec{e}_r + (1000\text{ ft/s}^2)t\vec{e}_\theta$, $\vec{a}_A = -(10,000\text{ ft/s}^3)t\vec{e}_r + (2000\text{ ft/s}^2)\vec{e}_\theta$
2.3.25	Case 1: $\vec{a}_A = -822\vec{e}_r\,\text{m/s}^2$; case 2: $\ddot{\theta} = -2.74 \times 10^5\,\text{rad/s}^2$, $t = 1.9 \times 10^{-3}\,\text{s}$
2.3.27	$\|\vec{v}\| = 3.52\,\text{m/s}$, $\vec{a} = -(6.42\vec{e}_r + 28.25\vec{e}_\theta)\ \text{m/s}^2$
2.3.29	$t = \sqrt{\frac{\pi}{2a}}$, $\vec{r}_B = \frac{b\pi}{2a}\vec{\jmath}$, $\vec{v}_B = -\pi b\sqrt{\frac{\pi}{2a}}\vec{\imath} + 2b\sqrt{\frac{\pi}{2a}}\vec{\jmath}$, $\vec{a}_{B/A} = -5\pi b\vec{\imath} + b(2-\pi^2)\vec{\jmath}$
2.3.31	$\vec{a} = -ra^2\omega_1^2\sin^2(\omega_1 t)\vec{e}_r + ra\omega_1^2\cos(\omega_1 t)\vec{e}_\theta + b\omega_2^2\cos(\omega_2 t)\vec{k}$
2.3.33	$\vec{v} = (17.58\vec{e}_\theta - 0.84\vec{e}_z)\,\text{ft/s}$, direction of velocity $= 2.7°$; $\|\vec{a}\| = -7.73\,\text{ft/s}^2$, $\vec{a}$ is in radial direction
2.4.1	$a_n = 69.2\,\text{ft/s}^2$
2.4.3	$\|\vec{a}_A\| = 12.5\,\text{m/s}^2$, $r_B = 0.919\,\text{m}$
2.4.5	$v_{\max} = 16.43\,\text{ft/s}$
2.4.7	$a_t = 5\,\text{m/s}^2$
Do It 2.4.1	$v_A = 3.13\,\text{ft/s}$, $a_{B,t} = 8.94\,\text{ft/s}^2$
2.4.9	$\vec{a}_P(3\,\text{s}) = (-1.20\vec{e}_t + 2.24\vec{e}_n)\,\text{ft/s}^2$
Do It 2.4.2	Car A will make the turn in $t_A = 11.8\,\text{s}$. Car B will slide off the road.
2.4.11	$\vec{v}_{before} = \vec{v}_{after} = v_b\vec{\imath}$
2.4.13	$x = \frac{1}{12}\,\text{ft}$, $v = \sqrt{2}\,\text{ft/s}$, $a_t = 6\sqrt{2}\,\text{ft/s}^2$, $a_n = 6\sqrt{2}\,\text{ft/s}^2$
Do It 2.4.3	$a_{BC} = 7.34\,\text{ft/s}^2$, $t_{\text{total}} = 6.85\,\text{s}$
2.4.15	$r_c(4\,\text{m}) = 22.36\,\text{m}$, $\|\vec{a}\| = 4.90\,\text{m/s}^2$

2.4.17	$a_n = 17.32\,\mathrm{ft/s^2}$, $v_t = 18.61\,\mathrm{ft/s}$, $\Delta y = 5.38\,\mathrm{ft}$, $t_{\mathrm{total}} = 1.156\,\mathrm{s}$
2.4.19	$t = \dfrac{87.8\,\mathrm{ft}}{29.3\,\mathrm{ft/s}} = 3\,\mathrm{s}$
2.4.21	$r_C = 899\,\mathrm{ft}$
Do It 2.4.4	$\vec{a}_E = (5.657\vec{\imath} + 5.657\vec{\jmath})\,\mathrm{m/s^2}$, $\vec{a}_R = (3.823\vec{e}_t + 5.669\vec{e}_n)$, $r_C = 5115\,\mathrm{km}$
2.4.23	$\vec{a} = [3.06\vec{\imath} + 6.89\vec{\jmath}]\ \mathrm{m/s^2}$
2.4.25	$r_C = 629\,\mathrm{m}$
2.4.27	$v_B = 14.14\,\mathrm{m/s}$, $a_{BC} = 13.33\,\mathrm{m/s^2}$, $v_D = 17.32\,\mathrm{m/s}$, $t_{\mathrm{total}} = 16.30\,\mathrm{s}$
Do It 2.5.1	$\vec{v}_{A/B} = 1360\vec{\imath}\ \mathrm{ft/s}$, $\vec{a}_{A/B} = (90\vec{\imath} + 4356\vec{\jmath})\,\mathrm{ft/s^2}$
2.5.1	$s = 40\,\mathrm{mi}$, $\theta = 21.8^\circ$, $v_{p/w} = 215\,\mathrm{mph}$
2.5.3	$\lVert\vec{v}_C\rVert = 33.6\,\mathrm{ft/s}$, $\lVert\vec{a}_C\rVert = 1.63\times 10^3\,\mathrm{ft/s^2}$
2.5.5	$\vec{v}_B = (2\dot{x} - \dot{y})\vec{\jmath}$
2.5.7	$\vec{v}_B = 2\vec{\jmath}\ \mathrm{m/s}$
2.5.9	$a_A = -400\,\mathrm{ft/s^2}$, angle with respect to the floor is 90°
2.5.11	$\lVert\vec{v}_A\rVert = 25.2\,\mathrm{mph}$, $\lVert\vec{a}_A\rVert = 146.3\,\mathrm{ft/s^2}$
2.5.13	$\vec{v}_{A/B} = (-30\vec{\imath} - 21.96\vec{\jmath})\,\mathrm{mph}$, $\vec{a}_{A/B} = -19.36\ \mathrm{ft/s^2}\,\vec{\imath}$
2.5.15	$d = 8.4\,\mathrm{ft}$, $\vec{v}_{rd/A} = (-30\vec{\imath} - 25\vec{\jmath})\,\mathrm{mph}$
Do It 2.5.2	$x = 22.7\,\mathrm{m}$
2.5.17	$\vec{v}_{B/BodyC} = \left(-\vec{\imath} - \frac{1}{\sqrt{3}}\vec{\jmath}\right)\mathrm{m/s}$
Do It 2.5.3	$\vec{v}_B = \frac{3}{4}\vec{\jmath}\ \mathrm{m/s}$
2.5.19	$\vec{v}_D = 0.133\vec{\jmath}\ \mathrm{m/s}$
2.5.21	$\vec{v}_D = -\frac{10}{3}\vec{\jmath}\ \mathrm{in/s}$
2.5.23	$\vec{a}_{mass} = \frac{4}{3}\vec{\jmath}\ \mathrm{ft/s^2}$
2.5.25	$\vec{v}_B = 25\vec{\jmath}\ \mathrm{in./s}$

2.5.27 $\vec{v}_B = \dot{x}\vec{\jmath} = \frac{2}{3}v_A\vec{\jmath}$

Do It 2.5.4 $\dot{y}_3 = -\frac{2}{7}v_1 + \frac{9}{7}v_2,\ \dot{y}_4 = \frac{1}{7}v_1 + \frac{6}{7}v_2$

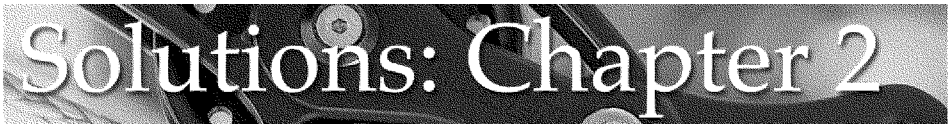
2.5.29 $$v_A = \frac{-v_R\sqrt{L^2 + x_A^2}}{x_A + 2\sqrt{L^2 + x_A^2}}$$

2.5.31 $v_A = -\frac{1}{2}\,\mathrm{ft/s}$

2.5.33 $v_a = 3.61\,\mathrm{m/s},\ s = 30\,\mathrm{m},\ \theta = 59.0°,\ \Delta t = 4.15\,\mathrm{s}$

2.5.35 $\vec{v}_A = \frac{1}{2}x\dot{x}(x^2 + h^2)^{-0.5}\vec{\jmath},$

$\vec{a}_A = \frac{1}{2}\left[\dot{x}^2h^2(x^2 + h^2)^{-1.5} + \ddot{x}x(x^2 + h^2)^{-0.5}\right]\vec{\jmath}$



DYNAMICS: Analysis and Design of Systems in Motion

Benson H. Tongue/

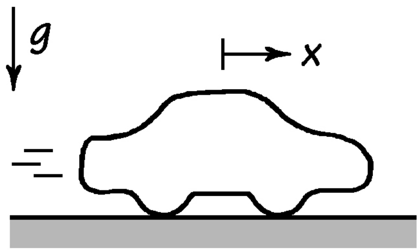
2.1 Straight-Line Motion

2.1.1

GOAL: Determine the distance and time needed for the car to reach its maximum speed.

GIVEN: The 2007 BMW Z4 Coupe 3.0si can accelerate from 0 to 60 mph in 5.6 s, and its maximum speed is $\dot{x}_{max} = 155$ mph. Assume that it accelerates from 0 to 60 mph at a constant rate and that this acceleration is maintained as the vehicle pushes toward its maximum speed.

DRAW:



FORMULATE EQUATIONS:

Since we're assuming that the car's acceleration is constant, we can say that

$$\ddot{x} = \frac{\Delta \dot{x}}{\Delta t} \tag{1}$$

$$\dot{x}^2 - \dot{x}_0^2 = 2\ddot{x}\,\Delta x \tag{2}$$

SOLVE:

To go from 0 to 60 mph in 5.6 s at a constant rate, the car's acceleration needs to be

$$(1) \Rightarrow \qquad \ddot{x} = \frac{(60\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)}{5.6\,\text{s}}$$

$$\ddot{x} = 15.7\,\text{ft/s}^2$$

If the car continues to accelerate at this rate, then it will reach its maximum speed after traveling

$$(2) \Rightarrow \qquad \Delta x_{max} = \frac{(\dot{x}_{max})^2}{2\ddot{x}}$$

$$\Delta x_{max} = \frac{\left[(155\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)\right]^2}{2(15.7\,\text{ft/s}^2)}$$

$$\boxed{\Delta x_{max} = 1.64 \times 10^3\,\text{ft} = 0.311\,\text{mi}}$$

The time it takes for the car to attain maximum speed is

(1) $\Rightarrow$

$$\Delta t_{\text{max}} = \frac{\dot{x}_{\text{max}}}{\ddot{x}}$$

$$\Delta t_{\text{max}} = \frac{(155\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)}{15.7\,\text{ft/s}^2}$$

$$\boxed{\Delta t_{\text{max}} = 14.5\,\text{s}}$$

2.1.2
GOAL: Plot the path of someone following a map's directions.
DRAW:

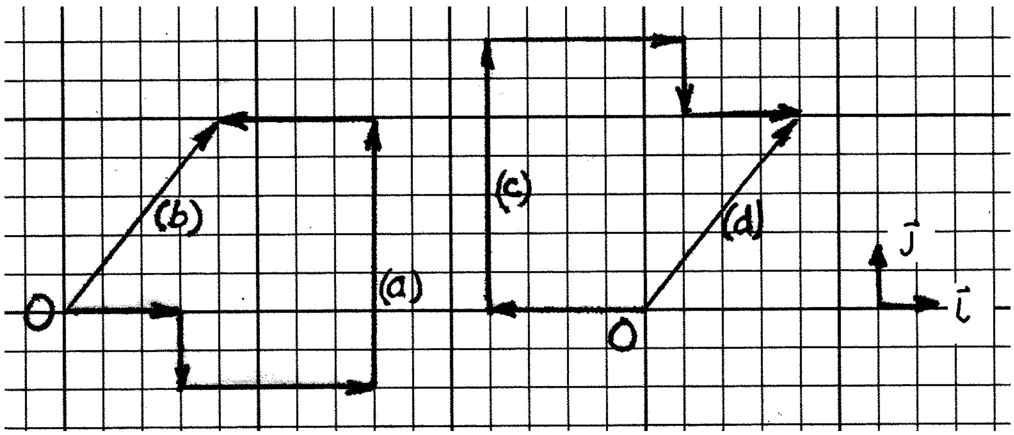
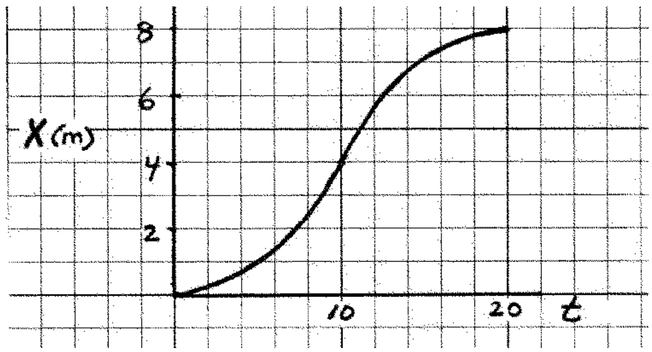


Figure 1: Paths from map

SOLVE: Figure 1 shows the two cases. (a) shows the map's original directions and (b) shows the overall change in position. (c) shows what we get by reversing the directions and (d) shows the starting position to finishing position vector. As can be seen from the figure, the overall start to finish vector is the same in each case. This illustrates how the order in which vector components are added won't affect the final answer.

2.1.3
GOAL:
Using the given information, find average speed and approximate instantaneous speed.
GIVEN: Graph of position vs time.
DRAW:

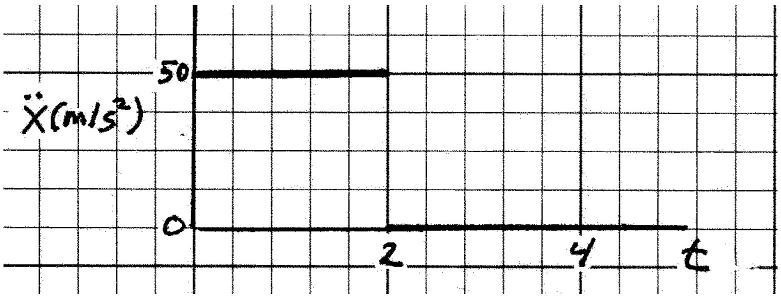


FORMULATE EQUATIONS: We'll use the graph and divide the distance traveled by the time taken in order to estimate the average speed and estimate the slope at a point in order to estimate the instantaneous speed.
SOLVE:
You can see from the graph of displacement versus time that the particle travels 8 meters in 20 seconds. Thus, $\overline{v} = \frac{8}{20}$ m/s $=\boxed{0.4 \text{ m/s}}$.

The maximum slope occurs at $t = 10$ seconds. Estimating the slope with a straight-edge provides an estimate for the maximum speed of $\boxed{1 \text{ m/s}}$.

The minimum slope occurs at $t = 0$ and $t = 20$ seconds. Estimating this slope provides you with an estimate for the minumum speed of $\boxed{0.1 \text{ m/s}}$.

2.1.4
GOAL: Find the appropriate initial speed so that the speed at $t = 2.5$ seconds is equal to zero.
GIVEN: Plot of acceleration vs. time.
DRAW:

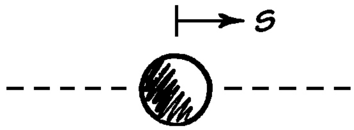


FORMULATE EQUATIONS: We'll need to integrate the acceleration to find speed and then adjust our initial condition so that at $t = 2.5\,\mathrm{s}$ it's equal to zero.
SOLVE:
Because the acceleration after $t = 2$ is equal to 0, the velocity does not change over the period $2 \leq t \leq 2.5$, so we can solve the problem on the period $0 \leq t \leq 2$. For this time interval, the acceleration is $50\mathrm{m/s^2}$. The velocity is therefore given by

$$v(t) = a\,t + v_0 \tag{1}$$

Using $v(2.5\,\mathrm{s}) = 0$, $a = 50$, and $t = 2$ in (1) and solving for v_0 shows that $\boxed{v_0\text{=-100 m/s}}$.

2.1.5
GOAL: Determine the ball bearing's speed v after traveling the given distance.
GIVEN: The ball bearing's acceleration is described by $\ddot{s} = as + be^{cs}$, where $a = 3\,\mathrm{s}^{-2}$, $b = 1\,\mathrm{ft/s}^2$, and $c = 0.2\,\mathrm{ft}^{-1}$. The ball bearing travels $d = 10\,\mathrm{ft}$, and it starts from rest.
DRAW:



FORMULATE EQUATIONS:
Since the ball bearing's acceleration is a function of its position,

$$\ddot{s} = \frac{d\dot{s}}{dt} = \frac{d\dot{s}}{ds}\cdot\frac{ds}{dt} = \frac{d\dot{s}}{ds}\dot{s}$$

$$\ddot{s}ds = \dot{s}d\dot{s} \tag{1}$$

SOLVE:

(1) $\Rightarrow$

$$(as + be^{cs})\,ds = \dot{s}d\dot{s}$$

$$\int_0^d (as + be^{cs})\,ds = \int_0^v \dot{s}d\dot{s}$$

$$\left(\frac{1}{2}as^2 + \frac{b}{c}e^{cs}\right)\Big|_0^d = \frac{1}{2}\dot{s}^2\Big|_0^v$$

$$\frac{1}{2}ad^2 + \frac{b}{c}\left(e^{cd} - 1\right) = \frac{1}{2}v^2$$

$$v = \sqrt{ad^2 + \frac{2b}{c}\left(e^{cd} - 1\right)}$$

$$v = \sqrt{(3\,\mathrm{s}^{-2})(10\,\mathrm{ft})^2 + \frac{2(1\,\mathrm{ft/s}^2)}{0.2\,\mathrm{ft}^{-1}}\left(e^{(0.2\,\mathrm{ft}^{-1})(10\,\mathrm{ft})} - 1\right)}$$

$$\boxed{v = 19.1\,\mathrm{ft/s} = 13.0\,\mathrm{mph}}$$

2.1.6

GOAL: Find the constant acceleration a_0 that brings a jet from 170 mph to 0 mph in 240 ft and the elapsed time Δt.

GIVEN: Distance needed to go from landing speed to zero.

FORMULATE EQUATIONS: Because the acceleration is constant we can use

$$v_2^2 - v_1^2 = 2a_0(x_2 - x_1) \tag{1}$$

$$v_2 - v_1 = a_0 \Delta t \tag{2}$$

where a_0 is the constant acceleration, Δt denotes the elapsed time, and the subscripts 1,2 indicate initial and final conditions, respectively.

SOLVE: First we'll convert 170 mph to ft/s:

$$\frac{(170\,\mathrm{mi})}{(1\,\mathrm{hr})}\frac{(5280\,\mathrm{ft})}{(1\,\mathrm{mi})}\frac{(1\,\mathrm{hr})}{(3600\,\mathrm{s})} = 249.3\,\mathrm{ft/s}$$

Using this, along with the known distance traveled, in (1) gives us

$$0 - (249.3\,\mathrm{ft/s})^2 = 2a_0(240\,\mathrm{ft})$$

$$\boxed{a_0 = -129.5\,\mathrm{ft/s^2}}$$

Because 1 g is equal to 32.2 ft/s^2 we have

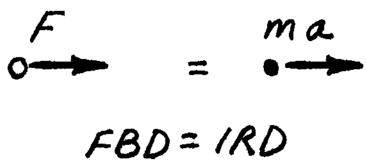
$$\boxed{a_0 = -129.5\,\mathrm{ft/s^2}\frac{(1\,g)}{(32.2\,\mathrm{ft/s^2})} = -4.02\,g}$$

We can then use (2) to find the time Δt taken for the jet to come to a halt:

$$0 - (249.3\,\mathrm{ft/s}) = (-129.5\,\mathrm{ft/s^2})\Delta t$$

$$\boxed{\Delta t = 1.93\,\mathrm{s}}$$

2.1.7
GOAL: Find the needed deck space to allow a fighter jet to take off from an aircraft carrier.
GIVEN: The jet is brought from 0 to 165 mph in 2.23 s.
DRAW:



ASSUME: We'll assume that the acceleration is constant.
FORMULATE EQUATIONS: We'll use the equation relating speed and acceleration, when starting from rest under a constant acceleration:

$$v = at$$

and the expression for distance traveled under a constant acceleration:

$$x = \frac{1}{2}at^2$$

SOLVE:
First we'll convert 165 mph to ft/s:

$$\frac{(165\,\text{mi})}{(1\,\text{hr})}\frac{(5280\,\text{ft})}{(1\,\text{mi})}\frac{(1\,\text{hr})}{(3600\,\text{s})} = 242\,\text{ft/s}$$

The acceleration is therefore

$$a = \frac{v}{t} = \frac{242\,\text{ft/s}}{2.23\,\text{s}} = 109\,\text{ft/s}^2$$

The needed space is given by

$$x = \frac{1}{2}(109\,\text{ft/s}^2)(2.23\,\text{s})^2$$

$$\boxed{x = 270\,\text{ft}}$$

2.1.8

GOAL: Construct plots of speed and acceleration for the given position function.

GIVEN: Differentiate to find speed and then differentiate again to find acceleration.

FORMULATE EQUATIONS:

Given: $$x(t) = \left(\frac{1}{3}\,\mathrm{m/s^3}\right)t^3 - (16\,\mathrm{m/s})t + 10\,\mathrm{m} \tag{1}$$

Differentiating (1): $$\dot{x} = (1\,\mathrm{m/s^3})t^2 - 16\,\mathrm{m/s} \tag{2}$$

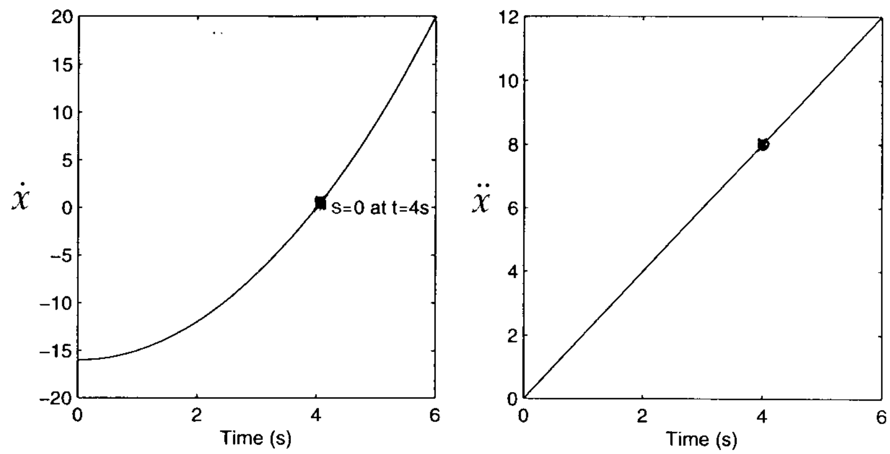
Differentiating (2): $$\ddot{x} = (2\,\mathrm{m/s^3})t \tag{3}$$

SOLVE:

Solving (2) for $\dot{x} = 0$ gives $t^2 = 16\,\mathrm{s}^2$ so $t = \pm 4\,\mathrm{s}$. Since $t = -4\,\mathrm{s}$ occurs before the beginning of the time interval, the correct answer is

$$\boxed{t = 4\,\mathrm{s}}.$$

The associated plots of speed and acceleration are given below, where $\dot{x}$ has units of m/s and $\ddot{x}$ has units of $\mathrm{m/s^2}$.



Do It 2.1.1
GOAL: Find the speed, $\dot{x}$, and the acceleration, $\ddot{x}$, at prescribed times. **GIVEN:** x as a function of time.
GOVERNING EQUATIONS: The position of the car is given by:

$$x(t) = 100\,\mathrm{ft} - (88\,\mathrm{ft/s})t + (10\,\mathrm{ft/s^2})t^2 \tag{1}$$

Differentiating with respect to time, we get:

$$\dot{x}(t) = -88\,\mathrm{ft/s} + (20\,\mathrm{ft/s^2})t \tag{2}$$

and

$$\ddot{x}(t) = 20\,\mathrm{ft/s^2} \tag{3}$$

SOLVE:

(2) $\Rightarrow$

$$\boxed{\dot{x}(2) = -88\,\mathrm{ft/s} + (20\,\mathrm{ft/s^2})(2\,\mathrm{s}) = -48\,\mathrm{ft/s}} \tag{4}$$

and

(3) $\Rightarrow$

$$\boxed{\ddot{x}(10) = 20\,\mathrm{ft/s^2}} \tag{5}$$

2.1.9
GOAL: Find the constant acceleration a_0 that brings a car from 0 to 60 mph in 5 seconds.
GIVEN: Time needed to go from zero to 60 mph.
FORMULATE EQUATIONS: Because the acceleration is constant we have

$$v(t) = v(0) + a_0 t \tag{1}$$

where a_0 is a constant acceleration.
SOLVE: First we'll convert 60 mph to ft/s:

$$\frac{(60\,\mathrm{mi})}{(1\,\mathrm{hr})}\frac{(5280\,\mathrm{ft})}{(1\,\mathrm{mi})}\frac{(1\,\mathrm{hr})}{(3600\,\mathrm{s})} = 88\,\mathrm{ft/s}$$

Using this in (1) gives us

$$88\,\mathrm{ft/s} = 0 + a_0(5\,\mathrm{s})$$

$$\boxed{a_0 = 17.6\,\mathrm{ft/s^2}}$$

Because 1 g is equal to 32.2 ft/s^2 we have

$$\boxed{a_0 = 17.6\,\mathrm{ft/s^2}\frac{(1\,g)}{(32.2\,\mathrm{ft/s^2})} = 0.55\,g}$$

2.1.10
GOAL: Determine the time to bring a car to a stop from an initial speed along with the distance over which stopping occurs.
GIVEN: Initial speed of car and the fact that the car decelerates at 1 g
FORMULATE EQUATIONS:
For a constant acceleration,

$$v(t) = v(0) + at$$

$$x(t) = x(0) + v(0)t + \frac{at^2}{2}$$

SOLVE:

$$70 \text{ mph } = \frac{70}{60}(88) = 102.\overline{6} \text{ ft/s}$$

$$v(t^*) = 102.\overline{6}\,\text{ft/s} - (32.2\,\text{ft/s}^2)t^* = 0 \Rightarrow \boxed{t^* = 3.19 \text{ s}}$$

And for the distance:

$$x(t^*) = 0 + (102.\overline{6}\,\text{ft/s})t^* - \frac{32.2\,\text{ft/s}^2}{2}t^{*2}$$

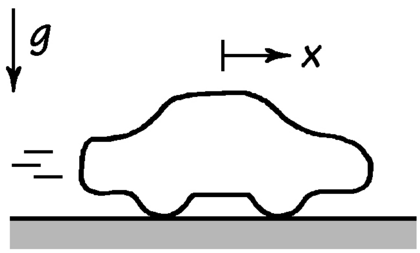
$$\boxed{\text{At } t^* = 3.19\,\text{s}, \; x = 164 \text{ ft}}$$

Do It 2.1.2

GOAL: Determine the distance the car needs to reach its maximum speed and the change in its 0-to-60 time such that it and a competitor's car reach their respective maximum speed in the same distance.

GIVEN: The 2008 Audi TT Coupe can accelerate from 0 to 60 mph in 6.5 s and has a maximum speed of $\dot{x}_{\max} = 147\,\text{mph}$. Assume that it accelerates from 0 to 60 mph at a constant rate and that this acceleration is maintained as the vehicle pushes toward its maximum speed.

DRAW:



FORMULATE EQUATIONS:

Since we're assuming that the car's acceleration is constant, we can say that

$$\ddot{x} = \frac{\Delta \dot{x}}{\Delta t} \tag{1}$$

$$\dot{x}^2 - \dot{x}_0^2 = 2\ddot{x}\,\Delta x \tag{2}$$

SOLVE:

To go from 0 to 60 mph in 6.5 s at a constant rate, the car's acceleration needs to be

(1) $\Rightarrow$
$$\ddot{x} = \frac{(60\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)}{6.5\,\text{s}}$$

$$\ddot{x} = 13.5\,\text{ft/s}^2$$

If the car continues to accelerate at this rate, then it will reach its maximum speed after traveling

(2) $\Rightarrow$
$$\Delta x_{\max} = \frac{(\dot{x}_{\max})^2}{2\ddot{x}}$$

$$\Delta x_{\max} = \frac{\left[(147\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)\right]^2}{2(13.5\,\text{ft/s}^2)}$$

$$\boxed{\Delta x_{\max} = 1.72 \times 10^3\,\text{ft} = 0.326\,\text{mi}}$$

We find that it takes a longer distance for the TT to reach its maximum speed as compared to its competitor. For the car to attain maximum speed in 0.312 mi like its competitor, its 0-to-60 time would need to be

$$(1) \rightarrow (2) \Rightarrow \qquad (\dot{x}_{\text{max}})^2 = \frac{2(\Delta\dot{x})(\Delta x)}{t^*}$$

$$t^* = \frac{2(\Delta\dot{x})(\Delta x)}{(\dot{x}_{\text{max}})^2}$$

$$t^* = \frac{2(60\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)(0.312\,\text{mi})\left(\frac{5280\,\text{ft}}{\text{mi}}\right)}{\left[(147\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)\right]^2}$$

$$t^* = 6.24\,\text{s}$$

Thus, the car's 0-to-60 time would need to decrease by

$$\Delta t = 6.5\,\text{s} - 6.24\,\text{s}$$

$$\boxed{\Delta t = 0.26\,\text{s}}$$

2.1.11
GOAL: Determine a piston's maximum speed and acceleration.
GIVEN: Position of piston as a function of time.
FORMULATE EQUATIONS: The governing equation of motion is given as

$$x(t) = \frac{3.53}{2}\sin\omega t$$

where x is given in inches and $\omega = 7000\,\mathrm{rpm} = 733\,\mathrm{rad/s}$.
SOLVE:

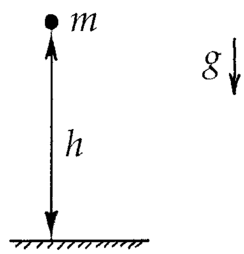
$$v(t) = \frac{d}{dt}x(t) = \left(\frac{3.53\,\mathrm{in}}{2}\right)\omega\cos\omega t$$

$$\boxed{v_{max} = \left(\tfrac{3.53\,\mathrm{in}}{2}\right)(733\,\mathrm{rad/s}) = 1.29\times10^3\,\mathrm{in/s}}$$

$$a(t) = \frac{d}{dt}v(t) = -\left(\frac{3.53\,\mathrm{in}}{2}\right)\omega^2\sin\omega t$$

$$\boxed{a_{max} = \left(\tfrac{3.53\,\mathrm{in}}{2}\right)(733\,\mathrm{rad/s})^2 = 9.48\times10^5\,\mathrm{in/s^2} = 2.45\times10^3 g}$$

2.1.12
GOAL: Find the height h for which a falling body will contact the ground at 35 mph.
GIVEN: Speed of contact.
DRAW:



ASSUME:

$$\begin{aligned} v_i &= 0 \\ v_f &= 35\,\text{mph} = 51.\overline{3}\,\text{ft/s} \\ a &= 32.2\,\text{ft/s}^2 \end{aligned} \tag{1}$$

FORMULATE EQUATIONS: We'll use the formula for the difference in speed due to a constant acceleration a over a distance h:

$$v_f^2 - v_i^2 = 2ah \tag{2}$$

SOLVE:

$$\begin{aligned} v_f^2 - v_i^2 &= 2gh \\ h &= \frac{v_f^2 - v_i^2}{2g} = \frac{(51.3\,\text{ft/s})^2 - 0}{2(32.2\,\text{ft/s}^2)} \\ h &= \boxed{40.9\text{ ft}} \end{aligned}$$

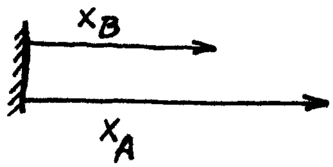
(1)→(2)⇒

2.1.13
GOAL: Find the percentage increase in drive force that will increase a car's maximum speed by 10%.
GIVEN: Air resistance $F_a = Av^2$ where A is a constant
ASSUME: When the car is going as fast as the engine can drive it, $F_d = F_a$
SOLVE:
From $F_{drive} = Av^2$ we see that if v increases by 10% we have

$$F = A(1.1v)^2 = 1.21Av^2 = 1.21F_{drive}$$

Thus we must increase the force generated by the engine by $\boxed{21\%}$ to support a 10% change in speed. This tells me that it is very difficult to increase a car's top speed because the force increases as the square of the speed.

2.1.14
GOAL: Find the constant speed needed for a pursuing cyclist to catch another cyclist.
GIVEN: Initial positions of the cyclists and the lead cyclist's speed.
DRAW:



FORMULATE EQUATIONS: We'll need to use the formula for position as a function of time due to a constant speed v:

$$x(t) = x(0) + vt$$

SOLVE: Bicyclist A has to travel one mile, or 5280 ft. 18 mph corresponds to 26.4 ft/s. Thus we have

$$5280\,\text{ft} = (26.4\,\text{ft/s})t \Rightarrow t = 200\,\text{s}$$

Bicyclist B has to travel an additional 750 ft, for a total of 6030 ft and has 200 s to do so. Thus we have

$$6030\,\text{ft} = v_B(200\,\text{s})$$

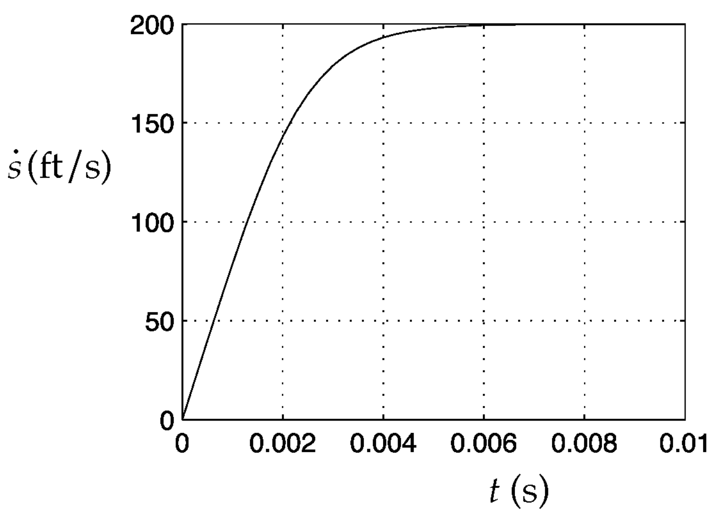
$$\boxed{v_B = 30.2\,\text{ft/s} = 20.6\,\text{mph}}$$

2.1.15
GOAL: Find the terminal speed of an object with a given acceleration and determine at what time it reaches 95 percent of terminal speed.
GIVEN: $a_0 = 80,000\,\mathrm{ft/s^2}$, $a_1 = 0.01\,\mathrm{s/\,ft^2}$
FORMULATE EQUATIONS:
The acceleration is given by

$$\ddot{s} = a_0 - a_1\dot{s}^3$$

SOLVE:
We can numerically integrate using MATLAB with initial conditions of $s(0) = 0$, $\dot{s}(0) = 0$. Using a time interval from $t = 0$ to $t = 0.01\,\mathrm{s}$ yields the following plot:
It can be seen from the plot that the terminal speed is

$$\boxed{v_{term} = 200\,\mathrm{ft/s}}$$



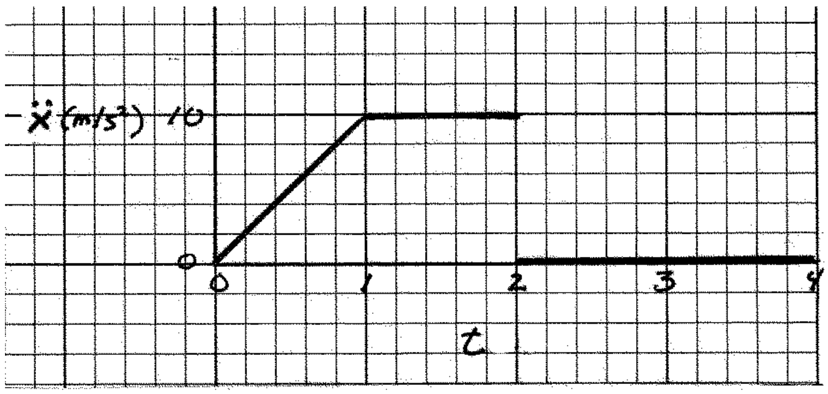
This result can be seen analytically as well. When v_{term} is reached, the acceleration $\ddot{s}$ is zero. Using this in our acceleration equation gives

$$0 = a_0 - a_1 v_{term}^3$$

which, when solved, returns the result $v_{term} = 200\,\mathrm{ft/s}$.
An examination of the output data allows the time at which the speed reaches 95 percent of its terminal value ($0.95(200\,\mathrm{ft/s}) = 190\,\mathrm{ft/s}$) to be determined as

$$\boxed{t_{term} = 0.00367\,\mathrm{s}}$$

2.1.16
GOAL: Find the position of a particle at $t = 3$ seconds given a plot of the particle's acceleration.
GIVEN: Plot of acceleration vs. time.
DRAW:



FORMULATE EQUATIONS: We'll need to integrate the acceleration to find speed and then integrate the speed to find position.
SOLVE:
The plot of acceleration versus time can be divided into three distinct regions on the time interval $0 \le t \le 3$. For the region where $0 \le t \le 1$, the acceleration is given by

$$\ddot{s} = 10t$$

Integrating for velocity gives

$$\dot{s}(t) = 5t^2 + v_0 \tag{1}$$

Integrating again for position provides

$$s(t) = \frac{5}{3}t^3 + v_0 + s_0 \tag{2}$$

Using the initial condition $v_0 = 0$ in (1) and evaluating at $t = 1$ gives $\dot{s}(1) = 5$. Using the initial conditions $s_0 = 1$ and $v_0 = 0$ in (2) yields $s(1) = \frac{8}{3}$.

For $1 < t \le 2$ the acceleration is constant;

$$\ddot{s} = 10$$

Re-initialize the time counter to begin at $t = 1$, treating this segment as if it's simply another initial condition problem for which the initial conditions start at $t = 1$.
Integrating for velocity gives

$$\dot{s}(t) = 10t + v_1 \tag{3}$$

Integrating again for position gives

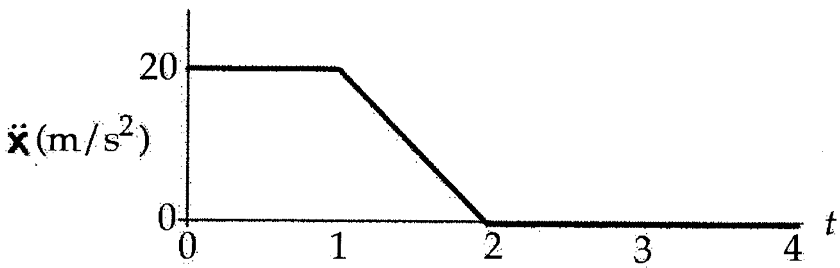
$$s(t) = 5t^2 + v_1 t + s_1 \tag{4}$$

Applying the conditions found over the first time interval into these equations, we find $\dot{s}(2) = 15$ and that $s(2) = \frac{38}{3}$.

For the final interval, $2 < t \leq 3$, the acceleration equals 0, so $\dot{s}(3) = v_2$ and $s(3) = \dot{s}(2)t + s_2$. Solving these equations shows that

$$\boxed{s(3)=27\tfrac{2}{3} \text{ m}}$$

2.1.17
GOAL: Determine $s(4) - s(0)$ given $\ddot{s}(t)$.
GIVEN: A plot of acceleration vs. time.
DRAW:



FORMULATE EQUATIONS: We'll have to integrate the acceleration to find speed and then integrate the speed to find position.
ASSUME: We have three phases of acceleration - constant from 0 to 1 s, linearly decreasing from 1 to 2 s and constant (and zero) beyond 2 s.
SOLVE: For $0 \leq t < 1\,\mathrm{s}$:

$$\dot{s}(t) = \dot{s}(0) + \int_0^t 20\,\mathrm{m/s^2}\,d\eta = -4\,\mathrm{m/s} + (20\,\mathrm{m/s^2})t$$

$$s(t) = s(0) - (4\,\mathrm{m/s})t + (10\,\mathrm{m/s^2})t^2$$

Thus $s(1\,\mathrm{s}) = s(0) + 6\,\mathrm{m},\ \dot{s}(1\,\mathrm{s}) = 16\,\mathrm{m/s}$
For $1\,\mathrm{s} \leq t < 2\,\mathrm{s}$:

$$\dot{s}(t) = \dot{s}(1\,\mathrm{s}) + \int_{1\mathrm{s}}^t (20\,\mathrm{m/s2})(2\,\mathrm{s} - \eta)d\eta = -14\,\mathrm{m/s} + (40\,\mathrm{m/s2})t - (10\,\mathrm{m/s3})t2$$

$$\begin{aligned} s(t) &= s(1\,\mathrm{s}) + \left(-(14\,\mathrm{m/s})\eta + (20\,\mathrm{m/s^2})\eta^2 - \left(\frac{10}{3}\,\mathrm{m/s^3}\right)t^3\right)\Big|_{1\mathrm{s}}^t \\ &= s(0) + \frac{10}{3}\,\mathrm{m} - (14\,\mathrm{m/s})t + (20\,\mathrm{m/s^2})t^2 - \left(\frac{10}{3}\,\mathrm{m/s^3}\right)t^3 \end{aligned}$$

Evaluating at $t = 2\,\mathrm{s}$ gives $s(2\,\mathrm{s}) = s(0) + 28\frac{2}{3}\,\mathrm{m}$ and $\dot{s}(2) = 26\,\mathrm{m/s}$.
Finally, for $t = 2\,\mathrm{s}$ and beyond we have zero acceleration. Thus the speed is constant at $\dot{s}(2\,\mathrm{s})$ and the displacement increases linearly.
For $2\,\mathrm{s} \leq t$:

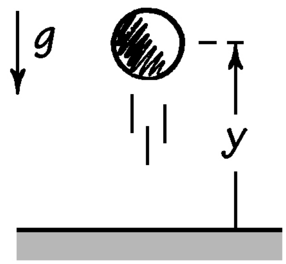
$$\dot{s}(t) = 26\,\mathrm{m/s}$$

$$s(t) = s(2\,\mathrm{s}) + \dot{s}(2\,\mathrm{s})(t - 2\,\mathrm{s}) = s(0) + 28\frac{2}{3}\,\mathrm{m} + (26\,\mathrm{m/s})(t - 2\,\mathrm{s})$$

$$s(4\,\mathrm{s}) = s(0) + 80\frac{2}{3}\,\mathrm{m}$$

$$\boxed{s(4\,\mathrm{s}) - s(0) = 80\tfrac{2}{3}\,\mathrm{m}}$$

Do It 2.1.3
GOAL: Determine the maximum height the ball reaches, and compare this to the maximum height achieved when air drag is neglected.
GIVEN: The ball is launched vertically with an initial speed of $\dot{y}_0 = 50\,\mathrm{m/s}$, and the air drag coefficient c_D is given by $c_D = 0.001\,\mathrm{m}^{-1}$.
DRAW:



FORMULATE EQUATIONS:
The ball's acceleration is governed by

$$\ddot{y} = -g - c_D \dot{y}^2 \tag{1}$$

The ball's acceleration is given in terms of its velocity $\dot{y}$, and so

$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{d\dot{y}}{dy} \cdot \frac{dy}{dt} = \frac{d\dot{y}}{dy}\dot{y}$$

$$dy = \frac{\dot{y}}{\ddot{y}} d\dot{y} \tag{2}$$

SOLVE:
We can simply integrate (1) to find the maximum height reached when air drag is considered:

(1) $\rightarrow$ (2), integrate $\Rightarrow$

$$\int_0^{y_{\max}} dy = \int_{\dot{y}_0}^{0} \frac{-\dot{y}}{g + c_D \dot{y}^2} d\dot{y}$$

$$y\Big|_0^{y_{\max}} = -\frac{1}{2c_D} \ln(g + c_D \dot{y}^2)\Big|_{\dot{y}_0}^{0}$$

$$y_{\max} = -\frac{1}{2c_D} \ln \left. \frac{g}{g + c_D \dot{y}_0^2} \right)$$

$$y_{\max} = -\frac{1}{2(0.001\,\mathrm{m}^{-1})} \ln \left. \frac{9.81\,\mathrm{m/s}^2}{9.81\,\mathrm{m/s}^2 + (0.001\,\mathrm{m}^{-1})(50\,\mathrm{m/s})^2} \right)$$

$$\boxed{y_{\max} = 114\,\mathrm{m}}$$

When air drag is neglected, the ball's acceleration is given by $\ddot{y} = -g$, and hence the maximum height achieved is

$$(2) \Rightarrow \qquad -g dy = \dot{y} d\dot{y}$$

$$-g \int_0^{y_{\max}} dy = \int_{\dot{y}_0}^{0} \dot{y} d\dot{y}$$

$$-gy\Big|_0^{y_{\max}} = \frac{1}{2}\dot{y}^2\Big|_{\dot{y}_0}^{0}$$

$$-gy_{\max} = -\frac{1}{2}\dot{y}_0^2$$

$$y_{\max} = \frac{\dot{y}_0^2}{2g}$$

$$y_{\max} = \frac{(50\,\mathrm{m/s})^2}{2(9.81\,\mathrm{m/s^2})}$$

$$\boxed{y_{\max} = 127\,\mathrm{m}}$$

The percent error e associated with neglecting air drag is

$$e = \frac{|114\,\mathrm{m} - 127\,\mathrm{m}|}{114\,\mathrm{m}} \times 100\%$$

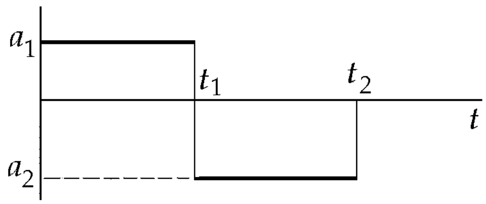
$$\boxed{e = 12.3\%}$$

2.1.18

GOAL: Find t_1 (time to switch from acceleration to braking), t_2 (time to stop braking), and maximum speed of a car traveling a set distance.

GIVEN: Constant acceleration (7.5 m/s^2) and braking (-10 m/s^2) and total distance traveled (105 *m*).

DRAW:



FORMULATE EQUATIONS: We'll use the displacement/velocity expressions for constant acceleration a_0, going from an initial time t_A to a final time t_B:

$$v(t_B) = v(t_A) + a_0[t_B - t_A] \tag{1}$$

$$x(t_B) = x(t_A) + v(t_A)[t_B - t_A] + \frac{1}{2}a_0[t_B - t_A]^2 \tag{2}$$

ASSUME: The speed will be greatest at time t_1, just before braking commences and the speed will be zero when the car has traveled 105 m.

SOLVE: Starting from zero, we reach a maximum speed at t_1:

$$v(t_1) = a_1 t_1$$

At this point its position is, from (2)

$$x(t_1) = \frac{1}{2}a_1 t_1^2$$

We can't evaluate yet because we don't know t_1.

Moving on, we can find the position and speed at t_2, given the initial position $x(t_1)$ and speed $v(t_1)$. Knowing that the speed must be zero at t_2 gives us

$$\begin{aligned} v(t_2) = 0 &= v(t_1) + a_2(t_2 - t_1) \\ &= a_1 t_1 + a_2(t_2 - t_1) \end{aligned}$$

$$a_1 t_1 = -a_2(t_2 - t_1)$$

$$(a_1 - a_2)t_1 = -a_2 t_2$$

$$t_2 = \frac{a_2 - a_1}{a_2} t_1 = b t_1 \tag{3}$$

where $b = \frac{a_2 - a_1}{a_2} = \frac{-10 - 7.5}{-10} = 1.75$.

We know that the car travels a distance $L = 105\,\text{m}$ and so we have

$$
\begin{aligned}
x(t_2) &= \frac{1}{2}a_1 t_1^2 + a_1 t_1 [t_2 - t_1] + \frac{1}{2}a_2 (t_2 - t_1)^2 \\
&= \frac{1}{2}a_1 t_1^2 + a_1 t_1^2 (b-1) + \frac{1}{2}a_2 (b-1)^2 t_1^2 \\
&= \left(\frac{1}{2}a_1 + (b-1)a_1 + (b-1)^2 \frac{a_2}{2}\right) t_1^2 = L
\end{aligned}
$$

Using the given values gives us

$$
\left(\frac{7.5\,\mathrm{m/s^2}}{2} + 0.75(7.5\,\mathrm{m/s^2}) - (0.75)^2(5\,\mathrm{m/s^2})\right) t_1^2 = 105\,\mathrm{m}
$$

$$
\boxed{t_1 = 4\,\mathrm{s}}
$$

We can now find the maximum speed:

$$
\boxed{v(t_1) = a_1 t_1 = (7.5\,\mathrm{m/s^2})(4\,\mathrm{s}) = 30\,\mathrm{m/s}}
$$

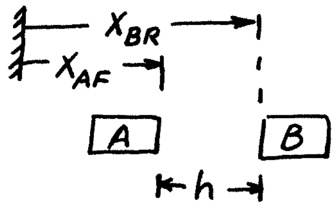
(3) $\Rightarrow$ $\boxed{t_2 = 1.75(4) = 7\,\mathrm{s}}$

Do It 2.1.4

GOAL: Find the time of collision between Car A to hit Car B and their relative speed at collision.

GIVEN: At $t = 0$ s Car A is traveling at a constant speed of 98 ft/s and Car B is 22.5 ft in front of Car A, traveling at 78 ft/s and decelerating at 20 ft/s^2. At $t = 0.5$ s Car A decelerates at a constant 30 ft/s^2. The separation of the two cars is given by h.

DRAW:



FORMULATE EQUATIONS: We'll use the general expressions for position and speed, given a constant acceleration and initial position and speed x_0, v_0:

$$x(t) = x_0 + v_0 t + \frac{at^2}{2}$$

$$v(t) = v_0 + at$$

SOLVE:

At $t = 0$ s let the position of Car A be $x_{AF} = 0$ ft and the position of Car B be $x_{BR} = 22.5$ ft. After 0.5 s have elapsed the positions and speeds are found from

$$x_{AF}(0.5\,\mathrm{s}) = (98\,\mathrm{ft/s})(0.5\,\mathrm{s}) = 49\,\mathrm{ft}$$

$$x_{BR}(0.5\,\mathrm{s}) = 22.5\,\mathrm{ft} + (78\,\mathrm{ft/s})(0.5\,\mathrm{s}) - 0.5(20\,\mathrm{ft/s^2})(0.5\,\mathrm{s})^2 = 59\,\mathrm{ft}$$

$$v_{AF}(0.5\,\mathrm{s}) = 98\,\mathrm{ft/s}$$

$$v_{BR}(0.5\,\mathrm{s}) = 78\,\mathrm{ft/s} - (20\,\mathrm{ft/s^2})(0.5\,\mathrm{s}) = 68\,\mathrm{ft/s}$$

At this point Car A begins to decelerate at $-30\,\mathrm{ft/s^2}$. For convenience we'll reset time to zero (t indicates time beyond the 0.5 s needed for Car A's braking to begin.)

$$x_{AF}(t) = 49\,\mathrm{ft} + (98\,\mathrm{ft/s})t + 0.5(-30\,\mathrm{ft/s^2})t^2$$

$$x_{BR}(t) = 59\,\mathrm{ft} + (68\,\mathrm{ft/s})t + 0.5(-20\,\mathrm{ft/s^2})t^2$$

The separation h is given by

$$h = x_{BR} - x_{AF} = 10\,\mathrm{ft} - (30\,\mathrm{ft/s})t + (5\,\mathrm{ft/s^2})t^2$$

and the collision occurs when $h = 0$. We're left with a quadratic to solve:

$$10\,\mathrm{ft} - (30\,\mathrm{ft/s})t + (5\,\mathrm{ft/s^2})t^2 = 0$$

$$t^2 - (6\,\mathrm{s})t + 2\,\mathrm{s}^2 = 0$$

The relevant solution to this equation is $t = 0.354\,\mathrm{s}$. Thus the elapsed time from when the driver of Car A first decides to brake is given by $0.5\,\mathrm{s} + 0.354\,\mathrm{s} = \boxed{0.854\,\mathrm{s}}$.
The collision speed is found from

$$-\dot{h} = (-68\,\mathrm{ft/s} + 98\,\mathrm{ft/s}) + (-30\,\mathrm{ft/s^2} + 20\,\mathrm{ft/s^2})(0.354\,\mathrm{s}) = 26.5\,\mathrm{ft/s}$$

Thus we have a collision speed of

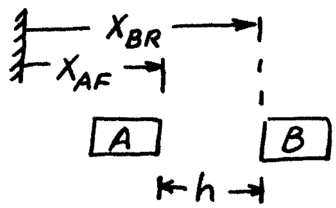
$$\boxed{26.5\,\mathrm{ft/s} = 18\,\mathrm{mph}}$$

2.1.19

GOAL: Determine whether Car A hits Car B and, if so, determine the relative speed of the collision.

GIVEN: Initially both cars are traveling at 98 ft/s. Car A is initially 30 ft behind Car B. Car B begins to decelerate at 35 ft/s^2 and 1 s later Car A begins to decelerate at 30 ft/s^2.

DRAW:



FORMULATE EQUATIONS:

We'll use the formulas for motion with constant acceleration a (initial position and speed given by x_0, v_0):

$$v(t) = v_0 + at$$

$$x(t) = x_0 + v_0 t + \frac{at^2}{2}$$

SOLVE:

x_{AF} indicates the front position of Car A and x_{BR} indicates the position of the rear of Car B. h indicates the separation between the two cars. Initially $x_{AF} = 0$ and $x_{BR} = 30$ ft

We'll break the analysis into two phases. First we'll determine the position and speed of the vehicles at $t = 1$ s. Next, we'll "reset" our time axis, defining $t = 0$ as the point at which vehicle A begins to decelerate. We'll then find the time to contact and the relative speed at contact.

At $t = 0$ we have

$$\begin{array}{ll} x_{AF} = 0, & x_{BR} = 30\,\text{ft} \\ v_A = 98,\ \text{ft/s} & v_B = 98\,\text{ft/s} \\ a_A = 0, & a_B = -35\,\text{ft/s}^2 \end{array}$$

At $t = 1$ s we have

$$\begin{array}{ll} x_{AF} = (98\,\text{ft/s})(1\,\text{s}) = 98\,\text{ft}, & x_{BR} = 30\,\text{ft} + (98\,\text{ft/s})(1\,\text{s}) - \dfrac{35\,\text{ft/s}^2(1\,\text{s})^2}{2} = 110.5\,\text{ft} \\ v_A = 98,\ \text{ft/s} & v_B = 98\,\text{ft/s} - (35\,\text{ft/s}^2)(1\,\text{s}) = 63\,\text{ft/s} \end{array}$$

Note that by the time the driver of Car A notices the problem ahead of him and initiates his braking (just 1 s delay), his distance from the car ahead of him has been reduced from 30 ft to $(110.5 - 98)$ ft $= 12.5$ ft.

We'll now "reset" our system, starting t from zero again but with the new values of position and speed, as well as having both cars decelerate (Car A at 30 ft/s^2 and Car B continuing at 35 ft/s^2).

$$x_{AF}(t) = 98\,\text{ft} + (98\,\text{ft/s})t - \frac{(30\,\text{ft/s}^2)t^2}{2}, \quad x_{BR}(t) = 110.5\,\text{ft} + (63\,\text{ft/s})t - \frac{(35\,\text{ft/s}^2)t^2}{2} \tag{1}$$

$$v_A(t) = 98\,\text{ft/s} - (30\,\text{ft/s}^2)t \qquad v_B(t) = 63\,\text{ft/s} - (35\,\text{ft/s}^2)t \tag{2}$$

Solving $x_{BR} - x_{AF} = 0$ will allow us to see whether a collision (separation of the two cars goes to zero) occurs.

$$(1) \Rightarrow \qquad 12.5\,\mathrm{ft} - (35\,\mathrm{ft/s})t^* - \frac{(5\,\mathrm{ft/s^2})t^{*2}}{2} = 0$$

This has only one positive solution: $\boxed{t^* = 0.348\,\mathrm{s}}$. Thus we do have a collision.
Using this value of time and (2) gives us a collision speed of

$$v_{coll} = v_A - v_B = 35\,\mathrm{ft/s} + (5\,\mathrm{ft/s^2})(0.348\,\mathrm{s}) = 36.7\,\mathrm{ft/s}$$

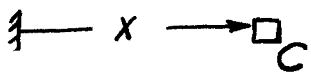
$$\boxed{v_{coll} = 36.7\,\mathrm{ft/s} = 25\,\mathrm{mph}}$$

Note that these figures are not unreasonable. People often travel far too close behind the car ahead of them and this shows that even a very brief lack of attention can easily lead to a collision.

2.1.20

GOAL: Determine the performance difference between manual braking and using Brake Assist.

GIVEN: In each case the car decelerates from 60 mph. Without Brake Assist the car decelerates at $20\,\mathrm{ft/s^2}$ for 1 s and then at $30\,\mathrm{ft/s^2}$ for the remaining time. With Brake Assist active the car decelerates at $30\,\mathrm{ft/s^2}$ the entire time. Let x represent the position of the car C.



FORMULATE EQUATIONS:

We'll use the formulas for motion with constant acceleration a (initial position and speed given by x_0, v_0):

$$v(t) = v_0 + at$$

$$x(t) = x_0 + v_0 t + \frac{at^2}{2}$$

SOLVE:

Case 1: Without Brake Assist. For this case we'll examine the car's response in two phases - from $t = 0\,\mathrm{s}$ to $t = 1\,\mathrm{s}$ and then for $t > 1\,\mathrm{s}$. We'll "reset" our time axis for the second phase, remembering that 1 s has already elapsed.

At $t = 0$ we have

$$\begin{aligned} x &= \ 0\,\mathrm{ft} \\ v &= \ 60\,\mathrm{mph} = 88\,\mathrm{ft/s} \\ a &= \ -20\,\mathrm{ft/s^2} \end{aligned}$$

At $t = 1\,\mathrm{s}$ we have

$$\begin{aligned} x &= (88\,\mathrm{ft/s})(1\,\mathrm{s}) - 0.5(20\,\mathrm{ft/s^2})(1\,\mathrm{s})^2 = 78\,\mathrm{ft} \\ v &= 88\,\mathrm{ft/s} - (20\,\mathrm{ft/s^2})(1\,\mathrm{s}) = 68\,\mathrm{ft/s} \end{aligned}$$

We'll now "reset" our system, starting t from zero again but with the new values of position and speed as initial conditions.

$$x(t) = 78\,\mathrm{ft} + (68\,\mathrm{ft/s})t - \frac{(30\,\mathrm{ft/s^2})t^2}{2} \tag{1}$$

$$v(t) = 68\,\mathrm{ft/s} - (30\,\mathrm{ft/s^2})t \tag{2}$$

Solving $v(t^*) = 0$ will tell us when the car's speed goes to zero:

(2) $\Rightarrow$ $$(30\,\mathrm{ft/s^2})t^* = 68\,\mathrm{ft/s} \Rightarrow t^* = 2.27\,\mathrm{s}$$

(1) $\Rightarrow$ $$x(2.27\,\mathrm{s}) = 78\,\mathrm{ft} + (68\,\mathrm{ft/s})(2.27\,\mathrm{s}) - \frac{(30\,\mathrm{ft/s^2})(2.27\,\mathrm{s})^2}{2} = 155\,\mathrm{ft}$$

Case 2: With Brake Assist we have

$$x(t) = (88\,\mathrm{ft/s})t - \frac{(30\,\mathrm{ft/s^2})t^2}{2} \tag{3}$$
$$v(t) = 88\,\mathrm{ft/s} - (30\,\mathrm{ft/s^2})t \tag{4}$$

Solving for $v(t^*) = 0$ gives us

(4) $\Rightarrow$ $$(30\,\mathrm{ft/s^2})t^* = 88\,\mathrm{ft/s} \Rightarrow t^* = 2.93\,\mathrm{s}$$

(3) $\Rightarrow$ $$x(2.93\,\mathrm{s}) = (88\,\mathrm{ft/s})(2.93\,\mathrm{s}) - \frac{(30\,\mathrm{ft/s^2})(2.93\,\mathrm{s})^2}{2} = 129\,\mathrm{ft}$$

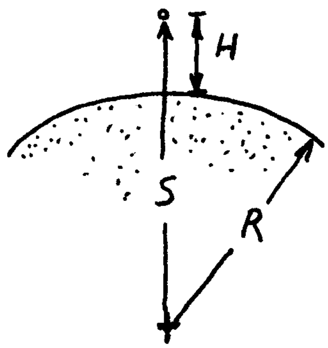
What we see from comparing these two results is that without Brake Assist the car traveled $\boxed{26\,\mathrm{ft}}$ farther, an increase of $\boxed{20\%}$.

2.1.21

GOAL: Determine the impact speed.

GIVEN: The battlestation falls from rest at a height $H = 400\,\text{mi}$. The planet has radius $R = 4000\,\text{mi}$ and a gravitational acceleration of $g_0 = 30\,\text{ft/s}^2$.

DRAW:



FORMULATE EQUATIONS:

The battlestation falls according to

$$\ddot{s} = -\frac{g_0 R^2}{s^2} \tag{1}$$

The acceleration is given in terms of the position s, so

$$\ddot{s} = \frac{d\dot{s}}{dt} = \frac{d\dot{s}}{ds}\cdot\frac{ds}{dt} = \frac{d\dot{s}}{ds}\dot{s}$$

$$\dot{s}d\dot{s} = \ddot{s}ds \tag{2}$$

SOLVE:

(1) → (2), integrate ⇒

$$\int_0^{\dot{s}_f} \dot{s}d\dot{s} = -\int_{H+R}^{R} \frac{g_0 R^2}{s^2} ds$$

$$\frac{1}{2}\dot{s}^2\Big|_0^{\dot{s}_f} = -g_0 R^2 \int_{H+R}^{R} \frac{ds}{s^2}$$

$$\frac{1}{2}(\dot{s}_f)^2 = \frac{g_0 R^2}{s}\Big|_{H+R}^{R}$$

$$\dot{s}_f = \sqrt{2 g_0 R^2 \left[\frac{1}{R} - \frac{1}{H+R}\right]}$$

$$\dot{s}_f = \sqrt{2(30\,\text{ft/s}^2)\left(\frac{\text{mi}}{5280\,\text{ft}}\right)\left(\frac{3600\,\text{s}}{\text{hr}}\right)^2 (4000\,\text{mi})^2 \left[\frac{1}{4000\,\text{mi}} - \frac{1}{400\,\text{mi} + 4000\,\text{mi}}\right]}$$

$$\boxed{\left|\dot{s}_f\right| = 7318\,\text{mph}}$$

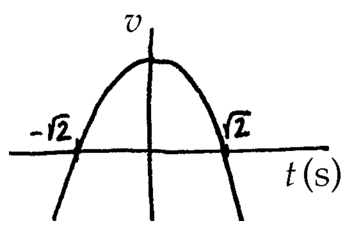
2.1.22
GOAL: Explain the significance of a negative time solution.
GIVEN: Position of a particle as a function of time.
FORMULATE EQUATIONS:

$$x(t) = 20\,\mathrm{m} + (180\,\mathrm{m/s})t - (30\,\mathrm{m/s^3})t^3$$

SOLVE: First we need to solve for the times at which the speed is zero:

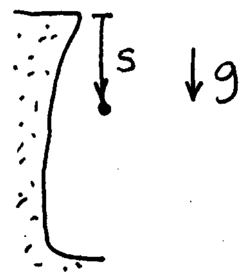
$$v(t) = \frac{d}{dt}x(t) = (180\,\mathrm{m/s}) - (90\,\mathrm{m/s^3})t^2 = 0$$

$$t^2 = 2\,\mathrm{s}^2 \Rightarrow t = \pm\sqrt{2}\,\mathrm{s}$$



The plot shows how speed varies with time. If the process could be started at some time before the first zero crossing then the particle would encounter two instants at which its speed was zero. If, however, a real system is being modeled for which the given $t = 0$ really corresponds to the physical time at which the process begins, then the first solution is unattainable, occurring as it does before the system is set into motion.

2.1.23
GOAL: Find a rock's impact velocity when dropped from 100 feet.
GIVEN: Downward acceleration is $g - c_d \dot{s}^2$
DRAW:



ASSUME: The rock experiences accelerations due only to that of gravity and drag.
FORMULATE EQUATIONS:
Measuring from the clifftop we have

$$s(0) = 0 \text{ and } \dot{s}(0) = 0$$

Because of the air resistance our ball's acceleration is

$$\ddot{s}(t) = 32.2\,\mathrm{ft/s^2} - (0.01\,\mathrm{ft^{-1}})\dot{s}^2$$

Using $\ddot{s}ds = \dot{s}d\dot{s}$ or

$$ds = \frac{\dot{s}d\dot{s}}{32.2\,\mathrm{m/s^2} - (0.01\,\mathrm{ft^{-1}})\dot{s}^2}$$

SOLVE:
Integrating gives

$$\int_0^{100\,\mathrm{m}} ds = \int_0^{\dot{s}_{impact}} \frac{\dot{s}d\dot{s}}{32.2\,\mathrm{ft/s^2} - (0.01\,\mathrm{ft^{-1}})\dot{s}^2}$$

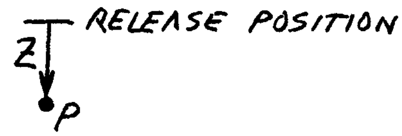
$$100\,\mathrm{m} = -\frac{1\,\mathrm{m}}{0.02}\ln(32.2\,\mathrm{ft/s^2} - (0.01\,\mathrm{m^{-1}})\dot{s}^2)\Big|_0^{\dot{s}_{impact}}$$

$$-2 = \ln\left[\frac{32.2\,\mathrm{ft/s^2} - (0.01\,\mathrm{m^{-1}})\dot{s}^2_{impact}}{32.2\,\mathrm{ft/s^2}}\right]$$

$$\boxed{\dot{s}_{impact} = 52.8\ \mathrm{ft/s}}$$

2.1.24

GOAL: Find the position of the probe at a prescribed value of the speed.
GIVEN: Acceleration as a function of the speed for a rectilinear motion.
DRAW:



FORMULATE EQUATIONS: For simplicity we'll introduce z as the coordinate which is zero at the release point and points down from there.
To get an equation for the probe's position z after release, use the general relationship

$$\dot{z}\,d\dot{z} = \ddot{z}\,dz \tag{1}$$

The particle acceleration is given by:

$$\ddot{z} = g - c\,\dot{z}^2 \tag{2}$$

where $g = 7.5\ \mathrm{m/s^2}$ and $c = 1.2\times 10^{-4}\ \mathrm{m^{-1}}$. Thus there is a gravitational acceleration pulling the probe down and a deceleration that depends upon the probe's speed $\dot{z}$.
SOLVE: The terminal speed, $\dot{z}_{term}$ is found when the probe's acceleration is equal to zero:

(2) $\Rightarrow$ $$\dot{z}_{term} = 250\,\mathrm{m/s}$$

Thus we see that, given enough time, the probe will move down at a speed of 245 m/s.
We want to find the distance travelled when $\dot{z} = 0.98\dot{z}_{term} = 0.98(250\,\mathrm{m/s}) = 245\,\mathrm{m/s}$.
To find z, we integrate (1):

$$\dot{z}\,d\dot{z} = (g - c\,\dot{z}^2)dz$$

$$\frac{\dot{z}\,d\dot{z}}{g - c\,\dot{z}^2} = dz$$

$$\frac{-2c\dot{z}\,d\dot{z}}{g - c\,\dot{z}^2} = -2cdz$$

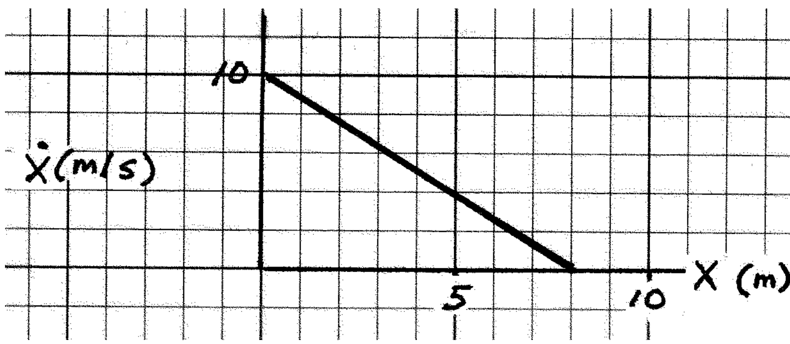
$$\frac{d(g - c\,\dot{z}^2)}{g - c\,\dot{z}^2} = -2cdz$$

$$\int_{\dot{z}_1=0}^{\dot{z}_2} \frac{d(g - c\,\dot{z}^2)}{g - c\,\dot{z}^2} = -2c\int_{z_1=0}^{z_2} dz$$

$$\ln\left(\frac{g - c\dot{z}_2^2}{g}\right) = -2cz_2 \Rightarrow \ln\left(\frac{7.5\,\mathrm{m/s^2} - (1.2\times 10^{-4}\,\mathrm{m^{-1}})(245\,\mathrm{m/s})^2}{7.5\,\mathrm{m/s^2}}\right) = -2(1.2\times 10^{-4}\,\mathrm{m^{-1}})z_2$$

$$\boxed{z_2 = 1.35\times 10^4\,\mathrm{m} = 13.5\ \mathrm{km}}$$

2.1.25
GOAL: Investigate how fast a particle reaches $s = 8$ when its velocity/displacement follows the graph given in the homework statement.
GIVEN: A plot of speed vs. position.
DRAW:



FORMULATE EQUATIONS: For this problem we don't have speed as a function of time but rather as a function of position. Thus our governing equation is a bit different than usual.
From the graph, we see that

$$\frac{ds}{dt} = -\frac{5}{4}s + 10$$

Separating variables yields

$$\frac{ds}{-\frac{5}{4}s + 10} = dt$$

SOLVE: Integrating gives us

$$\int_0^s \frac{ds}{-\frac{5}{4}s + 10} = \int_0^t dt$$

which provides the equation

$$-\frac{4}{5}\left(\ln\left(10 - \frac{5}{4}s\right)\right)\Bigg|_0^s = t\Bigg|_0^t$$

Analyzing over the bounds gives

$$-\frac{4}{5}\left(\ln\left(10 - \frac{5}{4}s\right)\right) + \frac{4}{5}(\ln 10) = t$$

Thus

$$t = -\frac{4}{5}\ln\left(\frac{10 - \frac{5}{4}s}{10}\right) \tag{1}$$

- To find the values of velocity that the particle will obtain, plug $s = 8$ into equation (1). The resulting time is $t = \infty$. Thus the particle will have velocities $\boxed{0 < \dot{s} \leq 10}$.

- Using 7.99 in (1) gives $t = 5.348$
  Using 7.999 into (1) gives $t = 7.190$
  Using 7.9999 into (1) gives $t = 9.032$
  Thus one can calculate (by subtraction) that $\boxed{\Delta t = 1.842\,\text{s}}$.

- To see where this constant jump is coming from, introduce a coordinate ϵ that runs to the left from $s = 8$:

$$s = 8 - \epsilon$$

Substituting this into our expression for t gives

$$\begin{aligned}
t &= -\frac{4}{5}\ln\left(\frac{10-\frac{5}{4}(8-\epsilon)}{10}\right) \\
&= -\frac{4}{5}\ln\left(\frac{\frac{5}{4}\epsilon}{10}\right) \\
&= -\frac{4}{5}\ln\left(\frac{5\epsilon}{40}\right) \\
&= -\frac{4}{5}\left(\ln\left(\frac{5}{40}\right)+\ln\left(\epsilon\right)\right) \\
&= -\frac{4}{5}\ln\left(\frac{5}{40}\right)-\frac{4}{5}\ln(\epsilon)
\end{aligned}$$

In our problem ϵ was 0.01, 0.001, etc. So let's let ϵ be equal to powers of 0.1:

$$\epsilon = 0.1^n \qquad n = 1, 2, 3, \cdots$$

Then

$$\begin{aligned}
t &= -\frac{4}{5}\ln\left(\frac{5}{40}\right)-\frac{4}{5}\ln(0.1^n) \\
&= -\frac{4}{5}\ln\left(\frac{5}{40}\right)-\frac{4n}{5}\ln(0.1)
\end{aligned}$$

The only thing changing is the second term as we move closer and closer to $s = 8$. And the change is equal to -$\frac{4}{5}\ln(.1)$, which, when evaluated gives us a time change of $\boxed{1.842}$, just as we saw numerically.

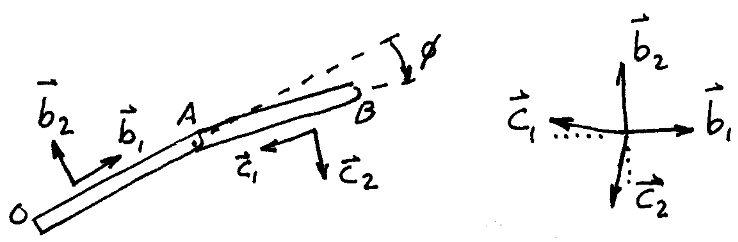
2.2 Cartesian Coordinates

2.2.1

GOAL: Determine the coordinate transformation between $\vec{c}_1,\ \vec{c}_2$ and $\vec{b}_1,\ \vec{b}_2$.

GIVEN: Structural configuration of the connected links.

DRAW:



SOLVE: In the illustration ϕ was made "small" so that the relationship between the unit vectors is more immediately apparent. Clearly $\vec{b}_1$ and $\vec{c}_1$ point in essentially opposite directions, as do $\vec{b}_2$ and $\vec{c}_2$. Hence the entry in the coordinate transformation array will be $-\cos\phi$. More specifically, we see that to get to the tip of $\vec{c}_1$ we need to go opposite from $\vec{b}_1$ and then a bit in the $\vec{b}_2$ direction. Thus we have

$$\vec{c}_1 = -\cos\phi\vec{b}_1 + \sin\phi\vec{b}_2$$

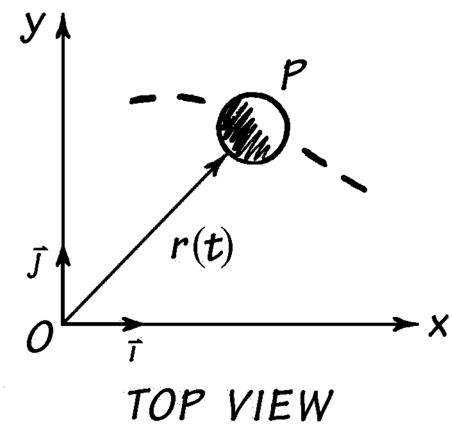
Similarly, to get to the tip of $\vec{c}_2$ means going opposite to $\vec{b}_2$ and then a little bit in the $-\vec{b}_1$ direction:

$$\vec{c}_2 = -\cos\phi\vec{b}_2 - \sin\phi\vec{b}_1$$

Putting these into a transformation array gives us

	$\vec{c}_1$	$\vec{c}_2$
$\vec{b}_1$	$-\cos\phi$	$-\sin\phi$
$\vec{b}_2$	$\sin\phi$	$-\cos\phi$

2.2.2
GOAL: Determine $\|\vec{r}(t)\|$, $\|\vec{v}(t)\|$, and $\|\vec{a}(t)\|$ at time $t = 2$ s.
GIVEN: The position of particle P in the horizontal plane is given by $\vec{r}(t) = 5t^2\vec{\imath} + 2te^{0.1t}\vec{\jmath}$ m.
DRAW:



FORMULATE EQUATIONS:
The velocity and acceleration vectors for particle P are defined as, respectively,

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} \tag{1}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} \tag{2}$$

SOLVE:
The distance from the origin to the particle at $t = 2$ s is given by

$$\|\vec{r}(t)\| = \sqrt{\vec{r}(t)\cdot\vec{r}(t)} = \sqrt{(5t^2)^2 + (2te^{0.1t})^2}\ \text{m}$$

$$\|\vec{r}(2)\| = \sqrt{[5(2)^2]^2 + \left[2(2)e^{0.1(2)}\right]^2}\ \text{m}$$

$$\boxed{\|\vec{r}(2)\| = 20.6\,\text{m}}$$

The particle's speed at the given instant is

(1) ⇒
$$\vec{v}(t) = 10t\vec{\imath} + 2e^{0.1t}\vec{\jmath} + 0.2te^{0.1t}\vec{\jmath}\ \text{m/s}$$

$$\vec{v}(t) = 10t\vec{\imath} + \left(2e^{0.1t} + 0.2te^{0.1t}\right)\vec{\jmath}\ \text{m/s}$$

$$\|\vec{v}(t)\| = \sqrt{\vec{v}(t)\cdot\vec{v}(t)} = \sqrt{(10t)^2 + (2e^{0.1t} + 0.2te^{0.1t})^2}\ \text{m/s}$$

$$\|\vec{v}(2)\| = \sqrt{[10(2)]^2 + \left[2e^{0.1(2)} + 0.2(2)e^{0.1(2)}\right]^2}\ \text{m/s}$$

$$\boxed{\|\vec{v}(2)\| = 20.2\,\mathrm{m/s}}$$

The acceleration that the particle experiences at $t = 2\,\mathrm{s}$ is

$$(2) \Rightarrow \quad \vec{a}(t) = 10\vec{\imath} + 0.2e^{0.1t}\vec{\jmath} + 0.2e^{0.1t}\vec{\jmath} + 0.02te^{0.1t}\vec{\jmath}\ \mathrm{m/s^2}$$

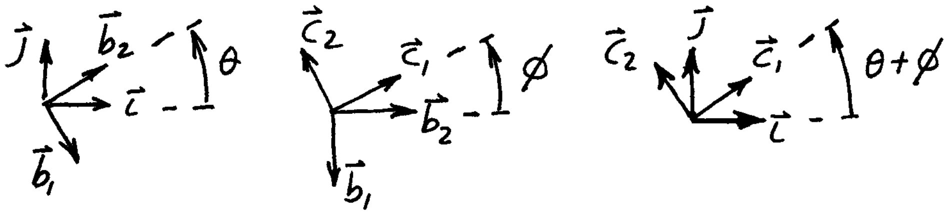
$$\vec{a}(t) = 10\vec{\imath} + \left(0.4e^{0.1t} + 0.02te^{0.1t}\right)\vec{\jmath}\ \mathrm{m/s^2}$$

$$\|\vec{a}(t)\| = \sqrt{\vec{a}(t)\cdot\vec{a}(t)} = \sqrt{100 + (0.4e^{0.1t} + 0.02te^{0.1t})^2}\ \mathrm{m/s^2}$$

$$\|\vec{a}(2)\| = \sqrt{100 + \left[0.4e^{0.1(2)} + 0.02(2)e^{0.1(2)}\right]^2}\ \mathrm{m/s^2}$$

$$\boxed{\|\vec{a}(2)\| = 10.0\,\mathrm{m/s^2}}$$

2.2.3
GOAL: Determine the coordinate transformation between $\vec{b}_1,\ \vec{b}_2$ and $\vec{c}_1,\ \vec{c}_2$ and between $\vec{\imath},\ \vec{\jmath}$ and $\vec{c}_1,\ \vec{c}_2$.
GIVEN: Angular configuration of the dial and indicator needle.
DRAW: All unit vector pairings are shown. The inclination of the $\vec{c}_i$ are made up of the rotation



of the dial plus the indicator needle's rotation with respect to the dial.
SOLVE: Our middle pairing allows us to form the transformation array between $\vec{b}_1, \vec{b}_2$ and $\vec{c}_1, \vec{c}_2$:
Putting these into a transformation array gives us

	$\vec{b}_1$	$\vec{b}_2$
$\vec{c}_1$	$-\sin\phi$	$\cos\phi$
$\vec{c}_2$	$-\cos\phi$	$-\sin\phi$

Using the rightmost pairing gives us our final transformation array:

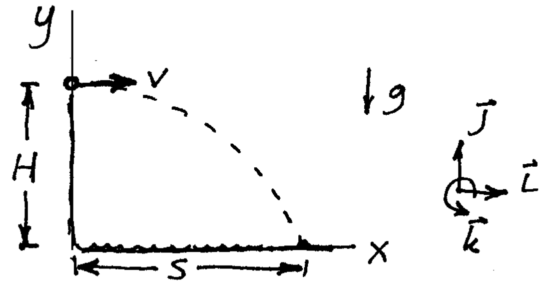
	$\vec{\imath}$	$\vec{\jmath}$
$\vec{c}_1$	$\cos(\theta+\phi)$	$\sin(\theta+\phi)$
$\vec{c}_2$	$-\sin(\theta+\phi)$	$\cos(\theta+\phi)$

2.2.4

GOAL: Determine the horizontal location before the island at which to drop the supplies.

GIVEN: System geometry and initial speed: $H = 150\,\text{ft}$, $v = 200\,\text{mph}$.

DRAW:



FORMULATE EQUATIONS:

The x position of the airlifted supplies is given by

$$x = vt \tag{1}$$

The y position of the supplies is given by

$$y = H - \frac{1}{2}gt^2 \tag{2}$$

SOLVE:

The total horizontal distance traveled is s, so (1) becomes

(1) $\Rightarrow$
$$s = vt \tag{3}$$

To find the time when the supplies land, set $y = 0$ in (2), so

(2) $\Rightarrow$
$$0 = H - \frac{1}{2}gt^2 \tag{4}$$

(4) $\Rightarrow$
$$t = \sqrt{\frac{2H}{g}} \tag{5}$$

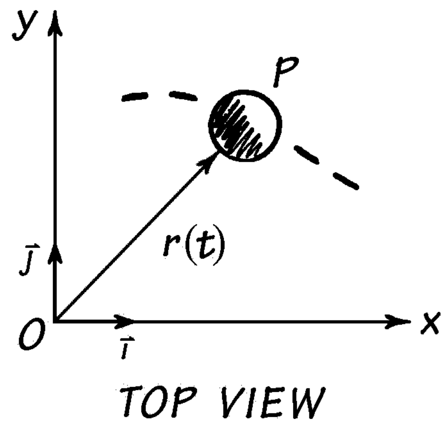
Eliminate time in (3) to find the distance:

(5) $\rightarrow$ (3) $\Rightarrow$
$$s = v\sqrt{\frac{2H}{g}}$$

$$s = (200\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)\sqrt{\frac{2(150\,\text{ft})}{32.2\,\text{ft/s}^2}}$$

$$\boxed{s = 895.4\,\text{ft}}$$

Do It 2.2.1
GOAL: Determine $\|\vec{r}(t)\|$, $\|\vec{v}(t)\|$, and $\|\vec{a}(t)\|$ at time $t = 1\,\text{s}$.
GIVEN: The position of particle P in the horizontal plane is described by $\vec{r}(t) = 0.3t^3\,\vec{\imath} + 0.4t^2 \sin 2t\,\vec{\jmath}$ ft.
DRAW:



FORMULATE EQUATIONS:
The velocity and acceleration vectors for particle P are defined as, respectively,

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} \tag{1}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} \tag{2}$$

SOLVE:
The particle's distance from the origin at $t = 1\,\text{s}$ is

$$\|\vec{r}(t)\| = \sqrt{\vec{r}(t) \cdot \vec{r}(t)} = \sqrt{(0.3t^3)^2 + (0.4t^2 \sin 2t)^2}\ \text{ft}$$

$$\|\vec{r}(1)\| = \sqrt{[0.3(1)^3]^2 + [0.4(1)^2 \sin(2 \cdot 1)]^2}\ \text{ft}$$

$$\boxed{\|\vec{r}(1)\| = 0.471\,\text{ft}}$$

The speed of the particle at the given instant is

(1) $\Rightarrow$
$$\vec{v}(t) = 0.9t^2\,\vec{\imath} + 0.8t \sin 2t\,\vec{\jmath} + 0.8t^2 \cos 2t\,\vec{\jmath}\ \text{ft/s}$$

$$\vec{v}(t) = 0.9t^2\,\vec{\imath} + \left(0.8t \sin 2t + 0.8t^2 \cos 2t\right)\vec{\jmath}\ \text{ft/s}$$

$$\|\vec{v}(t)\| = \sqrt{\vec{v}(t) \cdot \vec{v}(t)} = \sqrt{(0.9t^2)^2 + (0.8t \sin 2t + 0.8t^2 \cos 2t)^2}\ \text{ft/s}$$

$$\|\vec{v}(1)\| = \sqrt{[0.9(1)^2)]^2 + [0.8(1) \sin(2 \cdot 1) + 0.8(1)^2 \cos(2 \cdot 1)]^2}\ \text{ft/s}$$

$$\boxed{\|\vec{v}(1)\| = 0.983\,\text{ft/s}}$$

The particle's acceleration at $t = 1\,\text{s}$ is given by

(2) $\Rightarrow$ $$\vec{a}(t) = 1.8t\,\vec{\imath} + 0.8\sin 2t\,\vec{\jmath} + 1.6t\cos 2t\,\vec{\jmath} + 1.6t\cos 2t\,\vec{\jmath} - 1.6t^2\sin 2t\,\vec{\jmath}\ \text{ft/s}^2$$

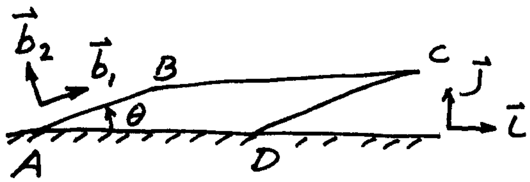
$$\vec{a}(t) = 1.8t\,\vec{\imath} + \left(0.8\sin 2t + 3.2t\cos 2t - 1.6t^2\sin 2t\right)\vec{\jmath}\ \text{ft/s}^2$$

$$\|\vec{a}(t)\| = \sqrt{\vec{a}(t)\cdot\vec{a}(t)} = \sqrt{(1.8t)^2 + (0.8\sin 2t + 3.2t\cos 2t - 1.6t^2\sin 2t)^2}\ \text{ft/s}^2$$

$$\|\vec{a}(1)\| = \sqrt{[1.8(1)]^2 + [0.8\sin(2\cdot 1) + 3.2(1)\cos(2\cdot 1) - 1.6(1)^2\sin(2\cdot 1)]^2}\ \text{ft/s}^2$$

$$\boxed{\|\vec{a}(1)\| = 2.73\,\text{ft/s}^2}$$

Do It 2.2.2
GOAL: Re-express a vector in terms of the $\vec{b}_1$, $\vec{b}_2$ frame.
GIVEN: Structural configuration of the connected links.
DRAW:



The transformation array is shown below.

	$\vec{\imath}$	$\vec{\jmath}$
$\vec{b}_1$	$\cos\theta$	$\sin\theta$
$\vec{b}_2$	$-\sin\theta$	$\cos\theta$

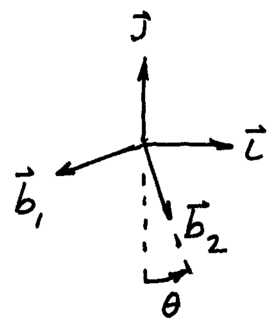
SOLVE:

$$\begin{aligned}
\vec{p} &= 4\vec{\imath} - 8\vec{\jmath} \\
&= 4(\cos\theta\vec{b}_1 - \sin\theta\vec{b}_2) - 8(\sin\theta\vec{b}_1 + \cos\theta\vec{b}_2) \\
&= (4\cos\theta - 8\sin\theta)\vec{b}_1 + (-4\sin\theta - 8\cos\theta)\vec{b}_2
\end{aligned}$$

Evaluating at $\theta = 130$ degrees yields

$$\boxed{\vec{p} = -8.70\vec{b}_1 + 2.08\vec{b}_2}$$

2.2.5
GOAL: Re-express a given vector in a new set of unit vectors.
GIVEN: Orientation of two sets of unit vectors on a movable link.
DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{b}_1$	$-\cos\theta$	$-\sin\theta$
$\vec{b}_2$	$\sin\theta$	$-\cos\theta$

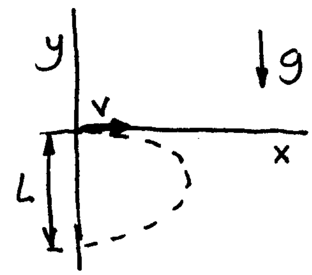
FORMULATE EQUATIONS:
SOLVE: Express $\vec{p}$ in terms of $\vec{b}_1$ and $\vec{b}_2$:

$$\vec{p} = 3\,\vec{\imath} + 4\,\vec{\jmath} \tag{1}$$
$$= 3\,(-\cos\,\theta\,\vec{b}_1 + \sin\,\theta\,\vec{b}_2) + 4\,(-\sin\,\theta\,\vec{b}_1 - \cos\,\theta\,\vec{b}_2) \tag{2}$$
$$= \vec{b}_1\,(-3\,\cos\,\theta - 4\sin\,\theta) + \vec{b}_2\,(3\,\sin\,\theta - 4\cos\,\theta) \tag{3}$$

For $\theta = 53°$, we get

$$\boxed{\vec{p} = -5.00\,\vec{b}_1 - 0.01\,\vec{b}_2} \tag{4}$$

2.2.6
GOAL: Find whether a magnet thrown off the end of a car will attach to the car, or fall to the ground.
GIVEN: Initial velocity of the magnet is $\vec{v} = \sqrt{5}\vec{\imath}$. The magnitude/car interaction as an acceleration is equal to $-a_m\vec{\imath}$ where $a_m = 10$ m/s^2. Also, the sheet metal extends a length, L, equal to 1 m.
DRAW:



ASSUME: Magnet follows a parabolic trajectory once it is thrown from the car.
SOLVE:

$$x = vt - a_m\frac{t^2}{2}$$

$$y = -g\frac{t^2}{2}$$

Solve for t for $x(t) = 0$

$$0 = vt^* - \frac{a_m t^2}{2} \Rightarrow t^*(v - \frac{a_m t^*}{2}) = 0$$

$$t^* = 0\,(\text{initial condition})$$

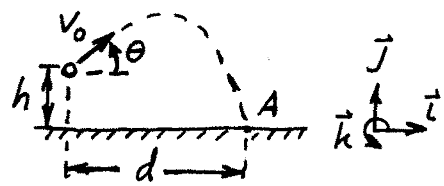
$$t^* = \frac{2v}{a_m}\,(\text{return time})$$

$$y(t^*) = -\frac{g}{2}\left(\frac{2v}{a_m}\right)^2 = -\frac{2gv^2}{a_m^2} = -\frac{2(9.81\,\text{m/s}^2)(5\,\text{m}^2/\,\text{s}^2)}{(10\,\text{m/s}^2)^2}$$

$$\boxed{y(t^*) = -0.98\,\text{m}}$$

So, just barely, the magnet hits the car (since missing would imply $|y(t^*)| > 1$ m)

Do It 2.2.3
GOAL: Determine the minimum launch angle and corresponding time of flight.
GIVEN: System geometry and initial speed: $d = 200\,\text{ft}$, $h = 3\,\text{ft}$, $v_0 = 100\,\text{ft/s}$.
DRAW:



FORMULATE EQUATIONS:
The x position of the pumpkin is given by

$$x = (v_0 \cos\theta)t \tag{1}$$

The y position of the pumpkin is given by

$$y = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

SOLVE:
The total horizontal distance traveled is d, so (1) becomes

(1) $\Rightarrow$
$$d = (v_0 \cos\theta)t \tag{3}$$

(3) $\Rightarrow$
$$t = \frac{d}{(v_0 \cos\theta)} \tag{4}$$

To find the launch angle, first set $y = 0$ in (2), so

(2) $\Rightarrow$
$$0 = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{5}$$

Then eliminate the time t:

(4) $\rightarrow$ (5) $\Rightarrow$
$$0 = h + \frac{\sin\theta}{\cos\theta}d - \frac{1}{2}g\left(\frac{d}{v_0\cos\theta}\right)^2$$

$$\begin{aligned}
0 &= h\cos^2\theta + d\sin\theta\cos\theta - \frac{gd^2}{2v_0^2} \\
0 &= h\cos^2\theta + \frac{1}{2}d\sin(2\theta) - \frac{gd^2}{2v_0^2} \\
0 &= 2h\cos^2\theta + d\sin(2\theta) - \frac{gd^2}{v_0^2}
\end{aligned}$$

$$\begin{aligned}
\frac{gd^2}{v_0^2} &= 2h\cos^2\theta + d\sin(2\theta) \\
\frac{(32.2\,\text{ft/s}^2)(200\,\text{ft})^2}{(100\,\text{ft/s})^2} &= 2(3\,\text{ft})\cos^2\theta + (200\,\text{ft})\sin(2\theta) \\
128.8\,\text{ft} &= 6\cos^2\theta\,\text{ft} + 200\sin(2\theta)\,\text{ft} \qquad (6)
\end{aligned}$$

Using MATLAB® to solve for the minimum angle that satisfies (6),

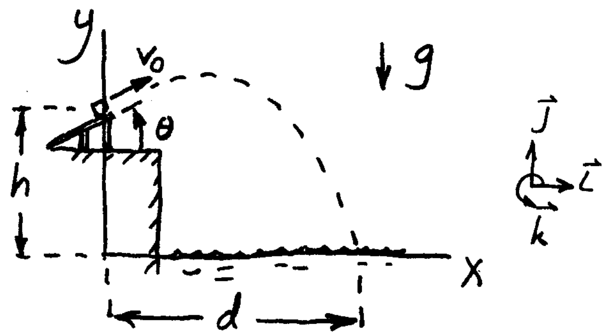
$$\boxed{\theta_{\text{min}} = 0.333\,\text{rad} = 19.06^\circ} \tag{7}$$

(7) → (4) ⇒

$$t = \frac{200\,\text{ft}}{(100\,\text{ft/s})(\cos(19.06^\circ))}$$

$$\boxed{t = 2.116\,\text{s}}$$

2.2.7
GOAL: Determine the time of flight, the horizontal distance traveled, and the maximum height achieved.
GIVEN: System geometry and initial speed: $h = 20\,\text{ft}$, $\theta = 45°$, $v_0 = 25\,\text{ft/s}$.
DRAW:



FORMULATE EQUATIONS:
The x position of the craft is given by

$$x = (v_0 \cos\theta)t \tag{1}$$

The y position of the craft is given by

$$y = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

SOLVE:
Find the time of flight by setting $y = 0$ in (2) and solving for t:

(2) $\Rightarrow$
$$0 = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{3}$$

(3) $\Rightarrow$
$$t = \frac{-v_0\sin\theta \pm \sqrt{(v_0\sin\theta)^2 + 2gh}}{-g}$$

$$\begin{aligned} t &= \frac{-(25\,\text{ft/s})\sin(45°) \pm \sqrt{((25\,\text{ft/s})\sin(45°))^2 + 2(32.2\,\text{ft/s}^2)(20\,\text{ft})}}{-32.2\,\text{ft/s}^2} \\ t &= -0.693\,\text{s},\ 1.791\,\text{s} \end{aligned}$$

$$\boxed{t = 1.791\,\text{s}} \tag{4}$$

Use the time of flight to solve for the distance traveled:

(4) $\rightarrow$ (1) $\Rightarrow$
$$x = (25\,\text{ft/s})\cos(45°)(1.791\,\text{s})$$

$$\boxed{x = 31.67\,\text{ft}}$$

Determine the maximum height achieved by first differentiating (2) and setting it equal to 0:

Differentiate (2) $= 0 \Rightarrow$
$$\frac{dy}{dt} = v_0\sin\theta - gt = 0 \tag{5}$$

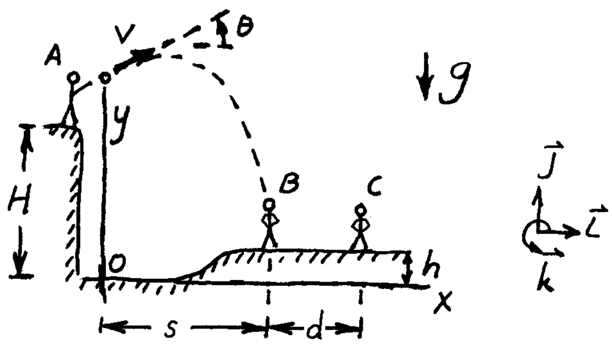
(5) $\Rightarrow$
$$t_{\text{peak}} = \frac{v_0 \sin\theta}{g} \tag{6}$$

(6) $\rightarrow$ (2) $\Rightarrow$
$$y_{\max} = h + \frac{(v_0 \sin\theta)^2}{g} - \frac{(v_0 \sin\theta)^2}{2g}$$

$$\begin{aligned} y_{\max} &= h + \frac{(v_0 \sin\theta)^2}{2g} \\ y_{\max} &= 20\,\text{ft} + \frac{((25\,\text{ft/s})\sin(45^\circ))^2}{2(32.2\,\text{ft/s}^2)} \end{aligned}$$

$$\boxed{y_{\max} = 24.85\,\text{ft}}$$

2.2.8
GOAL: Determine the separation distance and the constant running speed needed to catch the ball.
GIVEN: System geometry and initial speed: $H = 30\,\text{ft}$, $h = 10\,\text{ft}$, $d = 20\,\text{ft}$, $\theta = 30°$, $v = 20\,\text{ft/s}$.
DRAW:



FORMULATE EQUATIONS:
The x position of the ball is given by

$$x = (v\cos\theta)t \tag{1}$$

The y position of the ball is given by

$$y = H + (v\sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

The x position of the catcher is given by

$$x^* = x_0^* + v^*t \tag{3}$$

SOLVE:
The separation distance is s, so (1) becomes

$$(1) \Rightarrow \qquad s = (v\cos\theta)t \tag{4}$$

We need the time of flight to solve for s, so set $y = h$ in (2) and solve for t:

$$(2) \Rightarrow \qquad h = H + (v\sin\theta)t - \frac{1}{2}gt^2$$

$$0 = -\frac{1}{2}gt^2 + (v\sin\theta)t + (H - h) \tag{5}$$

$$(5) \Rightarrow \qquad t = \frac{-v\sin\theta \pm \sqrt{(v\sin\theta)^2 + 2g(H-h)}}{-g}$$

$$t = \frac{-(20\,\text{ft/s})\sin(30°) \pm \sqrt{((20\,\text{ft/s})\sin(30°))^2 + 2(32.2\,\text{ft/s}^2)(30\,\text{ft} - 10\,\text{ft})}}{-32.2\,\text{ft/s}^2}$$

$$t = -0.847\,\text{s},\ 1.468\,\text{s}$$

$$\boxed{t = 1.468\,\text{s}} \tag{6}$$

$(6) \rightarrow (4) \Rightarrow$
$$s = (20\,\text{ft/s})\cos(30^\circ)(1.468\,\text{s})$$

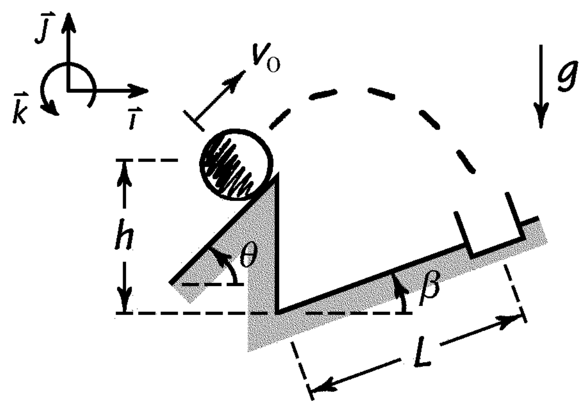
$$\boxed{s = 25.42\,\text{ft}}$$

To find the running speed,

$(3) \Rightarrow$
$$x^* - x_0^* = v^* t$$
$$\begin{aligned} -d &= v^* t \\ v^* &= -\frac{d}{t} \\ v^* &= -\frac{20\,\text{ft}}{1.468\,\text{s}} \end{aligned}$$

$$\boxed{v^* = -13.63\,\text{ft/s (running to the left)}}$$

2.2.9
GOAL: Determine the ball's launch speed and how long it is airborne.
GIVEN: The launch ramp is angled at $\theta = 45°$ with respect to the ground, and its end is located $h = 2\,\text{ft}$ above the base of the backboard, which is inclined at $\beta = 20°$ to the horizontal. The hole is $L = 3\,\text{ft}$ along the backboard from its base.
DRAW:



FORMULATE EQUATIONS:
The x position of the ball is given by

$$x = x_0 + \dot{x}_0 t \tag{1}$$

The y position of the ball is given by

$$y = y_0 + \dot{y}_0 t - \frac{1}{2} g t^2 \tag{2}$$

SOLVE:
We'll need to eliminate the time t to first find the launch speed v_0, and so from (1) we get that

(1) $\Rightarrow$
$$L\cos\beta = \left(v_0 \cos\theta\right) t$$

$$t = \frac{L\cos\beta}{v_0\cos\theta} \tag{3}$$

By eliminating the time t in (2), we find that the ball's launch speed v_0 must be

(3) $\rightarrow$ (2) $\Rightarrow$
$$L\sin\beta = h + \left(v_0 \sin\theta\right)\left(\frac{L\cos\beta}{v_0\cos\theta}\right) - \frac{1}{2} g \left(\frac{L\cos\beta}{v_0\cos\theta}\right)^2$$

$$(L\sin\beta - h)\cos^2\theta = L\cos\beta\,(\sin\theta\cos\theta) - \frac{gL^2\cos^2\beta}{2v_0^2}$$

$$\frac{gL^2\cos^2\beta}{2v_0^2} = (h - L\sin\beta)\cos^2\theta + \frac{1}{2} L\cos\beta\sin 2\theta$$

$$v_0 = \sqrt{\frac{gL^2 \cos^2\beta}{2\,(h - L\sin\beta)\cos^2\theta + L\cos\beta\sin 2\theta}}$$

$$v_0 = \sqrt{\frac{(32.2\,\mathrm{ft/s^2})(3\,\mathrm{ft})^2\cos^2(20^\circ)}{2\,[2\,\mathrm{ft} - (3\,\mathrm{ft})\sin(20^\circ)]\cos^2(45^\circ) + (3\,\mathrm{ft})\cos(20^\circ)\sin(2\cdot 45^\circ)}}$$

$$\boxed{v_0 = 8.21\,\mathrm{ft/s}}$$

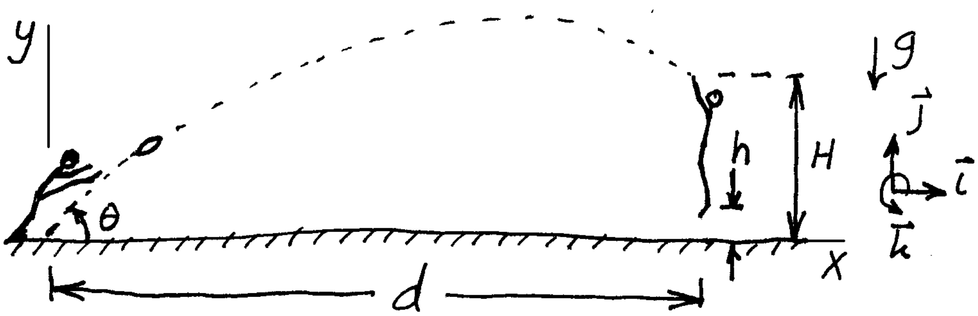
Therefore, the ball's time of flight is

$(3) \Rightarrow$
$$t = \frac{(3\,\mathrm{ft})\cos(20^\circ)}{(8.21\,\mathrm{ft/s})\cos(45^\circ)}$$

$$\boxed{t = 0.485\,\mathrm{s}}$$

2.2.10

GOAL Determine when the football player should jump to catch the ball and how far away he is.
GIVEN: System geometry, the football's initial speed, and the catcher's jump speed: $H = 8$ ft, $h = 1$ ft, $\theta = 50°$, $v_0 = 40$ ft/s, $v_j = 9$ ft/s.
DRAW:



FORMULATE EQUATIONS:
The x position of the football is given by

$$x = (v_0 \cos\theta)t \tag{1}$$

The y position of the football is given by

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

The y position of the catcher is given by

$$y_j = y_0 + v_j t_j - \frac{1}{2}gt_j^2 \tag{3}$$

SOLVE:
First find the total time taken for the football to reach the catching height H by setting $y = H$ in (2) and solving for t:

$$(2) \Rightarrow \quad H = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$0 = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t - H \tag{4}$$

$$(4) \Rightarrow \quad t = \frac{-v_0 \sin\theta \pm \sqrt{(v_0 \sin\theta)^2 - 2gH}}{-g}$$

$$\begin{aligned} t &= \frac{-(40\,\text{ft/s})\sin(50°) \pm \sqrt{((40\,\text{ft/s})\sin(50°))^2 - 2(32.2\,\text{ft/s}^2)(8\,\text{ft})}}{-32.2\,\text{ft/s}^2} \\ t &= 0.312\,\text{s},\ 1.591\,\text{s} \end{aligned}$$

$$t = 1.591\,\text{s when the ball reaches } H \text{ on the way down}$$

Next find the time it takes the player to jump and catch the football:

(3) $\Rightarrow$
$$y_j - y_0 = v_j t_j - \frac{1}{2} g t_j^2$$

$$\begin{aligned} h &= v_j t_j - \frac{1}{2} g t_j^2 \\ 0 &= -\frac{1}{2} g t_j^2 + v_j t_j - h \end{aligned} \tag{5}$$

(5) $\Rightarrow$
$$t_j = \frac{-v_j \pm \sqrt{v_j^2 - 2gh}}{-g}$$

$$\begin{aligned} t_j &= \frac{-9\,\mathrm{ft/s} \pm \sqrt{(9\,\mathrm{ft/s})^2 - 2(32.2\,\mathrm{ft/s^2})(1\,\mathrm{ft})}}{-32.2\,\mathrm{ft/s^2}} \\ t_j &= 0.153\,\mathrm{s},\ 0.406\,\mathrm{s} \end{aligned}$$

$$t_j = 0.153\,\mathrm{s} \text{ when the catcher reaches } H \text{ on the way up}$$

The catcher needs to jump at

$$t^* = t - t_j = 1.591\,\mathrm{s} - 0.153\,\mathrm{s}$$

$$\boxed{t^* = 1.438\,\mathrm{s}}$$

The distance the ball travels before being caught is d, so

(1) $\Rightarrow$
$$d = (v_0 \cos\theta) t$$

$$d = (40\,\mathrm{ft/s})(\cos(50°))(1.591\,\mathrm{s})$$

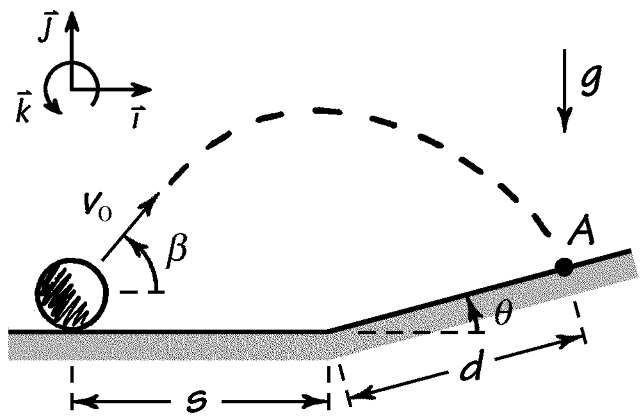
$$\boxed{d = 40.9\,\mathrm{ft}}$$

2.2.11

GOAL: Determine the ball's launch speed v_0 and time of flight.

GIVEN: The ball is launched toward point A, which is $d = 5\,\text{m}$ up an incline angled at $\theta = 15°$ with respect to the horizontal. The ball is projected at an angle $\beta = 50°$ to ground at a distance $s = 4\,\text{m}$ from the base of the incline.

DRAW:



FORMULATE EQUATIONS:

The x position of the ball is given by

$$x = x_0 + \dot{x}_0 t \tag{1}$$

The y position of the ball is given by

$$y = y_0 + \dot{y}_0 t - \frac{1}{2}gt^2 \tag{2}$$

SOLVE:

Solving for the launch speed v_0 first requires us to eliminate the time t, which we can express as

$$(1) \Rightarrow \qquad s + d\cos\theta = \left(v_0 \cos\beta\right) t$$

$$t = \frac{s + d\cos\theta}{v_0\cos\beta} \tag{3}$$

Eliminating the time t in (2) gives us the launch speed v_0 needed so that the ball lands at A:

$$(3) \rightarrow (2) \Rightarrow \qquad d\sin\theta = \left(v_0 \sin\beta\right)\left(\frac{s + d\cos\theta}{v_0\cos\beta}\right) - \frac{1}{2}g\quad\frac{s + d\cos\theta}{v_0\cos\beta}\Bigg)^2$$

$$d\sin\theta\cos^2\beta = (s + d\cos\theta)(\sin\beta\cos\beta) - \frac{g\,(s + d\cos\theta)^2}{2v_0^2}$$

$$\frac{g\,(s+d\cos\theta)^2}{2v_0^2} = \frac{1}{2}\,(s+d\cos\theta)\sin 2\beta - d\sin\theta\cos^2\beta$$

$$v_0 = \sqrt{\frac{g\,(s+d\cos\theta)^2}{(s+d\cos\theta)\sin 2\beta - 2d\sin\theta\cos^2\beta}}$$

$$v_0 = \sqrt{\frac{(9.81\,\mathrm{m/s^2})\,[4\,\mathrm{m}+(5\,\mathrm{m})\cos(15°)]^2}{[4\,\mathrm{m}+(5\,\mathrm{m})\cos(15°)]\sin(2\cdot 50°) - 2(5\,\mathrm{m})\sin(15°)\cos^2(50°)}}$$

$$\boxed{v_0 = 10.0\,\mathrm{m/s}}$$

Thus, the ball is airborne for

(3) $\Rightarrow$
$$t = \frac{4\,\mathrm{m}+(5\,\mathrm{m})\cos(15°)}{(10.0\,\mathrm{m/s})\cos(50°)}$$

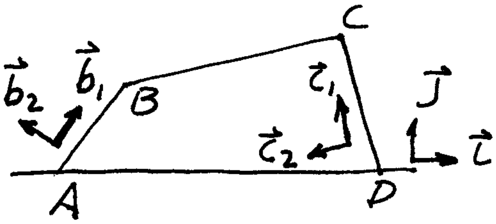
$$\boxed{t = 1.37\,\mathrm{s}}$$

2.2.12

GOAL: Show that the transformation array going from a set A to B to C can be expressed in terms of the first and last vector set along with the total angle change.

GIVEN: Physical configuration of the connected links.

DRAW:



The two transformation arrays are shown below

	$\vec{\imath}$	$\vec{\jmath}$
$\vec{b}_1$	$\cos\theta$	$\sin\theta$
$\vec{b}_2$	$-\sin\theta$	$\cos\theta$

	$\vec{\imath}$	$\vec{\jmath}$
$\vec{c}_1$	$\cos\phi$	$\sin\phi$
$\vec{c}_2$	$-\sin\phi$	$\cos\phi$

FORMULATE EQUATIONS:

$$\begin{aligned}
\vec{b}_1 &= \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} && (1)\\
\vec{b}_2 &= -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath} && (2)\\
\vec{\imath} &= \cos\phi\,\vec{c}_1 - \sin\phi\,\vec{c}_2 && (3)\\
\vec{\jmath} &= \sin\phi\,\vec{c}_1 + \cos\phi\,\vec{c}_2 && (4)
\end{aligned}$$

SOLVE:

We can use (3) and (4) to reexpress (1) and (2) in terms of $\vec{c}_1$ and $\vec{c}_2$:

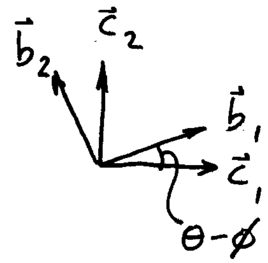
$$\begin{aligned}
\vec{b}_1 &= \cos\theta(\cos\phi\,\vec{c}_1 - \sin\phi\,\vec{c}_2) + \sin\theta(\sin\phi\,\vec{c}_1 + \cos\phi\,\vec{c}_2)\\
\vec{b}_2 &= -\sin\theta(\cos\phi\,\vec{c}_1 - \sin\phi\,\vec{c}_2) + \cos\theta(\sin\phi\,\vec{c}_1 + \cos\phi\,\vec{c}_2)
\end{aligned}$$

Next we can regroup terms and reexpress them in terms of a more compact trigonometric form:

$$\begin{aligned}
\vec{b}_1 &= (\cos\theta\cos\phi + \sin\theta\sin\phi)\vec{c}_1 + (\sin\theta\cos\phi - \cos\theta\sin\phi)\vec{c}_2\\
\vec{b}_2 &= -(\sin\theta\cos\phi - \cos\theta\sin\phi)\vec{c}_1 + (\cos\theta\cos\phi + \sin\theta\sin\phi)\vec{c}_2
\end{aligned}$$

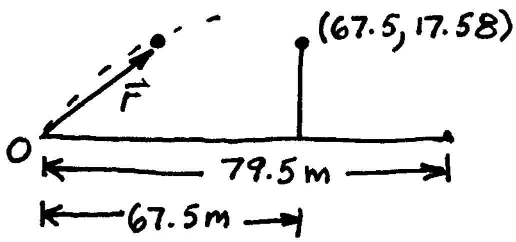
$$\begin{aligned}
\vec{b}_1 &= \cos(\theta-\phi)\vec{c}_1 + \sin(\theta-\phi)\vec{c}_2\\
\vec{b}_2 &= -\sin(\theta-\phi)\vec{c}_1 + \cos(\theta-\phi)\vec{c}_2
\end{aligned}$$

The final array transformation looks just like the original ones except it's in terms of the difference of two angles rather than in terms of a single angle. If I make superimpose the two unit vector sets $\vec{b}_1, \vec{b}_2$ and $\vec{c}_1, \vec{c}_2$, as done below, you can see that the same transformation is obtained.



	$\vec{c}_1$	$\vec{c}_2$
$\vec{b}_1$	$\cos(\theta-\phi)$	$\sin(\theta-\phi)$
$\vec{b}_2$	$-\sin(\theta-\phi)$	$\cos(\theta-\phi)$

2.2.13
GOAL: Determine launch angle
GIVEN: Dimensions of range and barrier.
DRAW:



ASSUME:

$$\vec{r}(t_1) = (67.5\vec{\imath} + 17.58\vec{\jmath})\,\text{m} \quad (1)$$
$$\vec{r}(t_2) = 79.5\vec{\imath}\,\text{m} \quad (2)$$
$$\vec{v}(0) = v(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath}) \quad (3)$$

FORMULATE EQUATIONS:

$$\vec{r}(t) = -\frac{1}{2}gt^2\vec{\jmath} + \vec{v}(0)t + \vec{r}(0) \quad (4)$$

SOLVE:

(1)→(4)⇒
$$(67.5\vec{\imath} + 17.58\vec{\jmath})\,\text{m} = \left(-\frac{g}{2}t_1^2\right)\vec{\jmath} + v(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath})t_1$$

$\vec{\imath}$:
$$\begin{aligned} v_0\cos\theta t_1 &= 67.5\,\text{m} \\ t_1 &= \frac{67.5\,\text{m}}{v_0\cos\theta} \end{aligned} \quad (5)$$

$\vec{\jmath}$:
$$\begin{aligned} v_0\sin\theta t_1 - \frac{g}{2}t_1^2 &= 17.58\,\text{m} \\ (67.5\,\text{m})\tan\theta - \frac{g}{2}\left(\frac{67.5\,\text{m}}{v_0\cos\theta}\right)^2 &= 17.58\,\text{m} \end{aligned} \quad (6)$$

(2)→(4)⇒
$$79.5\vec{\imath}\,\text{m} = \left(-\frac{g}{2}t_2^2\right)\vec{\jmath} + v(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath})t_2$$

$\vec{\imath}$:
$$\begin{aligned} v_0\cos\theta t_2 &= 79.5\,\text{m} \\ t_2 &= \frac{79.5\,\text{m}}{v_0\cos\theta} \end{aligned} \quad (7)$$

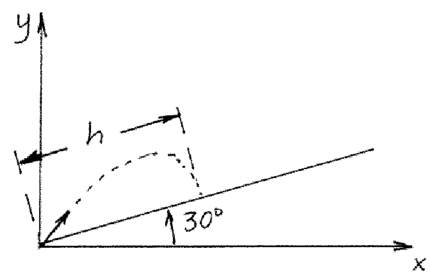
$\vec{\jmath}$:

$$
\begin{aligned}
v_0 \sin\theta t_2 - \frac{g}{2}t_2^2 &= 0 \\
(79.5\,\mathrm{m})\tan\theta - \frac{g}{2}\left(\frac{79.5\,\mathrm{m}}{v_0\cos\theta}\right)^2 &= 0 \qquad (8)
\end{aligned}
$$

(6)$\times 79.5^2$-(8)$\times 67.5^2 \Rightarrow$

$$
\begin{aligned}
\tan\theta\left(67.5(79.5)^2 - 79.5(67.5)^2\right) &= 17.58(79.5)^2 \\
\tan\theta &= \frac{17.58(79.5)^2}{67.5(79.5)^2 - 79.5(67.5)^2} \\
\theta &= \boxed{59.9\ \text{degrees}}
\end{aligned}
$$

2.2.14
GOAL: Find the position of impact for a pea projected at an angle onto a slope.
GIVEN: System configuration.
DRAW:



FORMULATE EQUATIONS:
The x position of the dried pea is given by the equation

$$x_p = v_0 t \cos\eta$$

The y position of the pea is given by the equation

$$y_p = v_0 t \sin\eta - \frac{1}{2}gt^2$$

The x and y positions of the intercept on the slope are

$$x_s = \frac{\sqrt{3}}{2}h$$

and

$$y_s = \frac{1}{2}h$$

SOLVE:
Setting $x_p = x_s$ and $y_p = y_s$ yields a set of simultaneous equations that can be solved for h. The equations are

$$v_0 t \cos\eta = \frac{\sqrt{3}}{2}h \tag{1}$$

and

$$v_0 t \sin\eta - \frac{1}{2}gt^2 = \frac{1}{2}h \tag{2}$$

Solving (1) for t gives

$$t = \frac{\sqrt{3}h}{2v_0 \cos\eta} \tag{3}$$

(3) $\Rightarrow$ (2) gives

$$\frac{\sqrt{3}v_0 h \sin\eta}{2v_0 \cos\eta} - \frac{3gh^2}{8v_0^2 \cos^2\eta} = \frac{1}{2}h$$

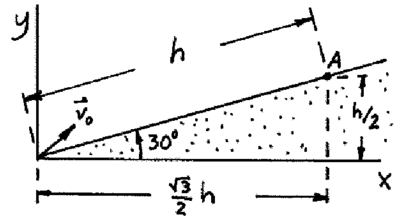
Solving for h gives

$$h = \left(\frac{\sqrt{3}\sin(\eta) - \cos(\eta)}{2\cos\eta} \right) \frac{8v_0^2 \cos^2\eta}{3g}$$

Simplification yields

$$\boxed{h = \frac{4v_0^2}{3g}\left(\sqrt{3}\sin\eta\cos\eta - \cos^2\eta\right)}$$

2.2.15
GOAL: Find where a cricket lands on an inclined slope
GIVEN: Angle of slope.
DRAW:



FORMULATE EQUATIONS:

$$x = v_0 t \cos\eta$$

$$y = v_0 t \sin\eta - \frac{1}{2}gt^2$$

$$\text{at } A: x = \frac{\sqrt{3}}{2}h, y = \frac{h}{2}$$

SOLVE:

$$\frac{\sqrt{3}}{2}h = v_0 t^* \cos\eta \qquad (1)$$

$$\frac{h}{2} = v_0 t^* \sin\eta - \frac{1}{2}g(t^*)^2 \qquad (2)$$

(1) ⇒

$$t = \frac{\sqrt{3}h}{2v_0\cos\eta} \qquad (3)$$

(3) → (2) ⇒

$$\frac{h}{2} = \frac{\sqrt{3}h\sin\eta}{2\cos\eta} - \frac{1}{2}g\frac{3h^2}{4v_0^2\cos^2\eta}$$

$$\sqrt{3}\sin\eta\cos\eta - (\cos\eta)^2 - \frac{3}{4}\frac{hg}{v_0^2} = 0$$

Let $f(\eta) = \sqrt{3}\sin\eta\cos\eta - \cos^2\eta - \frac{3}{4}\frac{hg}{v_0^2}$ and find the value(s) of η for which $f(\eta) = 0$
Using the Newton-Raphson procedure to solve for the roots gives:

$$\frac{df(\eta)}{d\eta} = \sqrt{3}\cos(2\eta) + 2\cos\eta\sin\eta = \sqrt{3}\cos(2\eta) + \sin(2\eta)$$

$$\eta_{i+1} = \eta_i - \frac{f}{\frac{df}{d\eta}} = \eta_i - \frac{\sqrt{3}}{\sqrt{3}\cos(2\eta) + \sin(2\eta)} \qquad (4)$$

Initial guess: $\eta_0 = 0.7$. Using (4) yields $\eta_1 = 0.7762$. Using (4) again yields $\eta_2 = 0.7852$. One last iteration gives $\eta_3 = 0.7854$.

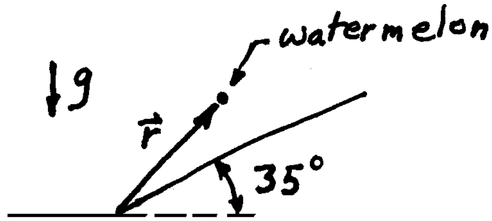
$$\boxed{\eta = 0.7854 \text{ rad} = 45°}$$

2.2.16

GOAL: Determine initial launch speed for a launched watermelon to impact a slope at a specified position.

GIVEN: Angle of slope and angle of launch.

DRAW:



FORMULATE EQUATIONS:

$$\vec{r}(t) = \left(-\frac{1}{2}gt^2\right)\vec{\jmath} + \vec{v}(0)t + \vec{r}(0) \tag{1}$$

ASSUME:

$$\begin{aligned} \vec{r}(t_f) &= 10(\cos(35°)\,\vec{\imath} + \sin(35°)\,\vec{\jmath})\,\text{ft} \\ \vec{v}(0) &= v_0(\cos(45°)\,\vec{\imath} + \sin(45°)\,\vec{\jmath})\,\text{ft/s} \end{aligned} \tag{2}$$

SOLVE:

(2)→(1)⇒

$$10(\cos(35°)\,\vec{\imath} + \sin(35°)\,\vec{\jmath})\,\text{ft} = \left(-\frac{1}{2}gt_f^2\right)\vec{\jmath} + v_0(\cos(45°)\,\vec{\imath} + \sin(45°)\,\vec{\jmath})t_f$$

$\vec{\imath}$:

$$\begin{aligned} 10\cos(35°)\,\text{ft} &= v_0\cos(45°)t_f \\ t_f &= \frac{10\cos(35°)}{v_0\cos(45°)}\,\text{sec} \end{aligned} \tag{3}$$

$\vec{\jmath}$:

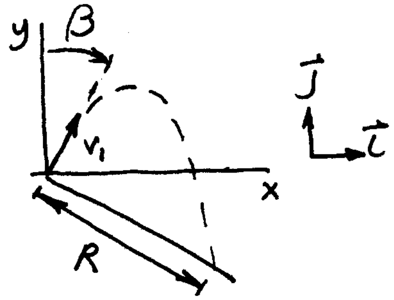
$$\begin{aligned} 10\sin(35°)\,\text{ft} &= v_0\sin(45°)t_f - \frac{1}{2}gt_f^2 \\ 10\sin(35°)\,\text{ft} &= v_0\sin(45°)\left(\frac{10\cos(35°)}{v_0\cos(45°)}\,\text{sec}\right) - \frac{1}{2}g\left(\frac{10\cos(35°)}{v_0\cos(45°)}\,\text{sec}\right)^2 \\ v_0 &= \sqrt{\frac{(-\frac{1}{2})(32.2\,\text{ft/s2})\left(\frac{10\cos(35°)}{\cos(45°)}\text{ft}\right)^2}{10(\sin(35°) - \tan(45°)\cos(35°))\,\text{ft}}} \\ v_0 &= \boxed{29.7\ \text{ft/s}} \end{aligned}$$

2.2.17

GOAL: Find how far along an inclined surface a tennis ball will impact when it rebounds at a given velocity.

GIVEN: Tennis ball's initial speed and slope's inclination.

DRAW:



FORMULATE EQUATIONS:

$$x = |v_1|(\sin\beta)t \tag{1}$$

$$y = |v_1|(\cos\beta)t - 4.905t^2 \tag{2}$$

SOLVE:

(1) → (2) ⇒

$$y = \frac{-4.905x^2}{|v_1|^2(\sin\beta)^2} + \frac{x}{\tan\beta} \tag{3}$$

A point on the sloped surface will satisfy

$$y = -(\tan\beta)x \tag{4}$$

(3), (4) ⇒

$$-(\tan\beta)x = \frac{-4.905x^2}{|\vec{v}_1|^2(\sin\beta)^2} + \frac{x}{\tan\beta}$$

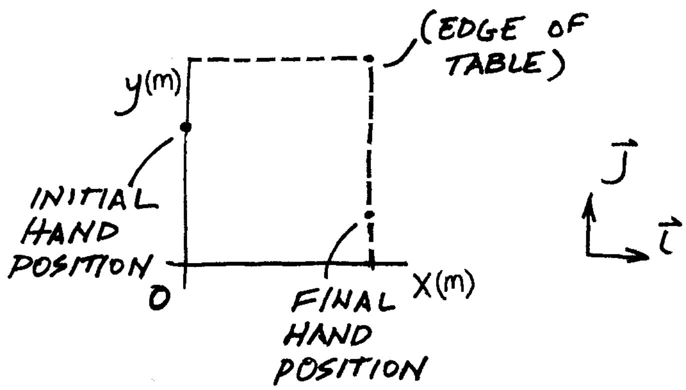
$$x\left[\frac{4.905x}{|\vec{v}_1|^2(\sin\beta)^2} - \left((\tan\beta) + \frac{1}{\tan\beta}\right)\right] = 0$$

$$x(.03139x - 2.3094) = 0$$

$$x = 73.57 \text{ m}$$

$$\boxed{R = \frac{73.57\,\text{m}}{\cos\beta} = 84.95 \text{ m}}$$

2.2.18
GOAL: Calculate required acceleration of hand to catch glass and speed of hand on contact with glass.
GIVEN: Positions at which glass falls and at which hand is initially.
DRAW:



FORMULATE EQUATIONS: Let $\vec{r}_G$ be the position vector of the glass and $\vec{r}_H$ be the position vector of the hand during the fall.

$$\vec{r}_G(t) = -\frac{1}{2}gt^2\vec{\jmath} + \vec{v}_G(0)t + \vec{r}_G(0) \tag{1}$$

$$\vec{r}_H(t) = \frac{1}{2}(t - 0.25\,\text{s})^2\vec{a}_H + \vec{r}_H(0) \tag{2}$$

ASSUME:

$$\vec{r}_G(0) = (2\vec{\imath} + 3\vec{\jmath})\,\text{ft} \tag{3}$$

$$\vec{r}_H(0) = 2\vec{\jmath}\,\text{ft} \tag{4}$$

$$\vec{r}_G(t_f) = \vec{r}_H(t_f) = (2\vec{\imath} + 0.5\vec{\jmath})\,\text{ft} \tag{5}$$

$$\vec{a}_H = a\frac{2\vec{\imath} - 1.5\vec{\jmath}}{\sqrt{2^2 + 1.5^2}} \tag{6}$$

SOLVE:

(1),(2)⇒
$$-\frac{1}{2}gt^2\vec{\jmath} + \vec{v}_G(0)t + \vec{r}_G(0) = \frac{1}{2}(t - 0.25\,\text{s})^2\vec{a}_H + \vec{r}_H(0) \tag{7}$$

(3),(4),(5),(6)→(8)⇒
$$-\frac{1}{2}gt^2\vec{\jmath} + (2\vec{\imath} + 3\vec{\jmath})\,\text{ft} = \frac{1}{2}(t - 0.25\,\text{s})^2\left(a\frac{2\vec{\imath} - 1.5\vec{\jmath}}{\sqrt{2^2 + 1.5^2}}\right) + (2\vec{\jmath}\,\text{ft}) \tag{8}$$

$\vec{\imath}$:
$$2\,\text{ft} = \frac{a(t - 0.25\,\text{s})^2}{\sqrt{2^2 + 1.5^2}} \tag{9}$$

$\vec{\jmath}$:
$$-\frac{1}{2}gt^2 + 3\,\text{ft} = -\frac{1.5}{2}\ \frac{a(t - 0.25\,\text{s})^2}{\sqrt{2^2 + 1.5^2}}\Bigg) + 2\,\text{ft} \tag{10}$$

(9)→(10)⇒

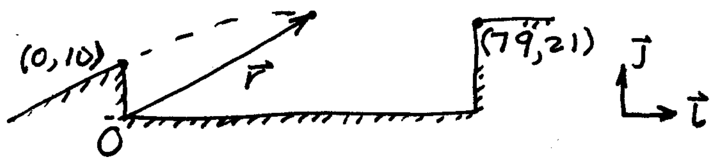
$$
\begin{aligned}
-\frac{g}{2}t^2 + 3\,\text{ft} &= (-1.5+2)\,\text{ft} \\
t &= \sqrt{\frac{2(-2.5\,\text{ft})}{-32.2\,\text{ft/s}^2}} = 0.394\,\text{s} \qquad (11)
\end{aligned}
$$

(11)→(9)⇒

$$
\begin{aligned}
a &= \frac{(2\,\text{ft})\sqrt{2^2+1.5^2}}{(0.394\,\text{s} - .25\,\text{s})^2} \\
a &= \boxed{241 \text{ ft/s}^2}
\end{aligned}
$$

$$
\begin{aligned}
\vec{v}_H &= \vec{a}_H t + \vec{v}_H(0) = (241\,\text{ft/s}^2)\left(\frac{2\vec{\imath} - 1.5\vec{\jmath}}{\sqrt{2^2+1.5^2}}\right)(0.394\,\text{s} - 0.25\,\text{s}) \\
|v_H| &= (241\,\text{ft/s}^2)[(0.394-0.25)\,\text{s}] = \boxed{34.7 \text{ ft/s}^2}
\end{aligned}
$$

2.2.19
GOAL: Calculate minimum speed v to complete stunt.
GIVEN: Positions at which the car leaves the ramp, angle of the ramp and target location.
DRAW:



FORMULATE EQUATIONS:

$$\vec{r}(t) = \frac{1}{2}t^2\vec{a} + \vec{v}(0)t + \vec{r}(0) \tag{1}$$

ASSUME:

$$\begin{aligned} \vec{v}(0) &= v(\cos(20°)\vec{\imath} + \sin(20°)\vec{\jmath} \\ \vec{r}(0) &= 10\vec{\jmath}\text{ ft} \\ \vec{r}(t_f) &= (79\vec{\imath} + 21\vec{\jmath})\text{ ft} \end{aligned} \tag{2}$$

SOLVE:

(2)→(1)⇒
$$-g\frac{1}{2}t^2\vec{\jmath} + v(\cos(20°)\vec{\imath} + \sin(20°)\vec{\jmath})t + 10\vec{\jmath}\text{ m} = 79\vec{\imath}\text{ m} + 21\vec{\jmath}\text{ m}$$

$\vec{\imath}$:
$$\begin{aligned} v\cos(20°)t &= 79\text{ ft} \\ t &= \frac{79\text{ ft}}{v\cos(20°)} \end{aligned} \tag{3}$$

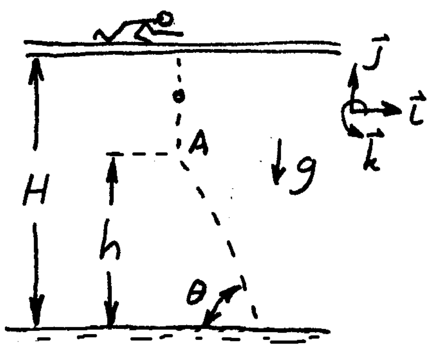
$\vec{\jmath}$:
$$\begin{aligned} 21\text{ ft} &= -\frac{g}{2}\left(\frac{79\text{ ft}}{v\cos(20°)}\right)^2 + v\sin(20°)\left(\frac{79\text{ ft}}{v\cos(20°)}\right) + 10\text{ ft} \\ v &= \frac{79\text{ ft}}{\cos(20°)}\sqrt{\frac{32.2\text{ ft/s}^2}{2((79\text{ ft})\tan(20°) - 11\text{ ft})}} \\ v &= 80.1\text{ ft/s} = \boxed{55\text{ mph}} \end{aligned}$$

2.2.20

GOAL: Determine the time it takes the ball to hit the water's surface, where it hits, the impact speed, and the angle of impact.

GIVEN: The ball starts from rest at $H = 70\,\text{ft}$. The wind speed is $v_w = 80\,\text{ft/s}$ at $h = 50\,\text{ft}$.

DRAW:



FORMULATE EQUATIONS:

The y position of the ball is given by

$$y = H - \frac{1}{2}gt^2 \tag{1}$$

The time it takes for the ball to fall from where the wind pushes it to the water's surface is

$$t^* = t - t_A \tag{2}$$

The horizontal distance traveled due to the wind is

$$s = v_w t^* \tag{3}$$

Since it starts from rest, the ball's final vertical speed is given by

$$\dot{y}_f^2 = 2gH \tag{4}$$

SOLVE:

Find the time to impact by setting $y = 0$ in (1):

(1) $\Rightarrow$
$$0 = H - \frac{1}{2}gt^2$$
$$t = \sqrt{\frac{2H}{g}}$$
$$t = \sqrt{\frac{2(70\,\text{ft})}{32.2\,\text{ft/s}^2}}$$
$$\boxed{t = 2.085\,\text{s}}$$

The time it takes to fall to where the wind pushes the ball (point A) is

(1) $\Rightarrow$
$$h = H - \frac{1}{2}gt_A^2$$

$$\begin{aligned}
t_A &= \sqrt{\frac{2(H-h)}{g}} \\
t_A &= \sqrt{\frac{2(70\,\text{ft} - 50\,\text{ft})}{32.2\,\text{ft/s}^2}} \\
t_A &= 1.115\,\text{s}
\end{aligned}$$

From point A to the water's surface,

(2) $\Rightarrow$
$$t^* = 2.085\,\text{s} - 1.115\,\text{s}$$
$$t^* = 0.971\,\text{s}$$

The distance traveled due to the wind is then

(3) $\Rightarrow$
$$s = (80\,\text{ft/s})(0.971\,\text{s})$$
$$\boxed{s = 77.6\,\text{ft}}$$

The impact speed is

$$v_f = \sqrt{\dot{x}_f^2 + \dot{y}_f^2} \tag{5}$$

The final horizontal speed is just the wind speed, so

(4) $\rightarrow$ (5) $\Rightarrow$
$$v_f = \sqrt{v_w^2 + 2gH}$$
$$v_f = \sqrt{(80\,\text{ft/s})^2 + 2(32.2\,\text{ft/s}^2)(70\,\text{ft})}$$
$$\boxed{|v_f| = 104.4\,\text{ft/s}}$$

The impact angle is

$$\theta = \tan^{-1}\left(\frac{\dot{y}_f}{\dot{x}_f}\right) \tag{6}$$

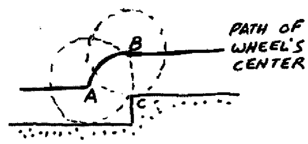
(4) $\rightarrow$ (6) $\Rightarrow$
$$\theta = \tan^{-1}\left(\frac{\sqrt{2gH}}{v_w}\right)$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{2(32.2\,\text{ft/s}^2)(70\,\text{ft})}}{80\,\text{ft/s}}\right)$$
$$\boxed{|\theta| = 40.0^\circ}$$

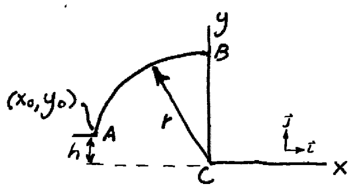
2.2.21

GOAL: Calculate the acceleration and jerk of the wheel's center in the vertical direction.

GIVEN: Size of wheel and bump.

DRAW





ASSUME: $\dot{x} = v_0,\ \ddot{x} = 0,\ \dot{r} = 0$

FORMULATE EQUATIONS: The relation between x, y, and r is:

$$x^2 + y^2 = r^2 \tag{1}$$

Rearranging (1) gives us

$$y^2 = r^2 - x^2 \tag{2}$$

$$y = \left|\left(r^2 - x^2\right)^{1/2}\right| \tag{3}$$

SOLVE: Differentiating (2) with respect to time yields:

$$2y\dot{y} = -2x\dot{x} \tag{4}$$

$$\dot{y} = \frac{-x\dot{x}}{y} \tag{5}$$

(3) → (5) ⇒
$$\dot{y} = \frac{-xv_0}{\left|(r^2 - x^2)^{1/2}\right|} \tag{6}$$

Differentiating (4) ⇒
$$\dot{y}^2 + y\ddot{y} = -\dot{x}^2 - x\ddot{x} \tag{7}$$

Because $\ddot{x} = 0$ and $\dot{x} = v_0$, (7) can be solved for $\ddot{y}$ to yield:

$$\ddot{y} = \frac{-{v_0}^2 - \dot{y}^2}{y} \tag{8}$$

(3), (6) → (8) ⇒
$$\ddot{y} = \frac{-{v_0}^2 - \dfrac{x^2{v_0}^2}{r^2 - x^2}}{\left|(r^2 - x^2)^{1/2}\right|} = \frac{-{v_0}^2 r^2}{\left|(r^2 - x^2)^{3/2}\right|} \tag{9}$$

$$\boxed{\ddot{y} = \frac{-{v_0}^2 r^2}{\left|(r^2 - x^2)^{3/2}\right|}}$$

Differentiating (7) $\Rightarrow$

$$2\dot{y}\ddot{y} + \dot{y}\ddot{y} + yy^{(3)} = 0 \tag{10}$$

where $y^{(3)}$ represents the third derivative of y with respect to time.
Solving (10) for $y^{(3)}$ $\Rightarrow$

$$y^{(3)} = -\frac{3\dot{y}\ddot{y}}{y} = -3\frac{1}{\left|(r^2 - x^2)^{1/2}\right|}\left(\frac{-xv_0}{\left|(r^2 - x^2)^{1/2}\right|}\right)\left(\frac{-{v_0}^2}{\left|(r^2 - x^2)^{3/2}\right|}\right) \tag{11}$$

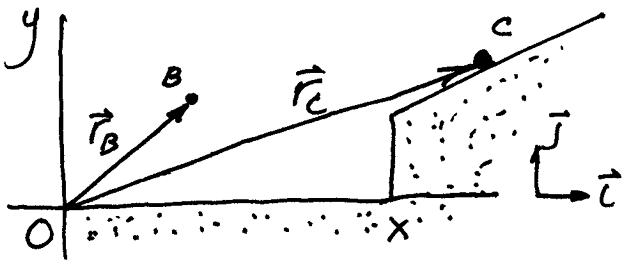
$$\boxed{y^{(3)} = \frac{-3{v_0}^3 xr^2}{\left|(r^2 - x^2)^{5/2}\right|}}$$

2.2.22

GOAL: Find an expression for the necessary launch angle θ in terms of ϕ, d, h, L, g and v_C.

GIVEN: System geometry.

DRAW:



FORMULATE EQUATIONS:

$$\vec{r}_B(t) = \left(-\frac{1}{2}gt^2\right)\vec{\jmath} + \vec{v}_B(0)t + \vec{r}_B(0) \tag{1}$$

$$\vec{r}_C(t) = \vec{v}_C(0)t + \vec{r}_C(0) \tag{2}$$

ASSUME:

$$\begin{aligned}
\vec{r}_B(0) &= 0 \\
\vec{v}_B(0) &= v_0(\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}) \\
\vec{r}_C(0) &= d\,\vec{\imath} + h\,\vec{\jmath} \\
\vec{v}_C(0) &= v_C(\cos\phi\,\vec{\imath} + \sin\phi\,\vec{\jmath}) \\
\vec{r}_B(t_f) &= \vec{r}_C(t_f) = (d + L\cos\phi)\,\vec{\imath} + (h + L\sin\phi)\,\vec{\jmath}
\end{aligned} \tag{3}$$

SOLVE:

(3)=(2)⇒

$$\vec{v}_C(0)t_f + \vec{r}_C(0) = = \vec{r}_C(0) + L(\cos\phi\,\vec{\imath} + \sin\phi\,\vec{\jmath})$$

$$v_C t_f = L \quad \Rightarrow \quad t_f = \frac{L}{v_C} \tag{4}$$

(3)=(1)⇒

$$\left(-\frac{1}{2}gt_f^2\right)\vec{\jmath} + \vec{v}_B(0)t_f + \vec{r}_B(0) = (d + L\cos\phi)\vec{\imath} + (h + L\sin\phi)\vec{\jmath}$$

$$\left(-\frac{1}{2}gt_f^2\right)\vec{\jmath} + v_0(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath})t_f = (d + L\cos\phi)\vec{\imath} + (h + L\sin\phi)\vec{\jmath} \tag{5}$$

$\vec{\imath}$:

$$v_0 t_f\cos\theta = d + L\cos\phi$$

$$v_0 = \frac{d + L\cos\phi}{t_f\cos\theta} = \frac{v_C(d + L\cos\phi)}{L\cos\theta} \tag{6}$$

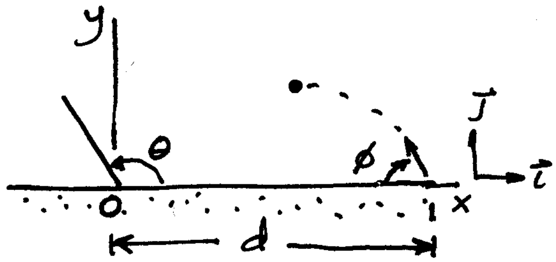
$\vec{\jmath}$:

$$-\frac{g}{2}t_f^2 + v_0 t_f\sin\theta = h + L\sin\phi \tag{7}$$

(4),(6)→(7)⇒

$$\begin{aligned}
-\frac{g}{2}\left(\frac{L}{v_C}\right)^2 + \left(\frac{v_C(d+L\cos\phi)}{L\cos\theta}\right)\left(\frac{L}{v_C}\right)\sin\theta &= h + L\sin\phi \\
\tan\theta(d+L\cos\phi) &= h + L\sin\phi + \frac{gL^2}{2v_C^2} \\
\theta &= \boxed{\tan^{-1}\left(\frac{h + L\sin\phi + \frac{gL^2}{2v_C^2}}{d+L\cos\phi}\right)}
\end{aligned}$$

2.2.23
GOAL: Find launch speed of mass so it strikes middle of bat when $\theta = \frac{\pi}{2}$.
GIVEN: Launch angle ϕ is fixed. Launching tube is a distance d from the bat.
DRAW:



FORMULATE EQUATIONS:

$$\ddot{\theta} = \alpha t$$
$$\dot{\theta} = \frac{\alpha t^2}{2} + \dot{\theta}(0)$$
$$\theta = \frac{\alpha t^3}{6} + \dot{\theta}(0)t + \theta(0) \tag{1}$$

$$\vec{r}(t) = -\frac{g}{2}t^2\vec{\jmath} + \vec{v}(0)t + \vec{r}(0) \tag{2}$$

ASSUME:

$$\begin{aligned} \theta(0) &= \pi \\ \dot{\theta}(0) &= 0 \\ \theta(t^*) &= \tfrac{\pi}{2} \end{aligned} \tag{3}$$

$$\begin{aligned} \vec{r}(0) &= d\vec{\imath} \\ \vec{v}(0) &= v_0(-\cos\phi\vec{\imath} + \sin\phi\vec{\jmath}) \\ \vec{r}(t^*) &= \tfrac{L}{2}\vec{\jmath} \end{aligned} \tag{4}$$

SOLVE:

(3)→(1)⇒

$$\frac{\alpha t^{*3}}{6} + \pi = \pi/2$$

$$t^* = \left(-\frac{3\pi}{\alpha}\right)^{\frac{1}{3}} \tag{5}$$

(4)→(2)⇒

$$-\frac{g}{2}t^{*2}\vec{\jmath} + v_0(-\cos\phi\vec{\imath} + \sin\phi\vec{\jmath})t^* + d\vec{\imath} = \frac{L}{2}\vec{\jmath}$$

$\vec{\imath}$:

$$-v_0t^*\cos\phi + d = 0$$

$$\cos\phi = \frac{d}{v_0t^*} \quad \Rightarrow \quad \sin\phi = \sqrt{1 - \left(\frac{d}{v_0t^*}\right)^2} \tag{6}$$

$\vec{\jmath}$:
$$-\frac{g}{2}t^{*2} + v_0 t^* \sin\phi = \frac{L}{2} \tag{7}$$

(6),(5)→(7)⇒
$$v_0 t^* \sqrt{1 - \left(\frac{d}{v_0 t^*}\right)^2} = \frac{L + gt^{*2}}{2}$$

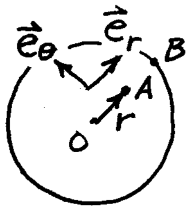
$$\left(v_0 t^*\right)^2 - d^2 = \left(\frac{L + gt^{*2}}{2}\right)^2$$

$$v_0^2 = \frac{\left(\frac{L+gt^{*2}}{2}\right)^2 + d^2}{t^{*2}}$$

$$\boxed{v_0 = \sqrt{\frac{\left(L + g\left(-\frac{3\pi}{\alpha}\right)^{\frac{2}{3}}\right)^2 + 4d^2}{4\left(-\frac{3\pi}{\alpha}\right)^{\frac{2}{3}}}}}$$

2.3 Polar and Cylindrical Coordinates

Do It 2.3.1
GOAL: Determine the absolute speed and acceleration of a running boy on a turntable.
GIVEN: The turntable is rotating at 0.4 rad/s and its radius is 10 m. Andy begins running from rest and accelerates at a constant rate of $3\,\mathrm{m/s^2}$.
DRAW



FORMULATE EQUATIONS:
We'll use our expressions for angular velocity and acceleration:

$$\vec{v}_A = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$\vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

as well as our formula for constant acceleration for a particle starting from rest (and $r = 0$):

$$\ddot{r}\,r = \frac{1}{2}\dot{r}^2 \tag{1}$$

ASSUME: Andy runs straight toward Ben.
SOLVE: After moving 10 m Andy is moving at

(1) $\Rightarrow$ $$\dot{r} = \sqrt{2(3\,\mathrm{m/s^2})(10\,\mathrm{m})} = 7.75\,\mathrm{m/s}$$

$$||\vec{v}_A|| = \sqrt{(r\dot{\theta})^2 + (\dot{r})^2} = \sqrt{[(10\,\mathrm{m})(0.4\,\mathrm{rad/s})]^2 + (7.75\,\mathrm{m/s})^2}$$

$$\boxed{||\vec{v}_A|| = 8.72\,\mathrm{m/s}}$$

$$||\vec{a}_A|| = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2} = \sqrt{[3\,\mathrm{m/s^2} - (10\,\mathrm{m})(0.4\,\mathrm{rad/s})^2]^2 + [2(7.75\,\mathrm{m/s})(0.4\,\mathrm{rad/s})]^2}$$

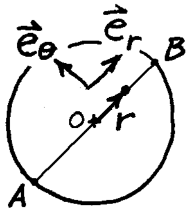
$$\boxed{||\vec{a}_A|| = 6.35\,\mathrm{m/s^2}}$$

2.3.1

GOAL: Determine the absolute speed and acceleration of a person running across a carousel when he reaches the edge.

GIVEN: The carousel is rotating at 0.6 rad/s (counter-clockwise) and its radius is 40 ft. When Bill reaches the center O he has a speed of 5 ft/s and an acceleration of 2 ft/s^2 ($t = 0$ is referenced to the point at which he reaches the center). At $t = 0$ the carousel begins to decelerate at a constant rate of 0.1 rad/s^2.

DRAW



FORMULATE EQUATIONS:

We'll use our expressions for angular velocity and acceleration

$$\vec{v}_A = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

$$\vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta \tag{2}$$

and our position, speed, constant acceleration relationships

$$a(t) = a_0, \quad v(t) = v_0 + a_0 t, \quad x(t) = x_0 + v_0 t + 0.5 a_0 t^2$$

SOLVE: First we solve for the elapsed time for Bill to go from O to B (radial motion):

$$40\,\text{ft} = (5\,\text{ft/s})t + 0.5(2\,\text{ft/s}^2)t^2$$

which has solution $t = 4.30\,\text{s}$. Evaluating the radial speed corresponding to this time gives us $\dot{r} = 5\,\text{ft/s} + (2\,\text{ft/s}^2)(4.30\,\text{s}) = 13.6\,\text{ft/s}$. At $t = 4.30\,\text{s}$ the carousel is rotating at $0.6\,\text{rad/s} - (0.1\,\text{rad/s}^2)(4.30\,\text{s}) = 0.170\,\text{rad/s}$.

(1) $\Rightarrow$
$$\vec{v}_{Bill} = [13.6\vec{e}_r + (40\,\text{ft})(0.170\,\text{rad/s})\vec{e}_\theta]\,\text{ft/s}$$

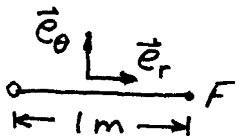
$$\boxed{\vec{v}_{Bill} = (13.6\vec{e}_r + 6.80\vec{e}_\theta)\,\text{ft/s}}$$

(2) $\Rightarrow$

$$\vec{a}_{Bill} = [(2\,\text{ft/s}^2 - (40\,\text{ft})(0.170\,\text{rad/s})^2)\vec{e}_r + ((40\,\text{ft})(-0.1\,\text{rad/s}^2) + 2(13.6\,\text{ft/s})(0.170\,\text{rad/s}))\vec{e}_\theta]\,\text{ft/s}^2$$

$$\boxed{\vec{a}_{Bill} = (0.845\vec{e}_r + 0.6225\vec{e}_\theta)\,\text{ft/s}^2}$$

2.3.2
GOAL: Find the acceleration of a fly on a fan blade.
GIVEN: Fan dimensions and rotation rate.
DRAW:



FORMULATE EQUATIONS:

$$\vec{a}_F = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta \quad (1)$$

ASSUME: The fly isn't moving outward and therefore $\dot{r} = \ddot{r} = 0$. The rotation rate is constant and therefore $\ddot{\theta} = 0$.
SOLVE: We're told that the fan blade rotates 2π radians in 1.3 seconds. Thus

$$\dot{\theta} = \frac{2\pi}{1.3}\,\text{rad/s}$$

$$\vec{a}_F = -r\dot{\theta}^2\vec{e}_r = -(1\,\text{m})\left(\frac{2\pi}{1.3}\,\text{rad/s}\right)^2\vec{e}_r$$

$$\boxed{\vec{a}_F = -23.4\vec{e}_r\,\text{m/s}^2}$$

2.3.3
GOAL: Find an example for which

$$\frac{d|\vec{r}_A|}{dt} = 0$$

and

$$|\vec{v}_A| = 1$$

FORMULATE EQUATIONS:

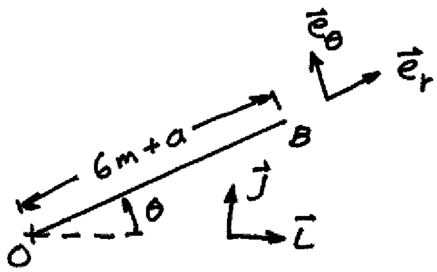
$$\vec{v}_A = \dot{r}_A \vec{e}_r + r_A \dot{\theta} \vec{e}_\theta \tag{1}$$

SOLVE: We must have $\frac{d|\vec{r}_A|}{dt} = 0$ and $|\vec{v}_A| = 1$. The first relation says r_A is constant ($\dot{r}_A = 0$). This means A must be moving in a circle about the origin and eliminates the $\dot{r}_A$ of (1), leaving us with

$$\vec{v}_A = r_A \dot{\theta} \vec{e}_\theta$$

$|\vec{v}_A| = 1$ implies the speed of rotation is constant ($|r\dot{\theta}| = 1$).

Do It 2.3.2
GOAL: Find the velocity and acceleration of the tip of a fire ladder.
GIVEN: System dimensions and rates.
DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

FORMULATE EQUATIONS:

$$\vec{v}_B = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

$$\vec{a}_B = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta \tag{2}$$

SOLVE:

(1) $\Rightarrow$
$$\vec{v}_B = (0.4\,\mathrm{m/s})\vec{e}_r + (8\,\mathrm{m})(0.05\,\mathrm{rad/s})\vec{e}_\theta$$

$$\boxed{\vec{v}_B = (0.4\vec{e}_r + 0.4\vec{e}_\theta)\,\mathrm{m/s}}$$

(2) $\Rightarrow$
$$\vec{a}_B = [0.1\,\mathrm{m/s^2} - (8\,\mathrm{m})(0.05\,\mathrm{rad/s})^2]\vec{e}_r + [(8\,\mathrm{m})(0.04\,\mathrm{rad/s^2}) + 2(0.4\,\mathrm{m/s})(0.05\,\mathrm{rad/s})]\vec{e}_\theta$$

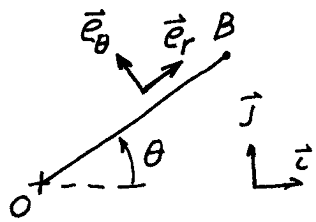
$$\boxed{\vec{a}_B = (0.08\vec{e}_r + 0.36\vec{e}_\theta)\,\mathrm{m/s^2}}$$

Using $\theta = \frac{\pi}{6}$ rad and the coordinate transformation array gives us

$$\boxed{\vec{v}_B = (0.146\vec{\imath} + 0.546\vec{\jmath})\,\mathrm{m/s}}$$

$$\boxed{\vec{a}_B = (-0.111\vec{\imath} + 0.352\vec{\jmath})\,\mathrm{m/s^2}}$$

2.3.4
GOAL: Find $|v_B|$ and $|a_B|$.
GIVEN: System geometry and rates
DRAW



FORMULATE EQUATIONS:

$$\vec{v}_B = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

$$\vec{a}_B = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta \tag{2}$$

SOLVE:

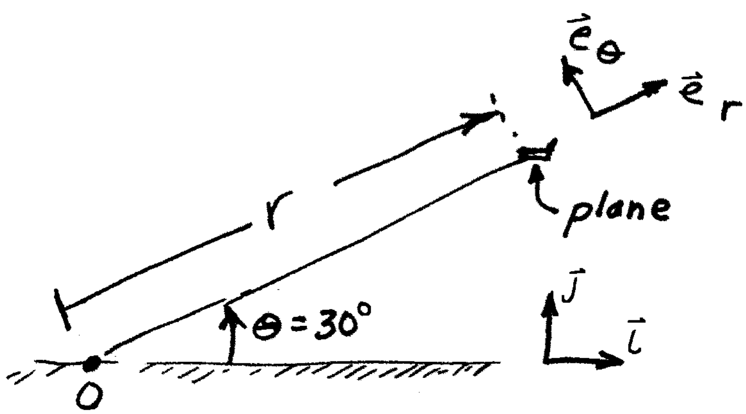
$$\vec{v}_B = (-0.0549 \text{ m/s})\vec{e}_r + (0.3 \text{ m})(-0.683 \text{ rad/s})\vec{e}_\theta$$

$$\boxed{|v_B| = \sqrt{0.0549^2 + 0.205^2} \text{ m/s} = 0.212 \text{ m/s}}$$

$$\vec{a}_B = [0.0375 \text{ m/s}^2 - (0.3 \text{ m})(-0.683 \text{ rad/s})^2]\vec{e}_r$$
$$+[(0.3 \text{ m})(-0.1585 \text{ rad/s}^2) + 2(-0.0549)(-0.683 \text{ rad/s})]\vec{e}_\theta$$
$$= [-0.102\vec{e}_r + 0.0274\vec{e}_\theta]\text{m/s}^2$$

$$\boxed{|a_B| = 0.106 \text{ m/s}^2}$$

2.3.5
GOAL:
A particle's velocity is given. Express the velocity in polar coordinates and determine $\dot{\theta}$.
GIVEN: System geometry and aircraft's speed.
DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\sqrt{3}/2$	$1/2$
$\vec{e}_\theta$	$-1/2$	$\sqrt{3}/2$

FORMULATE EQUATIONS:

$$\begin{aligned}\vec{v}_{plane} &= -1200\vec{\imath}\ \text{km/hr} = -333.\bar{3}\vec{\imath}\ \text{m/s} \\ &= -333.\bar{3}\ \left(\frac{\sqrt{3}}{2}\vec{e}_r - \frac{1}{2}\vec{e}_\theta\right)\ \text{m/s} \\ &= (-288.7\vec{e}_r + 166.\bar{6}\vec{e}_\theta)\,\text{m/s} \qquad (1)\end{aligned}$$

$\vec{v}_{plane}$ is also expressable as

$$\vec{v}_{plane} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \qquad (2)$$

SOLVE:

$(1), (2) \Rightarrow$

$$\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta = (-288.7\vec{e}_r + 166.\bar{6}\vec{e}_\theta)\,\text{m/s}$$
$$\dot{r}\vec{e}_r + (3000\,\text{m})\dot{\theta}\vec{e}_\theta = (-288.7\vec{e}_r + 166.\bar{6}\vec{e}_\theta)\,\text{m/s}$$

$\vec{e}_r$:

$$\dot{r} = -288.7\,\text{m/s} \qquad (3)$$

$\vec{e}_\theta$:

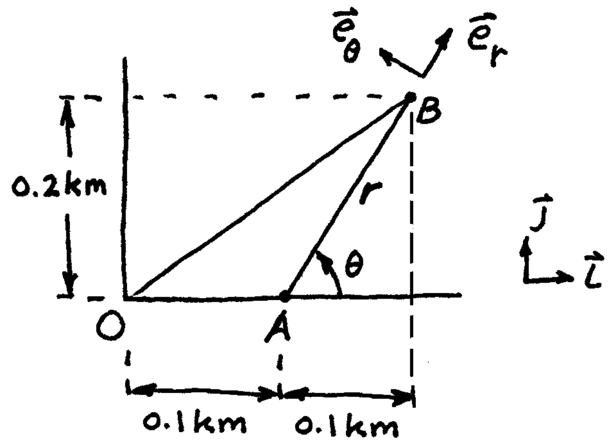
$$(3000\,\text{m})\dot{\theta} = 166.\bar{6}\,\text{m/s} \qquad (4)$$

$(4) \Rightarrow$

$$\boxed{\dot{\theta} = \frac{166.\bar{6}\,\text{m/s}}{3000\,\text{m}} = .05\ \text{rad/s}}$$

2.3.6

GOAL: Find the radial speed of car at B relative to point A in terms of v.
GIVEN: Dimensions to point A, dimensions to point B, $\angle AOB = 45°$
DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\cos 63.4°$	$\sin 63.4°$
$\vec{e}_\theta$	$-\sin 63.4°$	$\cos 63.4°$

FORMULATE EQUATIONS:

$$\vec{v}_B = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

SOLVE:

$$\theta = \arctan\frac{0.2\,\text{km}}{0.1\,\text{km}} = 63.4°$$

$$\vec{v}_B(\text{ expressed in } \vec{e}_r, \vec{e}_\theta) = \vec{v}_B(\text{ expressed in } \vec{\imath}, \vec{\jmath})$$

$$\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta = -v\frac{\sqrt{2}}{2}\vec{\imath} - v\frac{\sqrt{2}}{2}\vec{\jmath}$$

$$\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta = -v\frac{\sqrt{2}}{2}(\cos 63.4°\vec{e}_r - \sin 63.4°\vec{e}_\theta) - v\frac{\sqrt{2}}{2}(\sin 63.4°\vec{e}_r + \cos 63.4°\vec{e}_\theta)$$

$\vec{e}_r$:

$$\dot{r} = -v\frac{\sqrt{2}}{2}(\cos 63.4°) - v\frac{\sqrt{2}}{2}(\sin 63.4°) = -.949v$$

$$|\dot{r}_{B/A}| = |\dot{r}| = .949v$$

$$\boxed{|\dot{r}_{B/A}| = .949v}$$

2.3.7

GOAL: Why is $||\frac{d\vec{r}(t)}{dt}||$ generally not equal to $\frac{d}{dt}||\vec{r}(t)||$?

GIVEN: Two expressions involving a time derivative of $\vec{r}$.

FORMULATE EQUATIONS:

$$\vec{r}(t) = \vec{r} = r\vec{e}_r$$

polar velocity

$$\vec{v}_p = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

SOLVE:

note that all variables represent a function of time.

consider $\frac{d}{dt}||\vec{r}(t)||$:

$$||\vec{r}|| = ||r\vec{e}_r|| = r$$

$$\frac{d}{dt}||\vec{r}(t)|| = \frac{d}{dt}r = \dot{r}$$

now consider $||\frac{d\vec{r}(t)}{dt}||$:

$$\frac{d\vec{r}(t)}{dt} = \vec{v}_p = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$||\frac{d\vec{r}(t)}{dt}|| = ||\vec{v}_p|| = \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$$

$$\boxed{\dot{r} \neq \sqrt{\dot{r}^2 + (r\dot{\theta})^2}}$$

$||\frac{d\vec{r}(t)}{dt}||$ represents the magnitude of the polar velocity function, or the actual speed of a particle described over time. This speed compensates for both the speed at which a particle moves away from the origin and the speed at which a particle moves around the origin. $\frac{d}{dt}||\vec{r}(t)||$ is $\dot{r}$ or the radial speed describing the speed at which a particle moves away from the origin, but not around it.

2.3.8

GOAL: Determine when $\frac{d}{dt}|\vec{r}(t)|$ is equal to $|\frac{d}{dt}(\vec{r}(t))|$.

GIVEN: Two expressions involving $\vec{r}$.

FORMULATE EQUATIONS:

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

SOLVE:

$|\vec{r}|$ is the magnitude of $\vec{r}$, namely, r.

$\frac{dr}{dt}$ is the speed at which r changes, namely, $\dot{r}$.

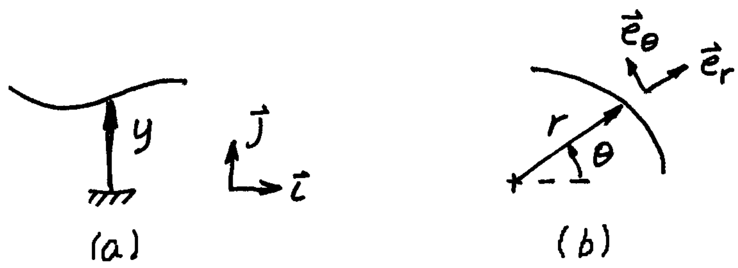
$\frac{d\vec{r}}{dt}$ is the velocity of the vector $\vec{r}$, including $\dot{r}$ and $r\theta$ terms:

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$$

$\dot{r}$ is equal to $\sqrt{\dot{r}^2 + (r\dot{\theta})^2}$ only when $\dot{\theta} = 0$. Thus, the particle must be moving outward but not around the origin. $\dot{r}$ has to be positive (moving away from the origin) for the signs to match.

2.3.9
GOAL: Determine if $\frac{d}{dt}(\underset{\sim}{r})$ is zero.
GIVEN: $\dot{r} = 3\,\text{ft/s}$ and $r\dot{\theta} = -3\,\text{ft/s}$.
FORMULATE EQUATIONS:
We'll use $\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$.
SOLVE:
Although $\dot{r}$ and $r\dot{\theta}$ are the magnitudes of the components of $\frac{d}{dt}(\underset{\sim}{r})$, these components are in orthogonal directions. $\frac{d}{dt}\vec{r}$ would equal $(3\vec{e}_r - 3\vec{e}_\theta)\,\text{ft/s}$, with a resultant magnitude of $3\sqrt{2}\,\text{ft/s}$. Thus the final answer to the question is "no."

2.3.10
GOAL: Compare the effect on acceleration of a particle moving horizontally as opposed to one moving in a circular manner, both with a sinusoidal variation of path.
GIVEN: In case (a) $y = r_0 - r_1 \cos(2\omega t)$ and in case (b) $\theta = 2\omega t$, $r = r_0 - r_1 \cos(\theta)$.
DRAW:



FORMULATE EQUATIONS: Acceleration in Cartesian and polar frames are given by

$$\vec{a}_A = \ddot{y}\vec{\jmath}$$

$$\vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

SOLVE:
Case (a): We can differentiate our expression for y twice, yielding

$$\boxed{\ddot{y} = 4\omega^2 r_1 \cos(2\omega t)}$$

Case (b): We can use our expression for θ in our expression for r, giving us

$$r = r_0 - r_1 \cos(2\omega t)$$

Now we need to consider the component of acceleration in the $\vec{e}_r$ direction:

$$a_r = \ddot{r} - r\dot{\theta}^2$$

Differentiating our expression for r twice yields

$$\ddot{r} = 4\omega^2 r_1 \cos(2\omega t)$$

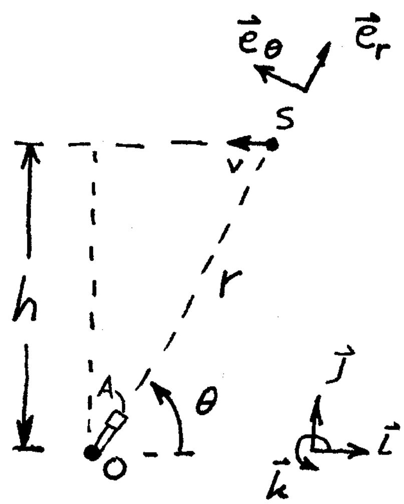
and differentiating θ gives us $\dot{\theta} = 2\omega$.
Put these results into our expression for a_r and we have

$$a_r = 4\omega^2 r_1 \cos(2\omega t) - 4\omega^2[r_0 - r_1 \cos(2\omega t)]$$

$$\boxed{a_r = -4r_0\omega^2 + 8\omega^2 r_1 \cos(2\omega t)}$$

We see two basic things. First, the fact that the particle in Case (b) is, on average, moving around a circle, there is a constant centripetal component $-4r_0\omega^2$. Interestingly, in addition to this, there is a further increase in the time varying acceleration as compared to the horizontal scenario (Case (a)). Rather than an acceleration magnitude of $4\omega^2 r_1$, the rotational case has a magnitude that's 100% larger: $8\omega^2 r_1$.

2.3.11
GOAL: Determine the angular speed of rotation needed to keep the image of a bird centered in a camera's viewfinder and calculate the total acceleration of the far end of the lens.
GIVEN: Distance to the end of the lens from the point of rotation is 17 inches. The viewfinder displays a 40 inch wide target when the target is 80 feet away. The swallow moves at a constant 40 mph along a straight line. The closest approach of the swallow to the camera is $h = 80\,\text{ft}$.
DRAW:



FORMULATE EQUATIONS: The equations for velocity and acceleration in polar coordinates are

$$\vec{v}_S = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

and

$$\vec{a}_S = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta \tag{2}$$

SOLVE: 40 mph is equivalent to 58.7 ft/s. At $\theta = 90^\circ$ $\vec{e}_r = \vec{\jmath}$ and $\vec{e}_\theta = -\vec{\imath}$. Using the provided parameters in (1) gives us

$$\vec{v}_S = -58.7\vec{\imath}\ \text{ft/s} = \left[\dot{r}\vec{e}_r + (80\,\text{ft})\dot{\theta}\vec{e}_\theta\right]$$

$$\dot{r} = 0, \quad \dot{\theta} = \frac{58.7\,\text{ft/s}}{80\,\text{ft}} = 0.733\,\text{rad/s}$$

$$\boxed{\dot{\theta} = 0.733\,\text{rad/s}}$$

Using this data along with the provided parameters in (2) yields

$$\vec{a}_S = 0 = \left[\ddot{r} - (80\,\text{ft})(0.733\,\text{rad/s})^2\right]\vec{e}_r + \left[2(0)(0.733\,\text{rad/s}) + (80\,\text{ft})\ddot{\theta}\right]\vec{e}_\theta$$

$$\ddot{r} = 43.0\,\text{ft/s}^2, \quad \ddot{\theta} = 0$$

Our equation for the acceleration of end of the lens is a bit different than the swallow's acceleration because the swallow is moving in a straight line whereas the lens is rotating in a circle. Hence for the lens both $\dot{r}$ and $\ddot{r}$ are zero. The acceleration is found from

$$\vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta = -\left(\frac{17\,\text{in}}{12\,\text{in/ ft}}\right)(0.733\,\text{rad/s})^2 + 0 = -0.762\vec{\jmath}\ \text{ft/s}^2$$

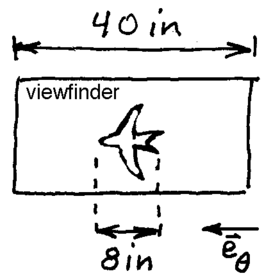
$$\boxed{\vec{a}_A = -0.762\vec{\jmath}\ \text{ft/s}^2}$$

2.3.12

GOAL: Determine the constant angular velocity of the camera that will cause the image of a swallow to disappear from the viewfinder in 0.5 s. Compare this angular speed to the angular speed needed to perfectly track the bird and determine the percent variation from the ideal case.

GIVEN: The camera's telephoto lens captures 40 inches of target image when the object is 80 feet away. The swallow is 8 inches long and initially centered. The bird is at $\theta = 90°$ and traveling at a constant 40 mph with a constant radius of 80 ft from the camera.

DRAW:



FORMULATE EQUATIONS: The equation for velocity in polar coordinates is

$$\vec{v}_m = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

SOLVE: Using the provided parameters in (1) gives us a nominal rotation rate:

$$\vec{v}_S = \frac{40\,\text{mph}}{60\,\text{mph}/(88\,\text{ft/s})}\vec{e}_\theta = 58.7\vec{e}_\theta\,\text{ft/s} = (80\,\text{ft})\dot{\theta}\vec{e}_\theta \;\Rightarrow\; \dot{\theta}_{\text{nominal}} = 0.733\,\text{rad/s}$$

If the camera is being rotated too slowly, the bird's image will move to the left in the viewfinder. It's initially centered and so in order to completely disappear from the viewfinder it needs to move $16\,\text{in} + 8\,\text{in} = 24\,\text{in} = 2\,\text{ft}$. The time for this to occur is given as 0.5 s. This implies a speed of $2\,\text{ft}/(0.5\,\text{s}) = 4\,\text{ft/s}$. The corresponding angular speed is

$$\dot{\theta} = \frac{4\,\text{ft/s}}{80\,\text{ft}} = 0.05\,\text{rad/s}$$

Thus the angular rate of the camera would be the nominal rate $\pm$ this variation. Assuming a camera pan that's too slow (the usual problem) we have $\dot{\theta} = 0.733\,\text{rad/s} - 0.05\,\text{rad/s} = 0.683\,\text{rad/s}$.

$$\boxed{\dot{\theta} = 0.683\,\text{rad/s}}$$

The percent variation from nominal is given by $100\left(\frac{0.05}{0.733}\right)$:

$$\boxed{\text{percent variation} = 6.8\%}$$

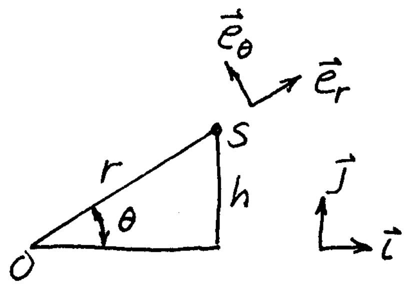
The implication is that it's not easy to keep a quickly moving object in the viewfinder, an observation that's very much supported by experience in the field.

2.3.13

GOAL: Calculate at what angle θ the camera's angular acceleration is a maximum and determine its angular acceleration at $\theta = 0$ and $\theta = 90°$.

GIVEN: The swallow moves at a constant 40 mph along a straight line. The closest approach of the swallow to the camera is $h = 80\,\text{ft}$.

DRAW:



$$\begin{array}{c|c|c|}
 & \vec{\imath} & \vec{\jmath} \\
\hline
\vec{e}_r & \cos\theta & \sin\theta \\
\hline
\vec{e}_\theta & -\sin\theta & \cos\theta \\
\hline
\end{array}$$

FORMULATE EQUATIONS: The equations for velocity and acceleration in polar coordinates are

$$\vec{v}_S = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

and

$$\vec{a}_S = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta \tag{2}$$

SOLVE:

(1) $\Rightarrow$
$$-v\vec{\imath} = -v[\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta] = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$\vec{e}_r$:
$$\dot{r} = -v\cos\theta \tag{3}$$

$\vec{e}_\theta$:
$$\dot{\theta} = \frac{v\sin\theta}{r} \tag{4}$$

(2) $\Rightarrow$
$$0 = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta$$

$\vec{e}_r$:
$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{5}$$

$\vec{e}_\theta$:
$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{6}$$

(3), (4), (6) $\Rightarrow$
$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} = \frac{2v^2\sin\theta\cos\theta}{r^2}$$

From geometry we see that $r = \frac{h}{\sin\theta}$ and hence we have

$$\ddot{\theta} = \frac{2v^2\sin^3\theta\cos\theta}{h^2} \tag{7}$$

We find where $\ddot{\theta}$ is maximized (or minimized) by setting $\frac{d\,\ddot{\theta}}{d\,\theta} = 0$:

$$\frac{d\,\ddot{\theta}}{d\,\theta} = \frac{6\sin^2\theta\cos^2\theta - 2\sin^4\theta}{h^2} = 0$$

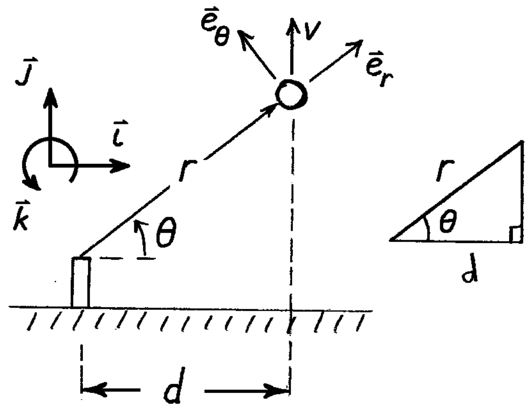
$$3\cos^2\theta = \sin^2\theta \;\Rightarrow\; \tan^2\theta = 3 \;\Rightarrow\; \tan\theta = \sqrt{3}$$

$$\boxed{\theta = 60^\circ}$$

Evaluating (7) at $\theta = 0$ gives us the same answer as for $\theta = 90^\circ$:

$$\boxed{\ddot{\theta} = 0}$$

2.3.14
GOAL: Determine r, $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$.
GIVEN: System geometry and battleship velocity: $d = 300\,\text{m}$, $\theta = 50°$, $\vec{v} = 25\vec{\jmath}$ m/s.
DRAW:



FORMULATE EQUATIONS:
The coordinate transformation array is

	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

The battleship velocity in terms of polar coordinates is

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

The battleship acceleration in terms of polar coordinates is

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta \tag{2}$$

SOLVE:
From the diagram we see that

$$\begin{aligned} d &= r\cos\theta \\ r &= \frac{d}{\cos\theta} \\ r &= \frac{300\,\text{m}}{\cos(50°)} \end{aligned}$$

$$\boxed{r = 466.7\,\text{m}} \tag{3}$$

To find $\dot{r}$ and $\dot{\theta}$,

(1) $\Rightarrow$ $$v\vec{\jmath} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$v \sin\theta \vec{e}_r + v\cos\theta \vec{e}_\theta = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$\vec{e}_r$:

$$\dot{r} = v\sin\theta$$

$$\dot{r} = (25\,\mathrm{m/s})\sin(50^\circ)$$

$$\boxed{\dot{r} = 19.15\,\mathrm{m/s}} \tag{4}$$

$\vec{e}_\theta$:

$$r\dot{\theta} = v\cos\theta$$

$$\dot{\theta} = \frac{v\cos\theta}{r} \tag{5}$$

(3) → (5) ⇒

$$\dot{\theta} = \frac{(25\,\mathrm{m/s})\cos(50^\circ)}{466.7\,\mathrm{m}}$$

$$\boxed{\dot{\theta} = 3.44\times 10^{-2}\,\mathrm{rad/s}} \tag{6}$$

To find $\ddot{r}$ and $\ddot{\theta}$,

(2) ⇒

$$\vec{0} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

$\vec{e}_r$:

$$\ddot{r} - r\dot{\theta}^2 = 0$$

$$\ddot{r} = r\dot{\theta}^2 \tag{7}$$

(3), (6) → (7) ⇒

$$\ddot{r} = (466.7\,\mathrm{m})(3.44\times 10^{-2}\,\mathrm{rad/s})^2$$

$$\boxed{\ddot{r} = 0.553\,\mathrm{m/s^2}}$$

$\vec{e}_\theta$:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

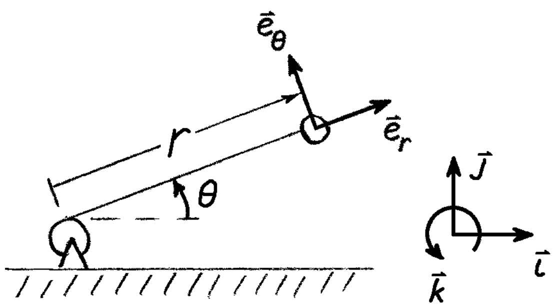
$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} \tag{8}$$

(3), (4), (6) → (8) ⇒

$$\ddot{\theta} = -\frac{2(19.15\,\mathrm{m/s})(3.44\times 10^{-2}\,\mathrm{rad/s})}{(466.7\,\mathrm{m})}$$

$$\boxed{\ddot{\theta} = -2.83\times 10^{-3}\,\mathrm{rad/s^2}}$$

2.3.15
GOAL: Determine $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$.
GIVEN: System geometry, and the exerciser's foot velocity and acceleration: $r = 2\,\text{ft}$, $\theta = 15°$, $\vec{v} = 4.33\vec{\imath} + 2.5\vec{\jmath}\ \text{ft/s}$, $\vec{a} = 0.87\vec{\imath} + 0.5\vec{\jmath}\ \text{ft/s}^2$.
DRAW:



FORMULATE EQUATIONS:
The coordinate transformation array is

	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

The velocity of the exerciser's foot in terms of polar coordinates is

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

The acceleration of the exerciser's foot in terms of polar coordinates is

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta \tag{2}$$

SOLVE:
To find $\dot{r}$ and $\dot{\theta}$,

(1) $\Rightarrow$
$$v_x\vec{\imath} + v_y\vec{\jmath} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$v_x(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta) + v_y(\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta) = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$\vec{e}_r$:
$$\dot{r} = v_x\cos\theta + v_y\sin\theta$$

$$\dot{r} = (4.33\,\text{ft/s})\cos(15°) + (2.5\,\text{ft/s})\sin(15°)$$

$$\boxed{\dot{r} = 4.830\,\text{ft/s}} \tag{3}$$

$\vec{e}_\theta$:
$$r\dot{\theta} = -v_x\sin\theta + v_y\cos\theta$$

$$\dot{\theta} = \frac{1}{r}[-v_x \sin\theta + v_y \cos\theta]$$

$$\dot{\theta} = \frac{1}{2\,\text{ft}}[-(4.33\,\text{ft/s})\sin(15°) + (2.5\,\text{ft/s})\cos(15°)]$$

$$\boxed{\dot{\theta} = 0.647\,\text{rad/s}} \tag{4}$$

To find $\ddot{r}$ and $\ddot{\theta}$,

(2) $\Rightarrow$

$$a_x\vec{\imath} + a_y\vec{\jmath} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

$$a_x(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta) + a_y(\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta) = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

$\vec{e}_r$:

$$\ddot{r} - r\dot{\theta}^2 = a_x\cos\theta + a_y\sin\theta$$

$$\ddot{r} = r\dot{\theta}^2 + a_x\cos\theta + a_y\sin\theta \tag{5}$$

(4) $\rightarrow$ (5) $\Rightarrow$

$$\ddot{r} = (2\,\text{ft})(0.647\,\text{rad/s})^2 + (0.87\,\text{ft/s}^2)\cos(15°) + (0.5\,\text{ft/s}^2)\sin(15°)$$

$$\boxed{\ddot{r} = 1.807\,\text{ft/s}^2}$$

$\vec{e}_\theta$:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = -a_x\sin\theta + a_y\cos\theta$$

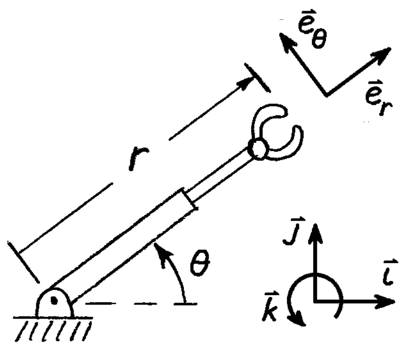
$$\ddot{\theta} = \frac{1}{r}[-2\dot{r}\dot{\theta} - a_x\sin\theta + a_y\cos\theta] \tag{6}$$

(3), (4) $\rightarrow$ (6) $\Rightarrow$

$$\ddot{\theta} = \frac{1}{2\,\text{ft}}[-2(4.830\,\text{ft/s})(0.647\,\text{rad/s}) - (0.87\,\text{ft/s}^2)\sin(15°) + (0.5\,\text{ft/s}^2)\cos(15°)]$$

$$\boxed{\ddot{\theta} = -2.996\,\text{rad/s}^2}$$

2.3.16
GOAL: Determine $\|\vec{v}\|$ and $\|\vec{a}\|$.
GIVEN: The radial position of the robotic arm's end effector is governed by $r(\theta) = 0.5 + 0.25[1 - \cos(2\theta)]$ m. At the given instant, $\theta = 45°$, $\dot{\theta} = 0.5$ rad/s, $\ddot{\theta} = 0$.
DRAW:



FORMULATE EQUATIONS:
The velocity of the end effector in terms of polar coordinates is

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{1}$$

The acceleration of the end effector in terms of polar coordinates is

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta \tag{2}$$

SOLVE:

$$r(45°) = 0.5 + 0.25[1 - \cos(2 \cdot 45°)]\,\text{m}$$

$$\boxed{r(45°) = 0.75\,\text{m}} \tag{3}$$

Differentiate $r(\theta) \Rightarrow$

$$\dot{r}(\theta) = 0.5\sin(2\theta)\dot{\theta}\,\text{m/s}$$

$$\dot{r}(45°) = 0.5\sin(2 \cdot 45°)(0.5)\,\text{m/s}$$

$$\boxed{\dot{r}(45°) = 0.25\,\text{m/s}} \tag{4}$$

Differentiate $\dot{r}(\theta) \Rightarrow$

$$\ddot{r}(\theta) = \cos(2\theta)\dot{\theta}^2 + 0.5\sin(2\theta)\ddot{\theta}\,\text{m/s}^2$$

$$\ddot{r}(45°) = \cos(2 \cdot 45°)(0.5)^2 + 0\,\text{m/s}^2$$

$$\boxed{\ddot{r}(45°) = 0\,\text{m/s}^2} \tag{5}$$

(1) $\Rightarrow$

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} \tag{6}$$

(3), (4) $\rightarrow$ (6) $\Rightarrow$

$$\|\vec{v}\| = \sqrt{(0.25\,\text{m/s})^2 + (0.75\,\text{m})^2(0.5\,\text{rad/s})^2}$$

$$\boxed{\|\vec{v}\| = 0.451\,\text{m/s}}$$

(2) $\Rightarrow$
$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta} + r\ddot{\theta})^2} \tag{7}$$

(3), (4), (5) $\rightarrow$ (7) $\Rightarrow$
$$\|\vec{a}\| = \sqrt{\left\{0 - (0.75\,\mathrm{m})(0.5\,\mathrm{rad/s})^2\right\}^2 + \left\{2(0.25\,\mathrm{m/s})(0.5\,\mathrm{rad/s}) + 0\right\}^2}$$

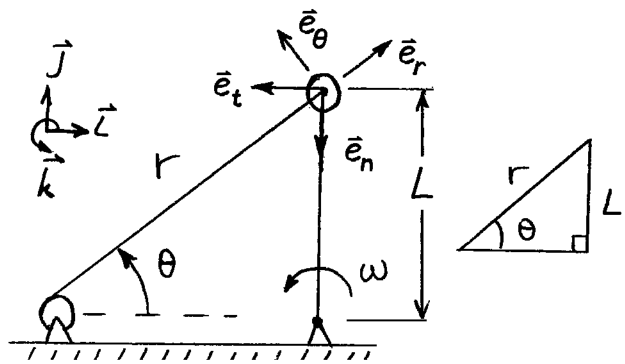
$$\boxed{\|\vec{a}\| = 0.313\,\mathrm{m/s^2}}$$

2.3.17

GOAL: Determine r, $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$.

GIVEN: System geometry and the pole's constant angular speed: $L = 20\,\text{ft}$, $\theta = 40°$, $\omega = 0.25\,\text{rad/s}$.

DRAW:



ASSUME: Neglect the mass of the pole, and analyze only the lumped mass on the end.

FORMULATE EQUATIONS:

The coordinate transformation array is

	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

With respect to the pole, the lumped mass's velocity and acceleration are, respectively,

$$\vec{v} = L\omega\vec{e}_t = -L\omega\vec{\imath} \tag{1}$$

$$\vec{a} = L\omega^2\vec{e}_n = -L\omega^2\vec{\jmath} \tag{2}$$

The velocity and acceleration of the lumped mass in terms of polar coordinates are, respectively,

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{3}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta \tag{4}$$

SOLVE:

From the diagram we see that

$$\begin{aligned} L &= r\sin\theta \\ r &= \frac{L}{\sin\theta} \\ r &= \frac{20\,\text{ft}}{\sin(40°)} \end{aligned}$$

$$\boxed{r = 31.11\,\text{ft}} \tag{5}$$

To find $\dot{r}$ and $\dot{\theta}$,

(1) = (3) $\Rightarrow$
$$-L\omega\vec{\imath} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$-L\omega(\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta) = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$\vec{e}_r$:
$$\dot{r} = -L\omega\cos\theta$$

$$\dot{r} = -(20\,\mathrm{ft})(0.25\,\mathrm{rad/s})\cos(40°)$$

$$\boxed{\dot{r} = -3.830\,\mathrm{ft/s}} \tag{6}$$

$\vec{e}_\theta$:
$$r\dot{\theta} = L\omega\sin\theta$$

$$\dot{\theta} = \frac{L\omega}{r}\sin\theta \tag{7}$$

(5) $\to$ (7) $\Rightarrow$
$$\dot{\theta} = \frac{(20\,\mathrm{ft})(0.25\,\mathrm{rad/s})}{31.11\,\mathrm{ft}}\sin(40°)$$

$$\boxed{\dot{\theta} = 0.103\,\mathrm{rad/s}} \tag{8}$$

To find $\ddot{r}$ and $\ddot{\theta}$,

(2) = (4) $\Rightarrow$
$$-L\omega^2\vec{\jmath} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

$$-L\omega^2(\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta) = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

$\vec{e}_r$:
$$\ddot{r} - r\dot{\theta}^2 = -L\omega^2\sin\theta$$

$$\ddot{r} = r\dot{\theta}^2 - L\omega^2\sin\theta \tag{9}$$

(5), (8) $\to$ (9) $\Rightarrow$
$$\ddot{r} = (31.11\,\mathrm{ft})(0.103\,\mathrm{rad/s})^2 - (20\,\mathrm{ft})(0.25\,\mathrm{rad/s})^2\sin(40°)$$

$$\boxed{\ddot{r} = -0.472\,\mathrm{ft/s^2}}$$

$\vec{e}_\theta$:
$$2\dot{r}\dot{\theta} + r\ddot{\theta} = -L\omega^2\cos\theta$$

$$\ddot{\theta} = -\frac{1}{r}[2\dot{r}\dot{\theta} + L\omega^2\cos\theta] \tag{10}$$

(5), (6), (8) $\to$ (10) $\Rightarrow$
$$\begin{aligned}\ddot{\theta} \;&=\; -\frac{1}{31.11\,\mathrm{ft}}[2(-3.830\,\mathrm{ft/s})(0.103\,\mathrm{rad/s}) \\ &\quad + (20\,\mathrm{ft})(0.25\,\mathrm{rad/s})^2\cos(40°)]\end{aligned}$$

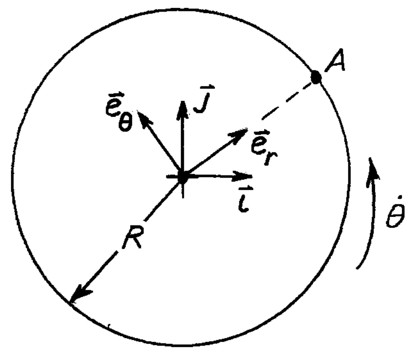
$$\boxed{\ddot{\theta} = -5.34\times10^{-3}\,\mathrm{rad/s^2}}$$

Do It 2.3.3

GOAL: Determine $\|\vec{v}_A\|$ and $\|\vec{a}_A\|$.

GIVEN: The merry-go-round has radius $R = 5\,\text{ft}$ and is spun to a constant $\dot{\theta} = 3\,\text{rad/s}$. The RC toy car is accelerated from rest at the center straight out to point A at a rate $\ddot{r} = 3\,\text{ft/s}^2$.

DRAW:



FORMULATE EQUATIONS:

With respect to the rotating merry-go-round, the RC toy car's velocity at point A is given by

$$\dot{r}_A^2 = 2\ddot{r}R \tag{1}$$

The velocity of the RC car at A in terms of polar coordinates is

$$\vec{v}_A = \dot{r}_A\vec{e}_r + R\dot{\theta}\vec{e}_\theta \tag{2}$$

The acceleration of the RC car at A in terms of polar coordinates is

$$\vec{a}_A = (\ddot{r} - R\dot{\theta}^2)\vec{e}_r + (2\dot{r}_A\dot{\theta})\vec{e}_\theta \tag{3}$$

SOLVE:

(2) $\Rightarrow$
$$\|\vec{v}_A\| = \sqrt{\vec{v}_A \cdot \vec{v}_A} = \sqrt{\dot{r}_A^2 + R^2\dot{\theta}^2} \tag{4}$$

(1) $\rightarrow$ (4) $\Rightarrow$
$$\|\vec{v}_A\| = \sqrt{2\ddot{r}R + R^2\dot{\theta}^2}$$

$$\|\vec{v}_A\| = \sqrt{2(3\,\text{ft/s}^2)(5\,\text{ft}) + (5\,\text{ft})^2(3\,\text{rad/s})^2}$$

$$\boxed{\|\vec{v}_A\| = 15.97\,\text{ft/s}}$$

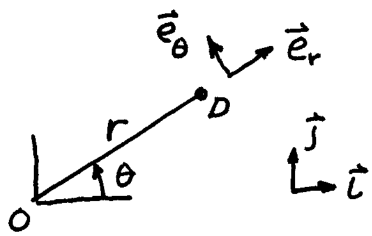
(3) $\Rightarrow$
$$\|\vec{a}_A\| = \sqrt{\vec{a}_A \cdot \vec{a}_A} = \sqrt{(\ddot{r} - R\dot{\theta}^2)^2 + 4\dot{r}_A^2\dot{\theta}^2} \tag{5}$$

(1) $\rightarrow$ (5) $\Rightarrow$
$$\|\vec{a}_A\| = \sqrt{(\ddot{r} - R\dot{\theta}^2)^2 + 8\ddot{r}R\dot{\theta}^2}$$

$$\|\vec{a}_A\| = \sqrt{\left((3\,\text{ft/s}^2) - (5\,\text{ft})(3\,\text{rad/s})^2\right)^2 + 8(3\,\text{ft/s}^2)(5\,\text{ft})(3\,\text{rad/s})^2}$$

$$\boxed{\|\vec{a}_A\| = 53.33\,\text{ft/s}^2}$$

2.3.18
GOAL: Find $\dot{r}, \ddot{r}, \dot{\theta}, \ddot{\theta}$ of the dolphin.
GIVEN: $r = 300\,\text{m}, v_r = 10\,\text{m/s}, v_\theta = 17.3\,\text{m/s}, a_r = 0\,\text{m/s}^2, a_\theta = 0\,\text{m/s}^2$
DRAW:



FORMULATE EQUATIONS:

polar velocity
$$\vec{v}_p = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$
$$\vec{v}_p = v_r\vec{e}_r + v_\theta\vec{e}_\theta$$

polar acceleration
$$\vec{a}_p = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$
$$\vec{a}_p = a_r\vec{e}_r + a_\theta\vec{e}_\theta$$

SOLVE:

v_r :
$$v_r = \dot{r} = 10\,\text{m/s}$$

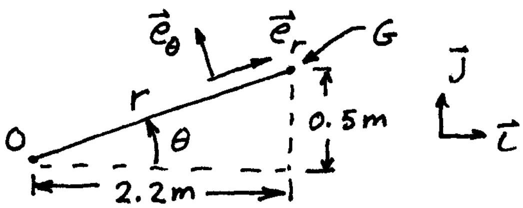
v_θ :
$$v_\theta = r\dot{\theta}$$
$$\dot{\theta} = \frac{v_\theta}{r} = \frac{17.3\,\text{m/s}}{300\,\text{m}} = 0.058\ \text{rad/s}$$

a_r :
$$a_r = \ddot{r} - r\dot{\theta}^2 = 0\,\text{m/s}^2$$
$$\ddot{r} = r\dot{\theta}^2 = (300\,\text{m})(0.058\ \text{rad/s})^2 = 0.998\,\text{m/s}^2$$

a_θ :
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0\,\text{m/s}^2$$
$$\ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r} = \frac{(-2)(10\,\text{m/s})(0.058\ \text{rad/s})}{300\,\text{m}} = -3.84\times10^{-3}\ \text{rad/s}^2$$

$$\boxed{\dot{r} = 10\,\text{m/s} \quad \dot{\theta} = 0.058\ \text{rad/s} \quad \ddot{r} = 0.998\,\text{m/s}^2 \quad \ddot{\theta} = -3.84\times10^{-3}\ \text{rad/s}^2}$$

2.3.19
GOAL: Find the gecko's speed.
GIVEN: System geometry and rate the thread exits the spinometer.
DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

ASSUME: The gecko moves vertically and therefore

$$\vec{v}_G = v_G \vec{\imath} \tag{1}$$

FORMULATE EQUATIONS:

$$\vec{v}_G = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{2}$$

$$\theta = \tan^{-1}\left(\frac{0.5}{2.2}\right) = 12.8^\circ \tag{3}$$

$$r = [.05^2 + 2.2^2]^{\frac{1}{2}}\,\mathrm{m} = 2.26\,\mathrm{m} \tag{4}$$

SOLVE: (1),(4) and $\dot{r} = 0.78\,\mathrm{m/s}$ give us:

$$\vec{v}_G = (0.78\,\mathrm{m/s})\vec{e}_r + (2.26\,\mathrm{m})\dot{\theta}\vec{e}_\theta = v_G\vec{\jmath}$$

$$(0.78\,\mathrm{m/s})(\cos(12.8^\circ)\vec{\imath} + \sin(12.8^\circ)\vec{\jmath}) + (2.26\,\mathrm{m})\dot{\theta}(-\sin(12.8^\circ)\vec{\imath} + \cos(12.8^\circ)\vec{\jmath}) = v_G\vec{\jmath}$$

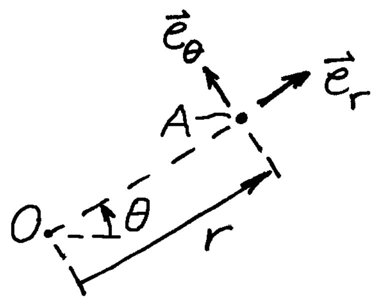
$$(0.761\,\mathrm{m/s} - 0.5\dot{\theta}\,\mathrm{m})\vec{\imath} + (0.173\,\mathrm{m/s} + 2.2\dot{\theta}\,\mathrm{m})\vec{\jmath} = v_G\vec{\jmath}$$

$$0.761\,\mathrm{m/s} - 0.5\dot{\theta}\,\mathrm{m} = 0 \Rightarrow \dot{\theta} = 1.52\,\mathrm{rad/s}$$

$$0.173\,\mathrm{m/s} + (2.2\,\mathrm{m})\dot{\theta} = v_G$$

$$\boxed{v_G = [0.173\,\mathrm{m/s} + (2.2\,\mathrm{m})(1.52\,\mathrm{rad/s})] = 3.52\,\mathrm{m/s}}$$

2.3.20
GOAL: Find magnitude of particle's velocity and acceleration.
GIVEN: Velocity of point A.
DRAW:



FORMULATE EQUATIONS:

Velocity:
$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta} \tag{1}$$

Acceleration:
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta \tag{2}$$

Differentiating equation for r:
$$\dot{r} = a - 3bt^2 \tag{3}$$

Differentiating equation for $\dot{r}$:
$$\ddot{r} = -6bt \tag{4}$$

Differentiating equation for θ:
$$\dot{\theta} = -ce^{-t} \tag{5}$$

Differentiating equation for $\dot{\theta}$:
$$\ddot{\theta} = ce^{-t} \tag{6}$$

SOLVE:

$(3),(5) \rightarrow (1) \Rightarrow$

$$\boxed{\|\vec{v}\| = \sqrt{(a - 3bt^2)^2 + (at - bt^3)^2\left(-ce^{-t}\right)^2}}$$

$(3) - (6) \rightarrow (2) \Rightarrow$

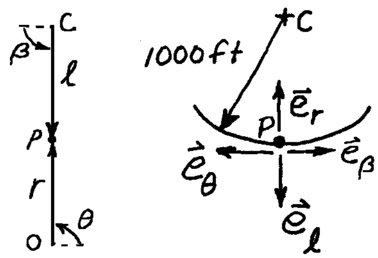
$$\boxed{\|\vec{a}\| = \sqrt{\left[-6bt - (at - bt^3)\left(-ce^{-t}\right)^2\right]^2 + \left[2\left(a - 3bt^2\right)\left(-ce^{-t}\right) + \left(at - bt^3\right)\left(ce^{-t}\right)\right]^2}}$$

2.3.21

GOAL: Determine $\ddot{r}$ for a plane in a circular loop.

GIVEN: Path of plane, position of plane and observer and speed/acceleration information.

DRAW



FORMULATE EQUATIONS: Define the velocity and acceleration in terms of the $\vec{e}_l$, $\vec{e}_\beta$ unit vectors, referenced to the center of the loop C:

$$\vec{v}_P = \dot{l}\vec{e}_l + l\dot{\beta}\vec{e}_\beta$$

$$\vec{a}_P = (\ddot{l} - l\dot{\theta}^2)\vec{e}_l + (l\ddot{\beta} + 2\dot{l}\dot{\beta})\vec{e}_\beta \tag{1}$$

as well as the $\vec{e}_r$, $\vec{e}_\theta$ unit vectors, referenced to the ground O:

$$\vec{v}_P = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$\vec{a}_P = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta \tag{2}$$

ASSUME: We're given the fact that the path is circular and so can deduce that

$$\dot{l} = \ddot{l} = 0 \tag{3}$$

SOLVE:

$(1), (2) \Rightarrow$
$$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta = (\ddot{l} - l\dot{\theta}^2)\vec{e}_l + (l\ddot{\beta} + 2\dot{l}\dot{\beta})\vec{e}_\beta \tag{4}$$

$(3), (4) \Rightarrow$
$$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta = -l\dot{\theta}^2\vec{e}_l + l\ddot{\beta}\vec{e}_\beta$$

Realizing that for the configuration under consideration $\vec{e}_\theta = -\vec{e}_\beta$ and $\vec{e}_r = -\vec{e}_l$ gives us

$$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta = l\dot{\theta}^2\vec{e}_r - l\ddot{\beta}\vec{e}_\theta$$

We're already given the fact that the acceleration is equal to 193.6 ft/s^2 in the $\vec{e}_r$ direction and thus have

$$\ddot{r} - r\dot{\theta}^2 = 193.6\,\text{ft/s}^2$$

$$\ddot{r} = r\dot{\theta}^2 + 193.6\,\text{ft/s}^2$$

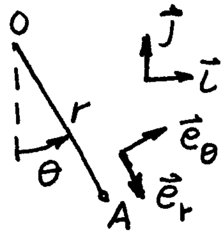
Both r and $\dot{\theta}$ are clearly non-zero and thus $\boxed{\ddot{r} \neq 193.6\,\text{ft/s}^2}$

2.3.22

GOAL: Find the velocity and acceleration of A which can only move vertically

GIVEN: $r = 2\,\text{m},\ \theta = 25°,\ \dot{r} = -2\,\text{m/s},\ \ddot{r} = 0\,\text{m/s}^2$

DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\cos 295°$	$\sin 295°$
$\vec{e}_\theta$	$-\sin 295°$	$\cos 295°$

FORMULATE EQUATIONS:

polar velocity
$$\vec{v}_p = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

polar acceleration
$$\vec{a}_p = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

SOLVE:

polar velocity
$$\vec{v}_p = \dot{r}(\cos 295°\,\vec{\imath} + \sin 295°\,\vec{\jmath}) + r\dot{\theta}(-\sin 295°\,\vec{\imath} + \cos 295°\,\vec{\jmath}) = v_A\vec{\jmath} + 0\vec{\imath}$$

$\vec{\imath}$:
$$\dot{r}\cos 295° - r\dot{\theta}\sin 295° = 0\,\text{m/s}$$

$$\dot{\theta} = \frac{\dot{r}\cos 295°}{r\sin 295°} = \frac{(-2\,\text{m/s})\cos 295°}{(2\,\text{m})\sin 295°} = 0.466\ \text{rad/s}$$

$\vec{\jmath}$:
$$\dot{r}\sin 295° + r\dot{\theta}\cos 295° = v_A$$

$$v_A = (-2\,\text{m/s})\sin 295° + (2\,\text{m})(0.466\ \text{rad/s})\cos 295° = 2.207\,\text{m/s}$$

polar accel
$$\vec{a}_p = (0\,\text{m/s}^2 - r\dot{\theta}^2)(\cos 295°\,\vec{\imath} + \sin 295°\,\vec{\jmath}) + (r\ddot{\theta} + 2\dot{r}\dot{\theta})(-\sin 295°\,\vec{\imath} + \cos 295°\,\vec{\jmath}) = a_A\vec{\jmath} + 0\vec{\imath}$$

$\vec{\imath}$:
$$-r\dot{\theta}^2\cos 295° - r\ddot{\theta}\sin 295° - 2\dot{r}\dot{\theta}\sin 295° = 0\,\text{m/s}^2$$

$$\ddot{\theta} = \frac{r\dot{\theta}^2\cos 295° + 2\dot{r}\dot{\theta}\sin 295°}{-r\sin 295°}$$

$$\ddot{\theta} = \frac{(2\,\mathrm{m})(0.466\ \mathrm{rad/s})^2\cos 295^\circ + (2)(-2\,\mathrm{m/s})(0.466\ \mathrm{rad/s})\sin 295^\circ}{-(2\,\mathrm{m})\sin 295^\circ} = 1.03\ \mathrm{rad/s^2}$$

$\vec{\jmath}$:
$$-r\dot{\theta}^2\sin 295^\circ + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cos 295^\circ = a_A$$

$$a_A = -(2\,\mathrm{m})(0.466\ \mathrm{rad/s})^2\sin 295^\circ + ((2\,\mathrm{m})(1.03\ \mathrm{rad/s^2}) + (2)(-2\,\mathrm{m/s})(0.466\ \mathrm{rad/s}))\cos 295^\circ = 0.480\,\mathrm{m/s^2}$$

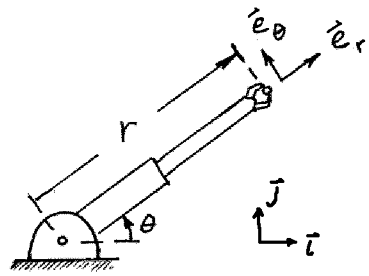
$$\boxed{\vec{v}_A = (2.207\vec{\jmath})\,\mathrm{m/s} \quad \vec{a}_A = (0.480\vec{\jmath})\,\mathrm{m/s^2}}$$

2.3.23

GOAL: Express the acceleration equation for the end of a robotic arm in terms of $\vec{\imath}$ and $\vec{\jmath}$, then calculate the angular velocity and extensional velocity of the arm.

GIVEN: Tip acceleration and assorted kinematic data.

DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

SOLVE:

The acceleration of the end of the robotic arm is

$$\vec{a} = (4\vec{e}_r + 5\vec{e}_\theta)\ \text{m/s}^2$$

Converting to the coordinates $\vec{\imath}$ and $\vec{\jmath}$ ($\theta = 45°$) gives

$$\vec{a} = (4\,\text{m/s}^2)\left(\frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{2}\vec{\jmath}\right) + (5\,\text{m/s}^2)\left(-\frac{\sqrt{2}}{2}\vec{\imath} + \frac{\sqrt{2}}{2}\vec{\jmath}\right)$$

Combining terms gives

$$\boxed{\vec{a} = \left(-\tfrac{\sqrt{2}}{2}\vec{\imath} + \tfrac{9\sqrt{2}}{2}\vec{\jmath}\right)\text{m/s}^2}$$

SOLVE:

To determine $\dot{\theta}$ and $\dot{r}$, start with the equation for acceleration in polar coordinates,

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta$$

Set that equal to the acceleration of the end of the arm

$$\left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta = (4\vec{e}_r + 5\vec{e}_\theta)\ \text{m/s}^2$$

Then, equate the coefficients. So, for $\vec{e}_r$

$$\ddot{r} - r\dot{\theta}^2 = 4\,\text{m/s}^2 \Rightarrow 10\,\text{m/s}^2 - (1.5\,\text{m})\dot{\theta}^2 = 4\,\text{m/s}^2$$

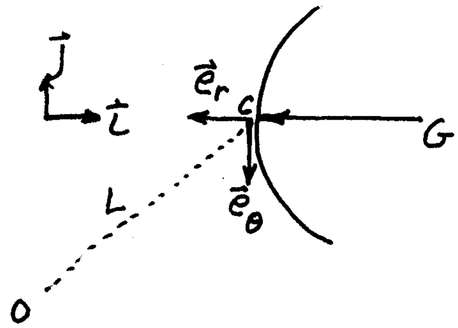
$$\boxed{\dot{\theta} = \pm 2\,\text{rad/s}}$$

Because there are two possible values of $\dot{\theta}$, there are two possible solutions for $\dot{r}$. Using the acceleration in the $\vec{e}_\theta$ direction gives us

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 5\,\mathrm{m/s^2} \Rightarrow (1.5\,\mathrm{m})(-0.5\,\mathrm{rad/s^2}) + 2\dot{r}(\pm 2\,\mathrm{rad/s}) = 5\,\mathrm{m/s^2}$$

$$\boxed{\dot{r} = \pm 1.44\,\mathrm{m/s}}$$

2.3.24
GOAL: Find a_c and $\ddot{L}$.
GIVEN: v_t, $\dot{v}_t$ and car's position.
DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	-1	0
$\vec{e}_\theta$	0	-1

FORMULATE EQUATIONS: The acceleration vector is

$$\vec{a}_c = (\ddot{r} - r\dot{\theta}^2)\,\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\vec{e}_\theta$$

$$r\dot{\theta} = v \quad \Rightarrow \quad \dot{\theta} = \frac{v}{r} = \frac{(50\,\text{mph})(\frac{88}{60})}{300\,\text{ft}} = 0.24 \text{ rad/s}$$

and

$$r\ddot{\theta} = 0.3(32.2\,\text{ft/s}^2) \quad \Rightarrow \quad \ddot{\theta} = \frac{0.3(32.2)}{300} = 0.0322 \text{ rad/s}^2$$

also

$$\dot{r} = \ddot{r} = 0$$

SOLVE: Calculate the acceleration

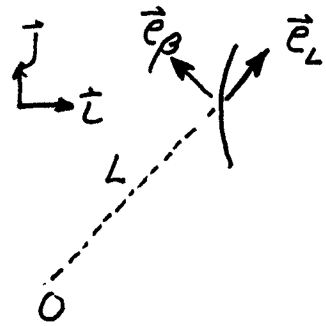
$$\vec{a}_c = -(300\,\text{ft})(0.24\,\text{rad/s})^2\,\vec{e}_r + (300\,\text{ft})(0.0322\,\text{rad/s}^2)\,\vec{e}_\theta$$

$$\boxed{\vec{a}_c = (-17.9\,\vec{e}_r + 9.66\,\vec{e}_\theta)\,\text{ft/s}^2 = (17.9\,\vec{\imath} - 9.66\,\vec{\jmath})\,\text{ft/s}^2}$$

	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\vec{e}_\theta$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

Next we'll find $\ddot{L}$. The velocity vector $\vec{v}_c$ is

$$\vec{v}_c = \dot{L}\,\vec{e}_L + L\dot{\beta}\,\vec{e}_\beta = -v\,\vec{\jmath} = -v(\frac{1}{\sqrt{2}}\,\vec{e}_L + \frac{1}{\sqrt{2}}\,\vec{e}_\beta)$$



$\vec{e}_L$:
$$\dot{L} = -\frac{v}{\sqrt{2}} = -\frac{73.\overline{3}\,\text{ft/s}}{\sqrt{2}} = -51.9\,\text{ft/s}$$

$\vec{e}_\beta$:
$$\dot{\beta} = -\frac{v}{\sqrt{2}(L)} = -\frac{73.\overline{3}\,\text{ft/s}}{\sqrt{2}(\sqrt{2}(300\,\text{ft}))} = -0.122\,\text{rad/s}$$

The acceleration vector $\vec{a}_c$ is

$$\vec{a}_c = (\ddot{L} - L\dot{\beta}^2)\,\vec{e}_L + (L\ddot{\beta} + 2\dot{L}\dot{\beta})\,\vec{e}_\beta = (17.9\,\vec{\imath} - 9.66\,\vec{\jmath})\,\text{ft/s}^2$$

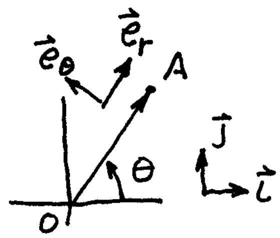
$\vec{e}_L$:
$$\ddot{L} - L\dot{\beta}^2 = \left(-\frac{9.66}{\sqrt{2}} + \frac{17.9}{\sqrt{2}}\right)\,\text{ft/s}^2$$

$\vec{e}_\beta$:
$$L\ddot{\beta} + 2\dot{L}\dot{\beta} = \left(-\frac{9.66}{\sqrt{2}} - \frac{17.9}{\sqrt{2}}\right)\,\text{ft/s}^2$$

SOLVE: Solve for $\ddot{L}$

$$\boxed{\ddot{L} = \sqrt{2}(300\,\text{ft})(0.122\,\text{rad/s})^2 - \tfrac{1}{\sqrt{2}}(9.66 - 17.9)\,\text{ft/s}^2 = 12.2\ \text{m/s}^2}$$

Do It 2.3.4
GOAL: Find velocity and acceleration functions of time that describe the motion of point A
GIVEN: $r = a\theta$, $a = 10$ ft/rad $\dot{\theta} = 10$ rad/s
DRAW:



ASSUME: $\ddot{\theta} = 0$
FORMULATE EQUATIONS:

polar velocity: $$\vec{v}_A = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

polar acceleration: $$\vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

SOLVE:

$$\theta = \int \dot{\theta} dt = 10 \text{ rad/s} \int dt = (10 \text{ rad/s})t$$

$$r = a\theta = (10 \text{ ft/rad})[(10 \text{ rad/s})t] = (100 \text{ ft/s})t$$

$$\dot{r} = 100 \text{ ft/s} \quad \ddot{r} = 0$$

polar velocity $$\vec{v}_A = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta = (100 \text{ ft/s})\vec{e}_r + (100 \text{ ft/s})t(10 \text{ rad/s})\vec{e}_\theta$$

$$\boxed{\vec{v}_A = (100 \text{ ft/s})\vec{e}_r + (1000 \text{ ft/s}^2)t\vec{e}_\theta}$$

polar accel $$\vec{a}_A = (0 - r\dot{\theta}^2)\vec{e}_r + (0 + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

$$\vec{a}_A = [-(100 \text{ ft/s})t(10 \text{ rad/s})^2]\vec{e}_r + (2)(100 \text{ ft/s})(10 \text{ rad/s})\vec{e}_\theta$$

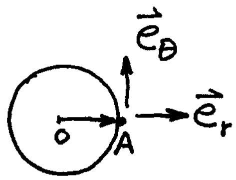
$$\boxed{\vec{a}_A = -(10,000 \text{ ft/s}^3)t\vec{e}_r + (2000 \text{ ft/s}^2)\vec{e}_\theta}$$

2.3.25

GOAL: (case 1) Find acceleration of A for a constant 5000 rpm. (case 2) Suppose that the $\vec{e}_\theta$ acceleration component was equal in magnitude to the previous $\vec{e}_r$ acceleration component. Find $\ddot{\theta}$ that yields this condition and the time required for drill to spin to rest with this acceleration.

GIVEN: $r = 0.003\,\text{m}$, $\dot{\theta} = 5000\,\text{rpm}$, $\ddot{\theta} = 0$ (for case 1)

DRAW:



FORMULATE EQUATIONS:

polar acceleration $$\vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

ASSUME: Because the point A is fixed on the periphery of the drill we have $\dot{r} = \ddot{r} = 0$.

SOLVE:

$$\dot{\theta} = 5000\,\text{rpm} = 524 \text{ rad/s}$$

case 1 $$\vec{a}_A = (0 - r\dot{\theta}^2)\vec{e}_r + (0+0)\vec{e}_\theta = (-(0.003\,\text{m})(524 \text{ rad/s})^2)\vec{e}_r = -822\vec{e}_r\,\text{m/s}^2$$

$$\boxed{\text{case 1:} \quad \vec{a}_A = -822\vec{e}_r\,\text{m/s}^2}$$

case 2 $$a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta}) = (r\ddot{\theta} + 0) = -822\,\text{m/s}^2$$

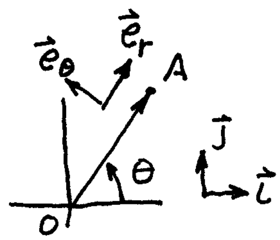
$$\ddot{\theta} = \frac{-822\,\text{m/s}^2}{0.003\,\text{m}} = -2.74\times10^5 \text{ rad/s}^2$$

$$\dot{\theta}(t) = \ddot{\theta}t + \dot{\theta}_0 = 0$$

$$t = \frac{-\dot{\theta}_0}{\ddot{\theta}} = \frac{-524 \text{ rad/s}}{-2.74\times10^5 \text{ rad/s}^2} = 1.9\times10^{-3}\,\text{s}$$

$$\boxed{\text{case 2:} \quad \ddot{\theta} = -2.74\times10^5 \text{ rad/s}^2, \quad t = 1.9\times10^{-3}\,\text{s}}$$

2.3.26
GOAL: Find velocity and acceleration of point A when $\theta = \frac{\pi}{2}$ rad.
GIVEN: $r_A = L(1 + \sin\theta)$, $\dot{\theta} = (2.0\ \text{rad/s}^2)t$, $\theta = 0$ when $t = 0$.
DRAW:



FORMULATE EQUATIONS:

polar velocity $$\vec{v}_p = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

polar acceleration $$\vec{a}_p = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

SOLVE:

$$\theta = \int \dot{\theta} dt = \int (2.0\ \text{rad/s}^2)t\ dt = (1.0\ \text{rad/s}^2)t^2 + \theta_0$$

$$\theta = 0 \ \text{ when } t = 0, \quad \theta_0 = 0, \quad \theta = (1.0\ \text{rad/s}^2)t^2$$

t: $$t = \sqrt{\frac{\theta}{1.0\ \text{rad/s}^2}} = \sqrt{\frac{\frac{\pi}{2}\,\text{rad}}{1.0\ \text{rad/s}^2}} = 1.25\,\text{s}$$

$\dot{\theta}$: $$\dot{\theta} = (2.0\ \text{rad/s}^2)t = (2.0\ \text{rad/s}^2)(1.25\,\text{s}) = 2.51\ \text{rad/s}$$

$\ddot{\theta}$: $$\ddot{\theta} = \frac{d}{dt}\dot{\theta} = \frac{d}{dt}(2.0\ \text{rad/s}^2)t = 2.0\ \text{rad/s}^2$$

r: $$r_A = L(1 + \sin\theta) = L(1 + \sin\frac{\pi}{2}) = 2L$$

$\dot{r}$: $$\dot{r} = \frac{d}{dt}L(1 + \sin\theta) = \dot{\theta}L\cos\theta = (2.51\ \text{rad/s})(L)(\cos\frac{\pi}{2}) = 0$$

$\ddot{r}$: $$\ddot{r} = \frac{d}{dt}\dot{\theta}L\cos\theta = \ddot{\theta}L\cos\theta - \dot{\theta}^2 L\sin\theta$$

$$\ddot{r} = (2.0\ \text{rad/s}^2)(L)(\cos\frac{\pi}{2}) - (2.51\ \text{rad/s})^2(L)(\sin\frac{\pi}{2}) = -6.28L\ \text{s}^{-2}$$

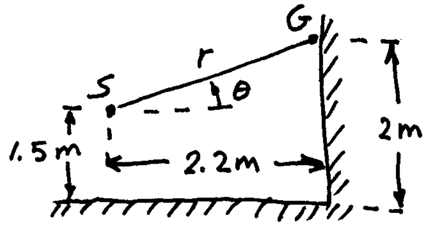
$\vec{v}_A$: $$\vec{v}_A = 0 + r\dot{\theta}\vec{e}_\theta = (2L)(2.51\ \text{rad/s})\vec{e}_\theta = 5.01L\,\text{s}^{-1}\ \vec{e}_\theta$$

$\vec{a}_A$: $$\vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 0)\vec{e}_\theta$$

$$\vec{a}_A = [(-6.28L\,\mathrm{s}^{-2}) - (2L)(2.51\ \mathrm{rad/s})^2]\vec{e}_r + [(2L)(2.0\ \mathrm{rad/s}^2)]\vec{e}_\theta = -18.85L\,\mathrm{s}^{-2}\,\vec{e}_r + 4L\,\mathrm{s}^{-2}\,\vec{e}_\theta$$

$$\boxed{\vec{v}_A = 5.01L\,\mathrm{s}^{-1}\,\vec{e}_\theta, \quad \vec{a}_A = -18.85L\,\mathrm{s}^{-2}\,\vec{e}_r + 4L\,\mathrm{s}^{-2}\,\vec{e}_\theta}$$

2.3.27
GOAL: Find gecko's speed and acceleration.
GIVEN: Thread feed rate and acceleration.
DRAW:



FORMULATE EQUATIONS:

Velocity $$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \quad (1)$$

Acceleration $$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta \quad (2)$$

The gecko is contrained to stay on the wall:

$$r\cos\theta = 2.2\,\mathrm{m}$$

Differentiating this constraint with respect to time yields:

$$\dot{r}\cos\theta - r\dot{\theta}\sin\theta = 0 \Rightarrow \dot{\theta} = \frac{\dot{r}\cos\theta}{r\sin\theta} \quad (3)$$

Differentiating again yields:

$$\ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r(\dot{\theta})^2\cos\theta = 0$$
$$\Rightarrow \ddot{\theta} = \frac{\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta}{r\sin\theta} \quad (4)$$

At the given point in time:

$$r = \sqrt{0.5^2 + 2.2^2}\ \mathrm{m} \quad (5)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{(2-1.5)\,\mathrm{m}}{2.2\,\mathrm{m}} = \frac{0.5}{2.2} \quad (6)$$

SOLVE:

(6),(3)→(1)⇒ $$\vec{v} = \dot{r}\left(\vec{e}_r + \frac{2.2}{0.5}\vec{e}_\theta\right)$$

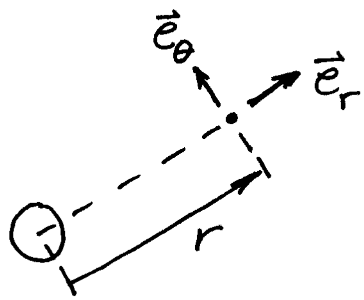
$$\boxed{\|\vec{v}\| = (0.78\,\mathrm{m/s})\sqrt{1 + \left(\frac{2.2}{0.5}^2\right)} = 3.52\,\mathrm{m/s}}$$

(3),(4),(5),(6)→ (2)⇒

$$\vec{a} = \left[-1.2\,\mathrm{m/s^2} - (2.26\,\mathrm{m})(1.52\,\mathrm{rad/s})^2\right]\vec{e}_r + \left[2(0.78\,\mathrm{m/s})(1.52\,\mathrm{rad/s}) + (2.26\,\mathrm{m})(-13.6\,\mathrm{rad/s^2})\right]\vec{e}_\theta$$

$$\boxed{\vec{a} = -\left(6.42\vec{e}_r + 28.25\vec{e}_\theta\right)\,\mathrm{m/s^2}}$$

2.3.28
GOAL: Find Xiyalian transport's velocity and acceleration as functions of time.
GIVEN: Radial position and tangential velocity.
DRAW:



FORMULATE EQUATIONS:

Velocity
$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \quad (1)$$

Acceleration
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta \quad (2)$$

SOLVE:
Differentiating the equation governing radial position,

$$\dot{r} = \frac{d}{dt}a(t_0 - t) = -a \quad (3)$$

Differentiating again:
$$\ddot{r} = 0 \quad (4)$$

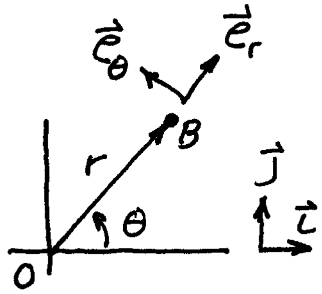
(1)⇒
$$\vec{v} \cdot \vec{e}_\theta = b \Rightarrow r\dot{\theta} = b \quad (5)$$

(5),(3)→(1)⇒
$$\boxed{\vec{v} = -a\vec{e}_r + b\vec{e}_\theta}$$

Differentiating (5)⇒
$$\frac{d}{dt}\left(r\dot{\theta}\right) = \dot{r}\dot{\theta} + r\ddot{\theta} = 0 \Rightarrow r\ddot{\theta} = -\dot{r}\dot{\theta} \quad (6)$$

(4), (5),(6) →(2) ⇒
$$\boxed{\vec{a} = -\frac{b^2}{a(t_0 - t)}\vec{e}_r - \frac{b}{t_0 - t}\vec{e}_\theta}$$

2.3.29
GOAL: Find time for B to reach $\frac{\pi}{2}$ rad and evaluate $\vec{r}_B$, $\vec{v}_B$ $\vec{a}_B$
GIVEN: r and θ as functions of time.
DRAW



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

FORMULATE EQUATIONS:

$$\vec{r}_{B_{/O}} = bt^2\vec{e}_r \tag{1}$$

$$\vec{v}_B = 2bt\vec{e}_r + bt^2\frac{d\vec{e}_r}{dt} = 2bt\vec{e}_r + bt^2\dot{\theta}\vec{e}_\theta \tag{2}$$

$$\begin{aligned}\vec{a}_B &= 2b\vec{e}_r + 2bt\frac{\vec{e}_r}{dt} + 2bt\dot{\theta}\vec{e}_\theta + bt^2\ddot{\theta}\vec{e}_\theta + bt^2\dot{\theta}\frac{\vec{e}_r}{dt}\\ &= 2b\vec{e}_r + 4bt\dot{\theta}\vec{e}_\theta + bt^2\ddot{\theta}\vec{e}_\theta - bt^2\dot{\theta}^2\vec{e}_r\\ &= (2b - bt^2\dot{\theta}^2)\vec{e}_r + (4bt\dot{\theta} + bt^2\ddot{\theta})\vec{e}_\theta \end{aligned}\tag{3}$$

$$\theta = at^2 \tag{4}$$

$$\dot{\theta} = 2at \tag{5}$$

$$\ddot{\theta} = 2a \tag{6}$$

$(2), (5) \Rightarrow$

$$\vec{v}_B = 2bt\vec{e}_r + 2abt^3\vec{e}_\theta \tag{7}$$

$(3), (5), (6) \Rightarrow$

$$\vec{a}_B = (2b - 4ba^2t^4)\vec{e}_r + 10abt^2\vec{e}_\theta \tag{8}$$

SOLVE:
Find t such that $\theta = \frac{\pi}{2}$:

$$\frac{\pi}{2} = at^2 \Rightarrow \boxed{t = \sqrt{\frac{\pi}{2a}}} \tag{9}$$

$(1) \Rightarrow$

$$\vec{r}_B = bt^2(\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath})$$

At $t=\sqrt{\frac{\pi}{2a}}$ and $\theta=\frac{\pi}{2}$ we have

$$\boxed{\vec{r}_B=\frac{b\pi}{2a}\vec{j}}$$

$(5),(7)\Rightarrow \qquad \vec{v}_B=(2bt\cos\theta-2abt^3\sin\theta)\vec{\imath}+(2bt\sin\theta+2abt^3\cos\theta)\vec{j}$

Evaluating at $t=\sqrt{\frac{\pi}{2a}}$, $\theta=\frac{\pi}{2}$ yields

$$\boxed{\vec{v}_B=-\pi b\sqrt{\frac{\pi}{2a}}\vec{\imath}+2b\sqrt{\frac{\pi}{2a}}\vec{j}}$$

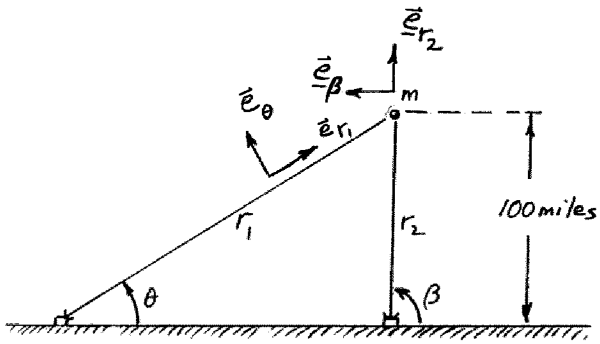
$(3),(5),(6)\Rightarrow \qquad \vec{a}_B=[(2b-4ba^2t^4)\cos\theta-10abt^2\sin\theta]\vec{\imath}+[(2b-4ba^2t^4)\sin\theta+10abt^2\cos\theta]\vec{j}$

At $t=\sqrt{\frac{\pi}{2a}}$, $\theta=\frac{\pi}{2}$ we have

$$\boxed{\vec{a}_{B/A}=-5\pi b\vec{\imath}+b(2-\pi^2)\vec{j}}$$

2.3.30

GOAL: Determine kinematic data of a satellite from ground information.
GIVEN: Position and assorted kinematic data.
DRAW:



	$\vec{e}_{r_1}$	$\vec{e}_\theta$
$\vec{e}_{r_2}$	$\cos(\theta-\beta)$	$-\sin(\theta-\beta)$
$\vec{e}_\beta$	$\sin(\theta-\beta)$	$\cos(\theta-\beta)$

FORMULATE EQUATIONS:
We'll be using the general formulas for velocity and acceleration in a polar frame, expressed in terms of the two sets of unit vectors:

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta$$

SOLVE:
Using the provided values gives

$$\vec{a} = \left(\ddot{r}_2 - (5280\,\mathrm{ft/mi})(100\,\mathrm{mi})(-4.88\times10^{-2}\,\mathrm{rad/s})^2\right)\vec{e}_{r_2} + (5280\,\mathrm{ft/mi})(100\,\mathrm{mi})\ddot{\beta}\vec{e}_\beta$$

$$\vec{a} = \left(\ddot{r}_2 - 1257\,\mathrm{ft/s^2}\right)\vec{e}_{r_2} + \left(5.28\times10^5\,\mathrm{ft}\right)\ddot{\beta}\vec{e}_\beta$$

The acceleration of the satellite is

$$\vec{a} = -30.7\vec{e}_{r_2}\,\mathrm{ft/s^2}$$

so

$$-30.7\vec{e}_{r_2}\,\mathrm{ft/s^2} = \vec{a} = \left(\ddot{r}_2 - 1257\,\mathrm{ft/s^2}\right)\vec{e}_{r_2} + \left(5.28\times10^5\,\mathrm{ft}\right)\ddot{\beta}\vec{e}_\beta$$

Equating coefficients gives for $\vec{e}_{r_2}$

$$\ddot{r}_2 - 1257\,\mathrm{ft/s^2} = -30.7\,\mathrm{ft/s^2}$$

$$\boxed{\ddot{r}_2 = 1227\,\mathrm{ft/s^2}}$$

and for $\vec{e}_\beta$

$$5.28\times10^5\ddot{\beta} = 0$$

$$\boxed{\ddot{\beta} = 0}$$

The first step to solve for the values with respect to Station A is to find the velocity of the satellite with respect to Station B. The velocity is

$$\vec{v}_B = \dot{r}_2\vec{e}_{r_2} + r_2\dot{\theta}\vec{e}_\beta$$

$$\vec{v}_B = 0\vec{e}_{r_2} + (5.28\times10^5\,\text{ft})(-4.88\times10^{-2}\,\text{rad/s})\vec{e}_\beta$$

$$\vec{v}_B = -2.58\times10^4\vec{e}_\beta\,\text{ft/s}$$

Applying a coordinate transform with $\theta = 45°$, $\beta = 90°$ gives

$$\vec{v}_A = (-2.58\times10^4\,\text{ft/s})\left(-\frac{\sqrt{2}}{2}\vec{e}_{r_1} + \frac{\sqrt{2}}{2}\vec{e}_\theta\right)$$

So

$$\vec{v}_A = [1.82\times10^4\vec{e}_{r_1} - 1.82\times10^4\vec{e}_\theta]\,\text{ft/s}$$

Now set this equation equal to the general equation for velocity in polar coordinates

$$\dot{r}_1\vec{e}_{r_1} + r_1\dot{\theta}\vec{e}_\theta = [1.82\times10^4\vec{e}_{r_1} - 1.82\times10^4\vec{e}_\theta]\,\text{ft/s} \tag{1}$$

Use the station's geometry to solve for r_1

$$r_1 = \frac{100\,\text{mi}}{\sin 45^o}$$

So

$$\boxed{r_1 = 7.47\times10^5\,\text{ft}}$$

Equating coefficients in (1) gives for $\vec{e}_{r_1}$

$$\boxed{\dot{r}_1 = 1.82\times10^4\,\text{ft/s}}$$

and for $\vec{e}_\theta$

$$r_1\dot{\theta} = -1.82\times10^4\,\text{ft/s}$$

So

$$\boxed{\dot{\theta} = -2.44\times10^{-2}\,\text{rad/s}}$$

Now, to solve for the rest of the variables, write the acceleration equations

$$\vec{a}_B = -30.7\vec{e}_{r_2}\,\text{ft/s}^2$$

Transforming to station A's coordinates

$$\vec{a}_A = (-21.7\vec{e}_{r_1} - 21.7\vec{e}_\theta)\,\text{ft/s}^2$$

Writing the general equation

$$\vec{a}_A = \left(\ddot{r}_1 - r_1\dot{\theta}^2\right)\vec{e}_{r_1} + \left(2\dot{r}_1\dot{\theta} + r_1\ddot{\theta}\right)\vec{e}_\theta$$

and equating coefficients for $\vec{e}_{r_1}$ gives

$$-21.7\,\text{ft/s}^2 = \ddot{r}_1 - (7.47\times10^5\,\text{ft})(-2.44\times10^{-2}\,\text{rad/s})^2$$

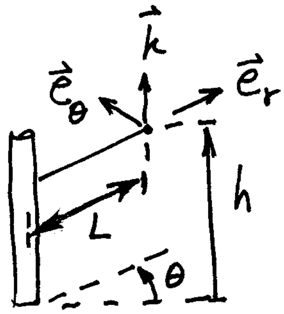
$$\boxed{\ddot{r}_1 = 423\,\text{ft/s}^2}$$

Equating for $\vec{e}_\theta$ gives

$$-21.7\,\text{ft/s}^2 = 2(1.82\times10^4\,\text{ft/s})(-2.44\times10^{-2}\,\text{rad/s}) + (7.47\times10^5\,\text{ft})\ddot{\theta}$$

$$\boxed{\ddot{\theta} = 1.16\times10^{-3}\,\text{rad/s}^2}$$

2.3.31
GOAL: Find acceleration of probe tip.
GIVEN: radial, angular, and vertical positions
DRAW:



FORMULATE EQUATIONS:

Acceleration
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta + \ddot{h}\vec{k} \tag{1}$$

Differentiating the equation governing radial position,

$$\frac{d}{dt}a(1 - \cos(\omega_1 t)) = \dot{\theta} = a\omega_1 \sin(\omega_1 t) \tag{2}$$

Differentiating again,
$$\frac{d}{dt}a\omega_1 \sin(\omega_1 t) = \ddot{\theta} = a\omega_1^2 \cos(\omega_1 t) \tag{3}$$

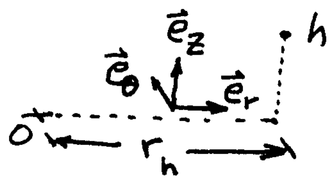
Differentiating the equation governing height twice,

$$\ddot{h} = b\omega_2^2 \cos(\omega_2 t) \tag{4}$$

SOLVE:

(2),(3),(4)→ (1)⇒
$$\boxed{\vec{a} = -ra^2\omega_1^2 \sin^2(\omega_1 t)\vec{e}_r + ra\omega_1^2 \cos(\omega_1 t)\vec{e}_\theta + b\omega_2^2 \cos(\omega_2 t)\vec{k}}$$

2.3.32
GOAL: Find the velocity and acceleration of a horse on a carousel.
GIVEN: System dimensions, position relation of height to rotation angle, and rotational speed as a function of time.
DRAW:



FORMULATE EQUATIONS:

$$\vec{v}_h = \dot{r}_h\vec{e}_r + r_h\dot{\theta}\vec{e}_\theta + \dot{z}\vec{e}_z \tag{1}$$

$$\vec{a}_h = (\ddot{r}_h - r_h\dot{\theta}^2)\vec{e}_r + (r_h\ddot{\theta} + 2\dot{r}_h\dot{\theta})\vec{e}_\theta + \ddot{z}\vec{e}_z \tag{2}$$

ASSUME: The horse can't move radially and thus

$$\dot{r}_h = \ddot{r}_h = 0 \tag{3}$$

SOLVE:

$$z = z_0 + z_1 \sin(\omega_2\theta) \tag{4}$$

$$\dot{\theta} = \omega_1 e^{-bt} \Rightarrow \theta = -\frac{\omega_1}{b}e^{-bt} \tag{5}$$

(4),(5) $\Rightarrow$

$$z = z_0 + z_1 \sin(-\frac{\omega_1\omega_2}{b}e^{-bt}) \tag{6}$$

(6) $\Rightarrow$

$$\dot{z} = z_1\omega_1\omega_2 e^{-bt}\cos(\frac{-\omega_1\omega_2}{b}e^{-bt}) \tag{7}$$

(7) $\Rightarrow$

$$\ddot{z} = -z_1 b\omega_1\omega_2 e^{-bt}\cos(\frac{-\omega_1\omega_2}{b}e^{-bt}) + z_1\omega_1^2\omega_2^2 e^{-2bt}\sin(\frac{-\omega_1\omega_2}{b}e^{-bt}) \tag{8}$$

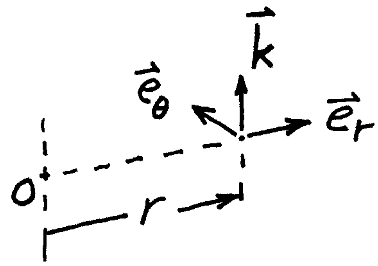
(3),(5),(7) $\rightarrow$ (1) $\Rightarrow$

$$\boxed{\vec{v}_h = r_h\omega_1 e^{-bt}\vec{e}_\theta + z_1\omega_1\omega_2 e^{-bt}\cos(\frac{-\omega_1\omega_2}{b}e^{-bt})\vec{e}_z}$$

(3),(5),(8) $\rightarrow$ (2) $\Rightarrow$

$$
\boxed{
\begin{aligned}
\vec{a}_h &= -r_h \omega_1^2 e^{-2bt} \vec{e}_r - r_h b \omega_1 e^{-bt} \vec{e}_\theta - \\
&\left[z_1 b \omega_1 \omega_2 e^{-bt} \cos\left(\frac{-\omega_1 \omega_2}{b} e^{-bt}\right) + z_1 \omega_1^2 \omega_2^2 e^{-2bt} \sin\left(\frac{-\omega_1 \omega_2}{b} e^{-bt}\right) \right] \vec{e}_z
\end{aligned}
}
$$

2.3.33
GOAL: Find magnitude and direction of thief's velocity and acceleration.
GIVEN: Magnitude of thief's speed and dimensions of the museum floor.
DRAW:



FORMULATE EQUATIONS:

Velocity
$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta} + \dot{h}\vec{e}_z \tag{1}$$

Acceleration
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta + \ddot{h}\vec{e}_z \tag{2}$$

The thief is constrained to remain on the floor,

Constraint
$$h = h_0 - \frac{12}{2\pi}\theta\,\text{ft} \tag{3}$$

where, h_0 is the initial height (in ft) and θ is the thief's angular position specified in radians. Differentiating (3) twice,

$$\dot{h} = -\frac{12}{2\pi}\dot{\theta}\,\text{ft/s} \tag{4}$$

$$\ddot{h} = -\frac{12}{2\pi}\ddot{\theta}\,\text{ft/s}^2 \tag{5}$$

SOLVE:

(4) → (1) ⇒
$$\vec{v} = r\dot{\theta}\vec{e}_\theta - \frac{12}{2\pi}\dot{\theta}\vec{e}_z\,\text{ft/s} \tag{6}$$

Speed is 12 mph ⇒
$$\sqrt{\vec{v}\cdot\vec{v}} = r^2\dot{\theta}^2 + \frac{144}{4\pi^2}\dot{\theta}^2 = 12\,\text{mph} \tag{7}$$

(7) ⇒
$$\dot{\theta} = 0.4395\,\text{rad/s} \tag{8}$$

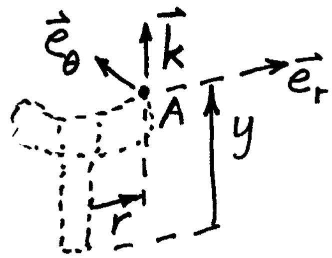
$$\boxed{\vec{v} = 17.58\vec{e}_\theta - 0.84\vec{e}_z\,\text{ft/s}}$$

$$\boxed{\text{direction of velocity} = \tan^{-1}\tfrac{0.84}{17.58} = 2.7^\circ}$$

(8), (5) → (2) ⇒
$$\vec{a} = -40\dot{\theta}^2\vec{e}_r\,\text{ft/s}^2$$

$$\boxed{\|\vec{a}\| = -7.73\,\text{ft/s}^2,\ \vec{a}\text{ is in radial direction}}$$

2.3.34
GOAL: Find a) revolutions of wingnut in given time, b) percentage increase in speed due to moving down.
GIVEN: vertical speed, constant angular velocity, and total acceleration of fixed point on wingnut.
DRAW:



FORMULATE EQUATIONS:
Because wingnut has constant angular velocity, it must also have constant vertical velocity due to constant thread pitch:

Vertical velocity
$$\dot{h} = \frac{0.9\,\text{in}}{1.2\,\text{s}} \tag{1}$$

Acceleration
$$\vec{a}_A = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta + \ddot{h}\vec{e}_z \tag{2}$$

SOLVE:

(2) $\Rightarrow$
$$\|\vec{a}_A\| = \| - r\dot{\theta}^2\vec{e}_r\| = 2.193 \times 10^3\,\text{in/s}^2 \tag{3}$$

(3) $\Rightarrow$
$$\dot{\theta} = 52.357\,\text{rad/s} \tag{4}$$

Integrating (4) $\Rightarrow$
$$\int_0^{1.2} \dot{\theta}dt = 52.357(1.2)\,\text{rad}$$

$$\boxed{\theta = 62.83\,\text{rad} = 10\,\text{revolutions}}$$

Speed at constant height $\Rightarrow$
$$\|\vec{v}_1\| = r\dot{\theta} \tag{5}$$

Speed at variable height $\Rightarrow$
$$\|\vec{v}_2\| = \sqrt{(r\dot{\theta})^2 + \dot{h}^2} \tag{6}$$

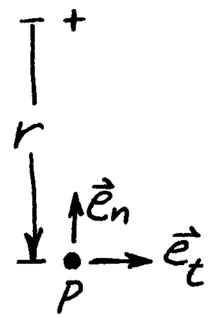
% increase in speed $\Rightarrow$
$$100 \times \frac{\|\vec{v}_2\| - \|\vec{v}_1\|}{\|\vec{v}_1\|} \tag{7}$$

(5), (6) $\rightarrow$ (7) $\Rightarrow$
$$\frac{\sqrt{[(0.8\,\text{in})(52.357\,\text{rad/s})]^2 + \left(\frac{0.9\,\text{in}}{1.2\,\text{s}}\right)^2} - (0.8\,\text{in})(52.36\,\text{rad/s})}{(0.8\,\text{in})(52.357\,\text{rad/s})}$$

$$\boxed{\%\text{ increase in speed} = 0.016\,\%}$$

2.4 Path Coordinates

2.4.1
GOAL: Find the normal acceleration felt by the airplane if the pilot pulls into an upward loop.
GIVEN: The radius of the loop is 1300 ft.
DRAW:



FORMULATE EQUATIONS: We'll use the formula for acceleration in path coordinates:

$$\vec{a}_P = a_t\vec{e}_t + a_n\vec{e}_n = \dot{v}\vec{e}_t + \frac{v^2}{r_c}\vec{e}_n$$

SOLVE:
The radius of the loop is the radius of curvature for this problem:

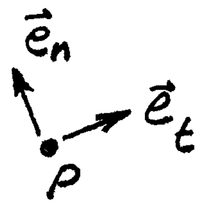
$$a_n = \frac{v^2}{r} = \frac{(300\,\text{ft/s})^2}{1300\,\text{ft}}$$

$$\boxed{a_n = 69.2\,\text{ft/s}^2}$$

2.4.2

GOAL: Calculate the speed of a car.

GIVEN: $\vec{a}_P = (-7.0\vec{e}_t + 5.0\vec{e}_n)\,\mathrm{m/s^2}$ and $r = 100\,\mathrm{m}$.



FORMULATE EQUATIONS: The acceleration vector written in the $(\vec{e}_t, \vec{e}_n)$ basis is

$$\vec{a} = \dot{v}\,\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n$$

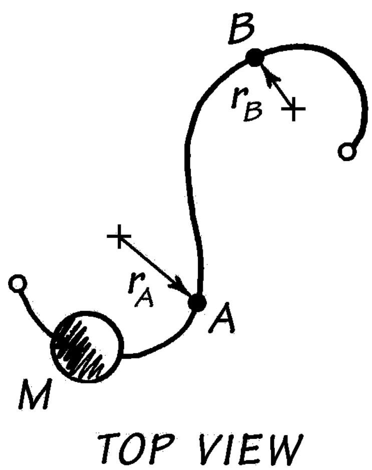
where r_C is the radius of curvature.

SOLVE: Given $a_n = 5\,\mathrm{m/s^2}$ and $r_C = 100\,\mathrm{m}$, we can solve for v :

$$\frac{v^2}{100\,\mathrm{m}} = 5\,\mathrm{m/s^2} \Rightarrow v = 22.4\,\mathrm{m/s}$$

$$\boxed{v = 22.4\,\mathrm{m/s}}$$

2.4.3
GOAL: Determine the total acceleration $\|\vec{a}_A\|$ at A and the radius of curvature r_B at B.
GIVEN: At point A, the radius of curvature is $r_A = 2\,\text{m}$, and the marble's speed is a constant $v_A = 5\,\text{m/s}$. At point B, the marble is traveling at $v_B = 3\,\text{m/s}$ with a total and tangential acceleration of $\|\vec{a}_B\| = 10\,\text{m/s}^2$ and $a_{B,t} = 2\,\text{m/s}^2$, respectively.
DRAW:



FORMULATE EQUATIONS:
The marble's acceleration in terms of path coordinates is given by

$$\vec{a} = a_t\vec{e}_t + \frac{v_t^2}{r}\vec{e}_n \tag{1}$$

SOLVE:
Since the marble is traveling at a constant speed at point A, its total and normal accelerations are the same, and hence

$$(1) \Rightarrow \qquad \vec{a}_A = \frac{v_A^2}{r_A}\vec{e}_n$$

$$\|\vec{a}_A\| = \sqrt{\vec{a}_A \cdot \vec{a}_A} = \frac{v_A^2}{r_A}$$

$$\|\vec{a}_A\| = \frac{(5\,\text{m/s})^2}{2\,\text{m}}$$

$$\boxed{\|\vec{a}_A\| = 12.5\,\text{m/s}^2}$$

To find the radius of curvature at B, we'll need to rearrange our expression for the marble's acceleration at that location:

$$(1) \Rightarrow \qquad \vec{a}_B = a_{B,t}\vec{e}_t + \frac{v_B^2}{r_B}\vec{e}_n$$

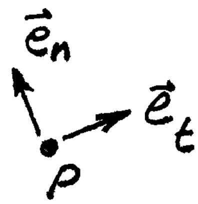
$$\|\vec{a}_B\|^2 = \vec{a}_B \cdot \vec{a}_B = \left(a_{B,t}\right)^2 + \left(\frac{v_B^2}{r_B}\right)^2$$

$$r_B = \frac{v_B^2}{\sqrt{\|\vec{a}_B\|^2 - \left(a_{B,t}\right)^2}}$$

$$r_B = \frac{(3\,\mathrm{m/s})^2}{\sqrt{(10\,\mathrm{m/s^2})^2 - (2\,\mathrm{m/s^2})^2}}$$

$$\boxed{r_B = 0.919\,\mathrm{m}}$$

2.4.4
GOAL: Calculate the speed of a race car.
GIVEN: $\vec{a}_P = (0\vec{e}_t + 0\vec{e}_n)\,\mathrm{m/s^2}$ and $r = 90\,\mathrm{m}$.



FORMULATE EQUATIONS: The acceleration vector written in the $(\vec{e}_t, \vec{e}_n)$ basis is

$$\vec{a} = \dot{v}\,\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n$$

where r_C is the radius of curvature.
SOLVE: Given $a_n = 0\,\mathrm{m/s^2}$ and $r_C = 90\,\mathrm{m}$, we can solve for v :

$$\frac{v^2}{90\,\mathrm{m}} = 0\,\mathrm{m/s^2} \Rightarrow v = 0\,\mathrm{m/s}$$

$$\boxed{v = 0}$$

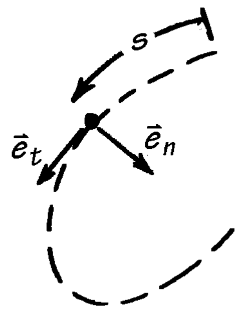
What's interesting here is that normally one expects a knowledge of acceleration to give no information about velocity. For instance, if a particle is moving at a constant speed of $(100\vec{\imath} + 50\vec{\jmath})\,\mathrm{m/s}$ it will have zero acceleration. Likewise it'll have zero acceleration when moving at a constant speed of $(-4\vec{\imath} + 8\vec{\jmath})\,\mathrm{m/s}$. Hence knowing that the acceleration is zero doesn't tell us anything about the velocity. For path coordinates, though, we see that a non-zero speed and positive radius of curvature implies a non-zero normal acceleration and the reverse holds as well: a zero normal acceleration implies zero speed (unless the radius of curvature is infinity, bringing us to pure rectilinear motion).

2.4.5

GOAL: Determine the constant v_{max} before the motorcycle begins to slip at $s = 20\,\text{ft}$.

GIVEN: The maximum allowable acceleration of the motorcycle before slip is $a_{\text{max}} = 30\,\text{ft/s}^2$. The path's radius of curvature is described by $r_c = 25 - 0.002s^3\,\text{ft}$. The tangential speed is constant.

DRAW:



FORMULATE EQUATIONS:

The acceleration of the motorcycle in terms of path coordinates is

$$\vec{a} = \frac{v^2}{r_c}\vec{e}_n \tag{1}$$

SOLVE:

(1) $\Rightarrow$

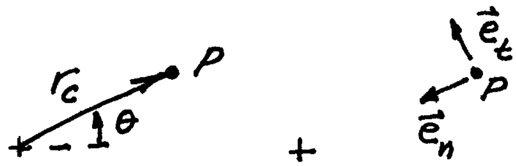
$$a_{\text{max}} = \frac{v_{\text{max}}^2}{r_c}$$

$$v_{\text{max}} = \sqrt{a_{\text{max}} r_c}$$

$$v_{\text{max}} = \sqrt{(30\,\text{ft/s}^2)\,\{25 - 0.002(20)^3\,\text{ft}\}}$$

$$\boxed{v_{\text{max}} = 16.43\,\text{ft/s}}$$

2.4.6
GOAL: Find the rotational speed of a Ferris wheel.
GIVEN: Magnitude of acceleration and dimensions.
DRAW:



FORMULATE EQUATIONS:

Velocity of P $$\vec{v}_P = r_C\dot{\theta}\vec{e}_t \tag{1}$$

Acceleration of P $$\vec{a}_P = a_t\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n \tag{2}$$

SOLVE:

constant rotational speed⇒ $$\vec{a}_P = \frac{v^2}{r_C}\vec{e}_n \tag{3}$$

(1) → (3) ⇒ $$\vec{a}_P = \frac{r_C^2\dot{\theta}^2}{r_C} = 0.33\,\text{ft/s}^2$$

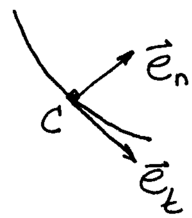
$$\boxed{\dot{\theta} = 0.105\,\text{rad/s} = 1\,\text{rev/min}}$$

2.4.7

GOAL: Determine $|a_t|$ for a car in a turn.

GIVEN: Radius of ramp is 180 m, $v = 30\,\mathrm{m/s}$, and the total acceleration magnitude is $7.07\,\mathrm{m/s^2}$.

DRAW



FORMULATE EQUATIONS:

$$\vec{a}_c = \dot{v}\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n$$

SOLVE:

$$|a_c| = 7.07\,\mathrm{m/s^2} \tag{1}$$

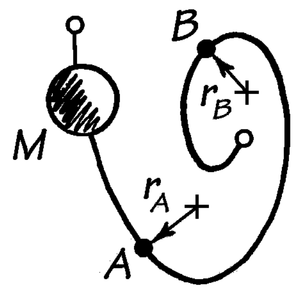
$$\vec{a}_c = a_t\vec{e}_t + \frac{(30\,\mathrm{m/s})^2}{180\,\mathrm{m}}\vec{e}_n \tag{2}$$

(1), (2) $\Rightarrow$ $$(7.07\,\mathrm{m/s^2})^2 = a_t^2 + (5\,\mathrm{m/s^2})^2$$

$$a_t^2 = 25.0\,(\mathrm{m/s^2})^2$$

$$\boxed{a_t = 5\,\mathrm{m/s^2}}$$

Do It 2.4.1
GOAL: Determine the marble's constant speed v_A at A and its tangential acceleration $a_{B,t}$ at B.
GIVEN: At point A, the radius of curvature is $r_A = 0.7\,\text{ft}$, and the marble's total acceleration is $\|\vec{a}_A\| = 14\,\text{ft/s}^2$. At point B, the marble is traveling at $v_B = 2.5\,\text{ft/s}$ with a total acceleration of $\|\vec{a}_B\| = 18\,\text{ft/s}^2$. The radius of curvature at B is $r_B = 0.4\,\text{ft}$.
DRAW:



TOP VIEW

FORMULATE EQUATIONS:
The marble's acceleration in terms of path coordinates is given by

$$\vec{a} = a_t \vec{e}_t + \frac{v_t^2}{r}\vec{e}_n \tag{1}$$

SOLVE:
We're told that the marble's speed at A is constant, and so its total acceleration is given by the acceleration normal to the path:

(1) $\Rightarrow$
$$\vec{a}_A = \frac{v_A^2}{r_A}\vec{e}_n$$

$$\|\vec{a}_A\| = \sqrt{\vec{a}_A \cdot \vec{a}_A} = \frac{v_A^2}{r_A}$$

$$v_A = \sqrt{\|\vec{a}_A\| r_A}$$

$$v_A = \sqrt{(14\,\text{ft/s}^2)(0.7\,\text{ft})}$$

$$\boxed{v_A = 3.13\,\text{ft/s}}$$

We can find the marble's tangential acceleration at B by rearranging our expression for its total acceleration at that point:

(1) $\Rightarrow$
$$\vec{a}_B = a_{B,t}\vec{e}_t + \frac{v_B^2}{r_B}\vec{e}_n$$

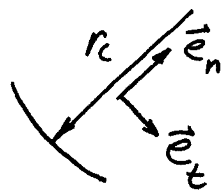
$$\|\vec{a}_B\|^2 = \vec{a}_B \cdot \vec{a}_B = \left(a_{B,t}\right)^2 + \left(\frac{v_B^2}{r_B}\right)^2$$

$$a_{B,t} = \sqrt{\|\vec{a}_B\|^2 - \left(\frac{v_B^2}{r_B}\right)^2}$$

$$a_{B,t} = \sqrt{(18\,\mathrm{ft/s^2})^2 - \left[\frac{(2.5\,\mathrm{ft/s})^2}{0.4\,\mathrm{ft}}\right]^2}$$

$$\boxed{a_{B,t} = 8.94\,\mathrm{ft/s^2}}$$

2.4.8
GOAL: Find r_{min} to support a maximum normal acceleration of 3.5g
GIVEN: Path and speed.
DRAW:



FORMULATE EQUATIONS:

$$\vec{a} = \dot{v}\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n$$

SOLVE:

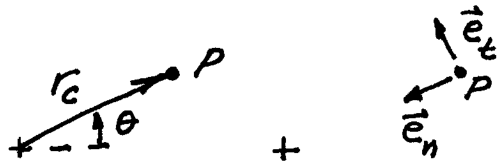
$$60\,\text{mph} = 88\,\text{ft/s}$$

$$a_n = \frac{v^2}{r_C} = \frac{(88\,\text{ft/s})^2}{r_C} = (3.5)(32.2\,\text{ft/s}^2)$$

$$r_C = 68.7\,\text{ft}$$

$$\boxed{r_{min} = r_C = 68.7\,\text{ft}}$$

2.4.9
GOAL: Find acceleration of point P at $t = 3\,\text{s}$.
GIVEN: Tangential acceleration, initial conditions, and dimensions.
DRAW:



FORMULATE EQUATIONS:

$a_t = \frac{dv}{dt} = ct \Rightarrow$

$$dv = c\,t\,dt \Rightarrow \int_0^t dv = \int_0^t \tau d\tau \Rightarrow v(t) - v(0) = \frac{c\,t^2}{2} \tag{1}$$

Acceleration of P

$$\vec{a}_P = a_t \vec{e}_t + \frac{v^2}{r_C}\vec{e}_n = ct\vec{e}_t + \frac{v(t)^2}{r_C}\vec{e}_n \tag{2}$$

SOLVE:

(1) and given conditions$\Rightarrow$

$$v(3\,\text{s}) = \frac{(-0.4\,\text{ft/s}^3)(3\,\text{s})^2}{2} + 10\,\text{ft/s} = 8.2\,\text{ft/s} \tag{3}$$

(3) $\rightarrow$ (2) $\Rightarrow$

$$\vec{a}_P(3\,\text{s}) = \left[(-0.4\,\text{ft/s}^3)(3\,\text{s})\vec{e}_t + \frac{(8.2\,\text{ft/s})^2}{30\,\text{ft}}\vec{e}_n\right]$$

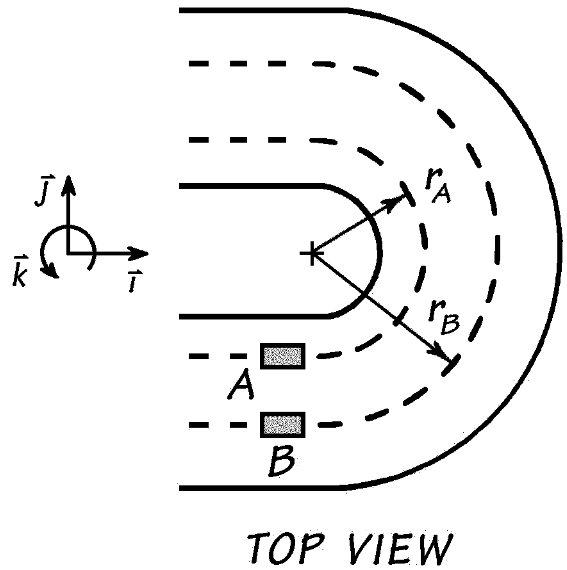
$$\boxed{\vec{a}_P(3\,\text{s}) = (-1.20\vec{e}_t + 2.24\vec{e}_n)\,\text{ft/s}^2}$$

Do It 2.4.2

GOAL: Determine which car (if any) will make it around the turn without sliding off the road and how long it takes to do so.

GIVEN: Car A is attempting to make a semicircular turn of radius $r_A = 90$ m at a constant speed of $v_A = 24$ m/s. Car B intends to travel at $v_B = 28$ m/s through a path of radius $r_B = 95$ m. The cars will slide off the road when their normal accelerations exceed $a_{n,\max} = 0.75g$.

DRAW:



FORMULATE EQUATIONS:

Since the speed of each car is constant through the turn, the acceleration can be expressed in terms of path coordinates as

$$\vec{a} = \frac{v_t^2}{r}\vec{e}_n \tag{1}$$

The time it takes for each car to travel through the turn with a constant speed is governed by

$$v = \frac{\Delta s}{\Delta t} \tag{2}$$

SOLVE:

We're told that loss of traction will occur when the normal acceleration exceeds $a_{n,\max} = 0.75g = 7.36\,\mathrm{m/s^2}$, so let's start by checking if car A will make it through the turn:

(1) $\Rightarrow$
$$\vec{a}_A = \frac{v_A^2}{r_A}\vec{e}_n$$

$$\|\vec{a}_A\| = \sqrt{\vec{a}_A \cdot \vec{a}_A} = \frac{v_A^2}{r_A}$$

$$\|\vec{a}_A\| = a_{A,n} = \frac{(24\,\text{m/s})^2}{90\,\text{m}}$$
$$a_{A,n} = 6.40\,\text{m/s}^2 < a_{n,\max}$$

Thus, we find that car A will maintain traction with the road, so it will make it through the turn in

(2) $\Rightarrow$
$$t_A = \frac{\pi r_A}{v_A}$$
$$t_A = \frac{\pi(90\,\text{m})}{24\,\text{m/s}}$$
$$t_A = 11.8\,\text{s}$$

$$\boxed{a_{A,n} < a_{n,\max} \quad \Rightarrow \quad \text{Car } A \text{ will make the turn in } t_A = 11.8\,\text{s}.}$$

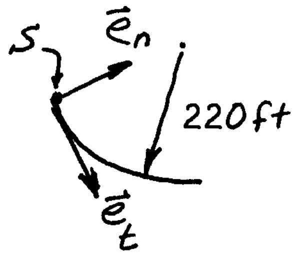
Let's now look at car B:

(1) $\Rightarrow$
$$\vec{a}_B = \frac{v_B^2}{r_B}\vec{e}_n$$
$$\|\vec{a}_B\| = \sqrt{\vec{a}_B \cdot \vec{a}_B} = \frac{v_B^2}{r_B}$$
$$\|\vec{a}_B\| = a_{B,n} = \frac{(28\,\text{m/s})^2}{95\,\text{m}}$$
$$a_{B,n} = 8.25\,\text{m/s}^2 > a_{n,\max}$$

Therefore, we see that car B will exceed slip conditions, and so it won't make it through the turn:

$$\boxed{a_{B,n} > a_{n,\max} \quad \Rightarrow \quad \text{Car } B \text{ will slide off the road.}}$$

2.4.10
GOAL: Find the acceleration magnitude of a skier, S.
GIVEN: Skier's path and speed.
DRAW



FORMULATE EQUATIONS:

$$\vec{a} = \dot{v}\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n$$

SOLVE: We need to find the skier's speed at B. The acceleration is constant and so we have

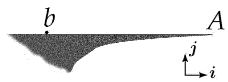
$$\frac{v^2}{2} = a\Delta s = (28\,\mathrm{ft/s^2})(200\,\mathrm{feet}) = 5,600\,(\mathrm{ft/s})^2$$

$$v = 105.8\,\mathrm{ft/s}$$

$$\vec{a}_S = 28\,\mathrm{ft/s^2}\,\vec{e}_t + \frac{(105.8\,\mathrm{ft/s})^2}{220\,\mathrm{ft}}\vec{e}_n = [28\vec{e}_t + 50.9\vec{e}_n]\,\mathrm{ft/s^2}$$

$$\boxed{|\vec{a}_S| = 58.1\,\mathrm{ft/s^2} = 1.80\,\mathrm{g}}$$

2.4.11
GOAL: Find direction and magnitude of velocity of point b immediately before and after reaching point A.
GIVEN: Constraint of constant speed before reaching point A, acceleration after reaching point A.
DRAW:



FORMULATE EQUATIONS:

Velocity of point b before reaching point A
$$\vec{v}_{before} = \vec{v}_b = v_b \vec{i} \tag{1}$$

Velocity of point b after reaching point A
$$\vec{v}_{after} = \vec{v}_b + \int_{t_A}^{t} \vec{a}dt \tag{2}$$

SOLVE:

Immediately after passing point A, $t \to t_A$ in (2)⇒
$$\vec{v}_{after} = \vec{v}_b + \int_{t_A}^{t_A} \vec{a}dt = \vec{v}_b$$

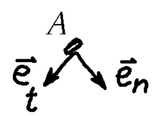
$$\boxed{\vec{v}_{before} = \vec{v}_{after} = v_b \vec{i}}$$

2.4.12

GOAL: Find the time at which a sports car's tangential acceleration is 10% of its normal acceleration.

GIVEN: The track has a radius of 100 m and the sports car's speed is given by $v = v_0\left(1 - e^{-at}\right)$ where $v_0 = 60\,\text{m/s}$ and $a = 0.3\,\text{s}^{-1}$.

DRAW:



FORMULATE EQUATIONS:

$$v_t = v_0\left(1 - e^{-at}\right) \tag{1}$$

$$a_t = \frac{dv_t}{dt} = av_0e^{-at} \tag{2}$$

$$a_n = \frac{(v_t)^2}{r} = \frac{v_0^2\left(1 - e^{-at}\right)^2}{r} \tag{3}$$

SOLVE:

We need to solve for when the tangential acceleration is 10% of the normal acceleration, hence $10a_t = a_n$. Using (1)-(3) gives us

$$10av_0e^{-at} = \frac{v_0^2\left(1 - e^{-at}\right)^2}{r}$$

Letting $x = e^{-at}$ and using the given parameter values yields the following quadratic equation

$$x^2 - 7x + 1 = 0$$

This has only one physical solution: $x = 0.146$. Thus we have

$$e^{-(0.3\,\text{s}^{-1})t} = 0.146$$

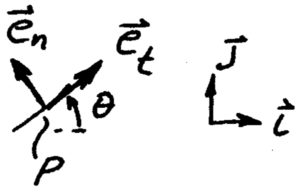
$$\boxed{t = 6.42\,\text{s}}$$

2.4.13

GOAL: Determine the normal and tangential acceleration of the point P when it has the same speed in both the $\vec{\imath}$ and $\vec{\jmath}$ directions.

GIVEN: $\dot{y} = bx$, with $b = 12\,\mathrm{s}^{-1}$.

DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_t$	$\cos\theta$	$\sin\theta$
$\vec{e}_n$	$-\sin\theta$	$\cos\theta$

FORMULATE EQUATIONS: We'll determine the acceleration in terms of $\vec{\imath}$,$\vec{\jmath}$ and then transform this into the path frame.

SOLVE:

Differentiating $y = ax^2$ with respect to time gives us

$$\dot{y} = 2ax\dot{x} \tag{1}$$

$\dot{x}$ and $\dot{y}$ will be equal if $2ax = 1$. Hence we have

$$\boxed{x = \frac{1}{2a} = \frac{1}{12\,\mathrm{ft}^{-1}} = \frac{1}{12}\,\mathrm{ft}}$$

as the position at which $\dot{x}$ and $\dot{y}$ are equal. Furthermore, we're given

$$\dot{y} = bx \tag{2}$$

(1), (2) $\Rightarrow$
$$2ax\dot{x} = bx \Rightarrow \dot{x} = \frac{b}{2a} = \frac{12\,\mathrm{s}^{-1}}{12\,\mathrm{ft}^{-1}} = 1.0\,\mathrm{ft/s} \tag{3}$$

If $\dot{x}$ and $\dot{y}$ are equal the $\vec{e}_t$ unit vector is oriented up by 45° (since it point along the velocity vector).

(2) $\Rightarrow$
$$\dot{y} = bx = (12\,\mathrm{s}^{-1})(\frac{1}{12}\,\mathrm{ft}) = 1.0\,\mathrm{ft/s} \tag{4}$$

and

$$\boxed{v = \sqrt{(1.0\,\mathrm{ft/s})^2 + (1.0\,\mathrm{ft/s})^2} = \sqrt{2}\,\mathrm{ft/s}}$$

(1) $\Rightarrow$
$$\ddot{y} = 2a\dot{x}^2 + 2ax\ddot{x} \tag{5}$$

(2) $\Rightarrow$
$$\ddot{y} = b\dot{x} \tag{6}$$

(5), (6) $\Rightarrow$
$$(12\,\mathrm{ft}^{-1})(1.0\,\mathrm{ft/s})^2 + \ddot{x} = (12\,\mathrm{s}^{-1})(1.0\,\mathrm{ft/s}) \Rightarrow \ddot{x} = 0 \tag{7}$$

(6) $\Rightarrow$
$$\ddot{y} = (12\,\mathrm{s}^{-1})(1.0\,\mathrm{ft/s}) = 12\,\mathrm{ft/s}^2 \tag{8}$$

$$(7),(8) \Rightarrow \qquad a_t\left(\frac{1}{\sqrt{2}}\vec{\imath}+\frac{1}{\sqrt{2}}\vec{\jmath}\right)\,\mathrm{ft/s^2}+a_n\left(-\frac{1}{\sqrt{2}}\vec{\imath}+\frac{1}{\sqrt{2}}\vec{\jmath}\right)\,\mathrm{ft/s^2}=12\vec{\jmath}\,\mathrm{ft/s^2} \tag{9}$$

$$\vec{\imath}: \qquad \frac{a_t}{\sqrt{2}}-\frac{a_n}{\sqrt{2}}=0 \tag{10}$$

$$\vec{\jmath}: \qquad \frac{a_t}{\sqrt{2}}+\frac{a_n}{\sqrt{2}}=12\,\mathrm{ft/s^2} \tag{11}$$

$$\boxed{a_t=6\sqrt{2}\,\mathrm{ft/s^2}}$$

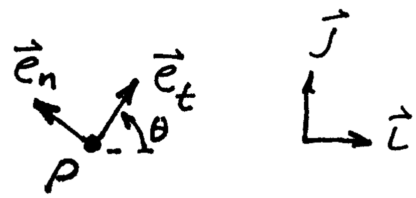
$$\boxed{a_n=6\sqrt{2}\,\mathrm{ft/s^2}}$$

2.4.14

GOAL: Determine the speed of a moving particle when it is at $x = \frac{\pi}{2\lambda}$.

GIVEN: $y = a[1 - \cos(\lambda x)]$ with $\lambda = \frac{1}{320}\,\text{ft}^{-1}$ and $a = 80\,\text{ft}$. When at $x = \frac{\pi}{2\lambda}$, $a_P = (8.0\vec{\imath} + 2.0\vec{\jmath})\,\text{ft/s}^2$.

DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_t$	$\cos\theta$	$\sin\theta$
$\vec{e}_n$	$-\sin\theta$	$\cos\theta$

FORMULATE EQUATIONS: We'll determine the acceleration in terms of $\vec{\imath}$,$\vec{\jmath}$ and then transform this into the path frame using

$$a_t\vec{e}_t + a_n\vec{e}_n = a_x\vec{\imath} + a_y\vec{\jmath}$$

SOLVE: To find the angle θ we can calculate $\frac{dy}{dx}$ at $x = \frac{\pi}{2\lambda}$:

$$\frac{dy}{dx} = a\lambda\sin\left[\lambda\left(\frac{\pi}{2\lambda}\right)\right] = (80\,\text{ft})(\frac{1}{320}\,\text{ft}^{-1}) = 0.25$$

$$\frac{dy}{dx} = 0.25 \Rightarrow \theta = 14.0^\circ$$

$$(8.0\vec{\imath} + 2.0\vec{\jmath})\,\text{ft/s}^2 = a_t\vec{e}_t + a_n\vec{e}_n = a_t\,(0.970\vec{\imath} + 0.243\vec{\jmath}) + a_n\,(-0.243\vec{\imath} + 0.970\vec{\jmath})$$

$\vec{\imath}$: $$8.0\,\text{ft/s}^2 = 0.970a_t - 0.243a_n$$

$\vec{\jmath}$: $$2.0\,\text{ft/s}^2 = 0.243a_t + 0.970a_n$$

Solving these two equations gives us $a_n = 0$. Since the normal acceleration is given by $a_n = \frac{v^2}{r_C}$ we can have a zero value of this acceleration component through having a speed of zero or an infinite radius of curvature. The radius of curvature at the give point is found from

$$r_C = \frac{\left[1 + (\frac{dy}{dx})^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}$$

Evaluating the second derivative of y at $x = \frac{\pi}{2\lambda}$ gives us

$$\frac{d^2y}{dx^2} = a\lambda^2\cos\left[\lambda\left(\frac{\pi}{2\lambda}\right)\right] = 0$$

Thus we see that our radius of curvature is infinite. We must therefore conclude that we *can't* determine v because any finite v will produce a zero normal acceleration component due to the infinite radius of curvature.

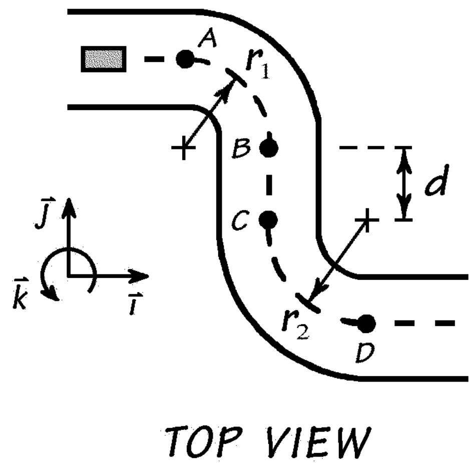
$$\boxed{v \text{ cannot be determined}}$$

Do It 2.4.3

GOAL: Determine the maximum constant acceleration in the straightaway from B to C so that the car just makes it through the bend from C to D without losing traction, and find the corresponding time to go from A to D.

GIVEN: From A to B, the car travels at a constant speed of $v_A = 25\,\text{mph}$ through a 90° arc of radius $r_1 = 70\,\text{ft}$. The straightaway from B to C is $d = 40\,\text{ft}$ long, and the bend from C to D is a 90° arc with a radius of $r_2 = 80\,\text{ft}$. The car's speed is constant from C to D. The car will slide off the road when its normal acceleration exceeds $a_{n,\max} = 0.75g$.

DRAW:



FORMULATE EQUATIONS:

Since the car's speed is constant through both bends, its acceleration during the turns can be expressed in terms of path coordinates as

$$\vec{a} = \frac{v_t^2}{r}\vec{e}_n \tag{1}$$

The time it takes for the car to travel through each turn with a constant speed is governed by

$$v = \frac{\Delta s}{\Delta t} \tag{2}$$

When the car is accelerating at a constant rate in the straightaway, we can say that

$$v^2 - v_0^2 = 2a\Delta s \tag{3}$$

$$a = \frac{\Delta v}{\Delta t} \tag{4}$$

SOLVE:

The car will lose traction when its normal acceleration exceeds $a_{n,\max} = 0.75g = 24.2\,\text{ft/s}^2$, so let's first verify that it will make it through the first turn from A to B:

(1) ⇒
$$\vec{a}_{AB} = \frac{v_A^2}{r_1}\vec{e}_n$$

$$\|\vec{a}_{AB}\| = \sqrt{\vec{a}_{AB}\cdot\vec{a}_{AB}} = \frac{v_A^2}{r_1}$$

$$\|\vec{a}_{AB}\| = a_{AB,n} = \frac{\left[(25\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)\right]^2}{70\,\text{ft}}$$

$$a_{AB,n} = 19.2\,\text{ft/s}^2 < a_{n,\max}$$

We find that the car indeed makes it through the first bend, and thus the time it takes to go from A to B is

(2) ⇒
$$t_{AB} = \frac{\pi r_1}{2v_A}$$

$$t_{AB} = \frac{\pi(70\,\text{ft})}{2(25\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)}$$

$$t_{AB} = 3.0\,\text{s}$$

We can find the maximum constant speed at which the car can travel through the bend from C to D by letting $\|\vec{a}_{CD}\| = a_{CD,n} = a_{n,\max}$:

(1) ⇒
$$a_{n,\max} = \frac{v_C^2}{r_2}$$

$$v_C = \sqrt{a_{n,\max} r_2}$$

$$v_C = \sqrt{0.75(32.2\,\text{ft/s}^2)(80\,\text{ft})}$$

$$v_C = 44.0\,\text{ft/s} = 30.0\,\text{mph}$$

Therefore, the time it takes to go from C to D is given by

(2) ⇒
$$t_{CD} = \frac{\pi r_2}{2v_C}$$

$$t_{CD} = \frac{\pi(80\,\text{ft})}{2(44.0\,\text{ft/s})}$$

$$t_{CD} = 2.86\,\text{s}$$

Now that we have the car's speed at C (and we already know that the speeds at A and B are the same), we can determine the constant acceleration in the straightaway:

(3) ⇒
$$a_{BC} = \frac{v_C^2 - v_B^2}{2d}$$

$$a_{BC} = \frac{(44.0\,\text{ft/s})^2 - \left[(25\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)\right]^2}{2(40\,\text{ft})}$$

$$\boxed{a_{BC} = 7.34\,\text{ft/s}^2}$$

The corresponding time of travel is

(4) $\Rightarrow$
$$t_{BC} = \frac{v_C - v_B}{a_{BC}}$$

$$t_{BC} = \frac{44.0\,\text{ft/s} - (25\,\text{mph})\left(\frac{\text{hr}}{3600\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{\text{mi}}\right)}{7.34\,\text{ft/s}^2}$$

$$t_{BC} = 0.99\,\text{s}$$

Lastly, the total time it takes to go from A to D is

$$t_{\text{total}} = t_{AB} + t_{BC} + t_{CD}$$

$$t_{\text{total}} = 3.0\,\text{s} + 2.86\,\text{s} + 0.99\,\text{s}$$

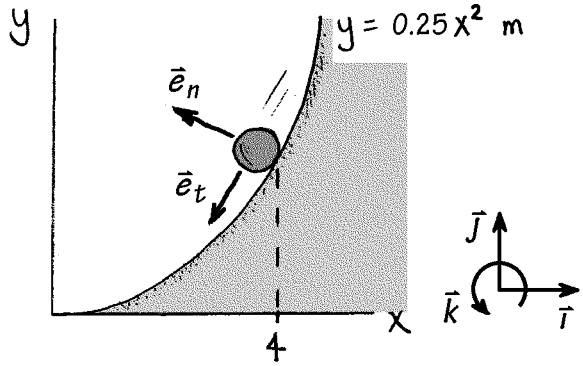
$$\boxed{t_{\text{total}} = 6.85\,\text{s}}$$

2.4.15

GOAL: Determine $\|\vec{a}\|$ at the given instant.

GIVEN: The snowball has a tangential speed $v_t = 10\,\text{m/s}$ and tangential acceleration $a_t = 2\,\text{m/s}^2$ at $x = 4\,\text{m}$. The hill profile is described by $y = 0.25x^2\,\text{m}$.

DRAW:



FORMULATE EQUATIONS:

The acceleration of the snowball in terms of path coordinates is

$$\vec{a} = a_t \vec{e}_t + \frac{v_t^2}{r_c}\vec{e}_n \tag{1}$$

The radius of curvature is

$$r_c = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|} \tag{2}$$

SOLVE:

Differentiate $y(x) \Rightarrow$

$$\frac{dy}{dx} = 0.5x \tag{3}$$

Differentiate $\frac{dy}{dx} \Rightarrow$

$$\frac{d^2y}{dx^2} = 0.5 \tag{4}$$

(3), (4) $\rightarrow$ (2) $\Rightarrow$

$$r_c(x) = \frac{[1 + (0.5x)^2]^{\frac{3}{2}}}{0.5}\,\text{m}$$

$$r_c(4\,\text{m}) = \frac{[1 + (0.5 \cdot 4)^2]^{\frac{3}{2}}}{0.5}\,\text{m}$$

$$\boxed{r_c(4\,\text{m}) = 22.36\,\text{m}} \tag{5}$$

(1) ⇒
$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_t^2 + \left(\frac{v_t^2}{r_c}\right)^2} \tag{6}$$

(5) → (6) ⇒
$$\|\vec{a}\| = \sqrt{(2\,\mathrm{m/s^2})^2 + \left(\frac{(10\,\mathrm{m/s})^2}{22.36\,\mathrm{m}}\right)^2}$$

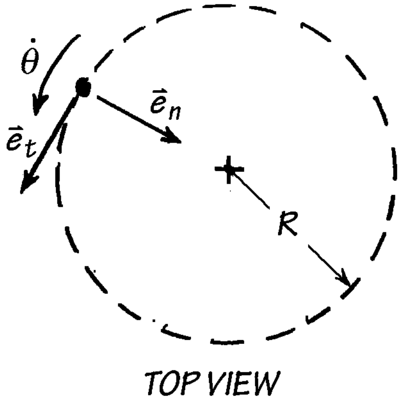
$$\boxed{\|\vec{a}\| = 4.90\,\mathrm{m/s^2}}$$

2.4.16

GOAL: Determine $\|\vec{a}\|$ in g's, $\dot{\theta}$, $\ddot{\theta}$.

GIVEN: The end of the test tube in the centrifuge has tangential velocity $v_t = 10\,\text{ft/s}$ and tangential acceleration $a_t = 10\,\text{ft/s}^2$ at the given instant. The radius of the path is $R = 0.75\,\text{ft}$.

DRAW:



FORMULATE EQUATIONS:

The acceleration of the end of the test tube in terms of path coordinates is

$$\vec{a} = a_t \vec{e}_t + \frac{v_t^2}{R}\vec{e}_n \tag{1}$$

SOLVE:

(1) ⇒

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_t^2 + \left(\frac{v_t^2}{R}\right)^2}$$

$$\|\vec{a}\| = \sqrt{(10\,\text{ft/s}^2)^2 + \left(\frac{(10\,\text{ft/s})^2}{0.75\,\text{ft}}\right)^2}$$

$$\boxed{\|\vec{a}\| = 133.7\,\text{ft/s}^2 = 4.152g}$$

(1) ⇒

$$\frac{v_t^2}{R} = R\dot{\theta}^2$$

$$\dot{\theta} = \frac{v_t}{R}$$

$$\dot{\theta} = \frac{10\,\text{ft/s}}{0.75\,\text{ft}}$$

$$\boxed{\dot{\theta} = 13.33\,\text{rad/s}}$$

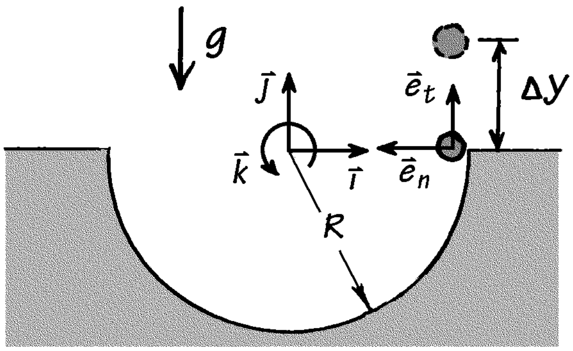
(1) ⇒

$$a_t = R\ddot{\theta}$$

$$\ddot{\theta} = \frac{a_t}{R}$$
$$\ddot{\theta} = \frac{10\,\mathrm{ft/s^2}}{0.75\,\mathrm{ft}}$$

$$\boxed{\ddot{\theta} = 13.33\,\mathrm{rad/s^2}}$$

2.4.17
GOAL: Determine a_n and v_t before the snowboarder becomes airborne. Also find the maximum height achieved and the total time in the air.
GIVEN: Before becoming airborne, the snowboarder has $\|\vec{a}\| = 20\,\text{ft/s}^2$ and $a_t = 10\,\text{ft/s}^2$. The halfpipe has a radius $R = 20\,\text{ft}$.
DRAW:



FORMULATE EQUATIONS:
The acceleration of the snowboarder at the edge of the halfpipe in terms of path coordinates is

$$\vec{a} = a_t\vec{e}_t + a_n\vec{e}_n = a_t\vec{e}_t + \frac{v_t^2}{R}\vec{e}_n \tag{1}$$

The maximum height achieved is governed by

$$v_t^2 = 2g\Delta y \tag{2}$$

The time to reach the maximum height is governed by

$$v_t = g\Delta t \tag{3}$$

SOLVE:

(1) ⇒
$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_t^2 + a_n^2}$$

$$\begin{aligned} a_n &= \sqrt{\|\vec{a}\|^2 - a_t^2} \\ a_n &= \sqrt{(20\,\text{ft/s}^2)^2 - (10\,\text{ft/s}^2)^2} \end{aligned}$$

$$\boxed{a_n = 17.32\,\text{ft/s}^2} \tag{4}$$

(1) ⇒
$$a_n = \frac{v_t^2}{R}$$

$$v_t = \sqrt{a_n R} \tag{5}$$

(4) → (5) ⇒
$$v_t = \sqrt{(17.32\,\text{ft/s}^2)(20\,\text{ft})}$$

$$\boxed{v_t = 18.61\,\text{ft/s}} \tag{6}$$

(2) ⇒
$$\Delta y = \frac{v_t^2}{2g} \tag{7}$$

(6) → (7) ⇒
$$\Delta y = \frac{(18.61\,\text{ft/s})^2}{2(32.2\,\text{ft/s}^2)}$$

$$\boxed{\Delta y = 5.38\,\text{ft}}$$

(3) ⇒
$$\Delta t = \frac{v_t}{g} \tag{8}$$

(6) → (8) ⇒
$$\Delta t = \frac{18.61\,\text{ft/s}}{32.2\,\text{ft/s}^2}$$

$$\Delta t = 0.578\,\text{s}$$

The total time in the air is simply twice the time to the maximum height, so

$$\begin{aligned} t_{\text{total}} &= 2\Delta t \\ t_{\text{total}} &= 2(0.578\,\text{s}) \end{aligned}$$

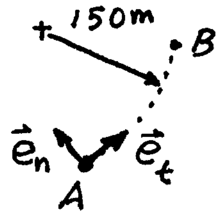
$$\boxed{t_{\text{total}} = 1.156\,\text{s}}$$

2.4.18

GOAL: Find constant tangential acceleration needed for car's total acceleration to be $8\,\mathrm{m/s^2}$ at point B.

GIVEN: Initial conditions and dimensions.

DRAW:



FORMULATE EQUATIONS:

$a\,dx = v\,dv$ with constant acceleration$\Rightarrow$

$$a\left(s_B - s_0\right) = \frac{1}{2}\left(v(s_B)^2 - v(s_0)^2\right) \quad (1)$$

Acceleration at B

$$\vec{a}_B = a\,\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n \quad (2)$$

SOLVE:

(1) and initial conditions$\Rightarrow$

$$\left[v(x_B)\right]^2 = 2a(100\,\mathrm{m})$$

$$v(x_B) = \sqrt{(200\,\mathrm{m})a} \quad (3)$$

(2) $\Rightarrow$

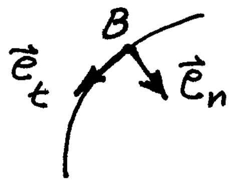
$$\|\vec{a}_B\| = \sqrt{a^2 + \frac{v(x_B)^4}{r_C^2}} = 8\,\mathrm{m/s} \quad (4)$$

(3), (4) $\Rightarrow$

$$a^2 = (8\,\mathrm{m/s^2})^2 - \frac{(200\,\mathrm{m})^2 a^2}{(150\,\mathrm{m})^2} \quad (5)$$

$$\boxed{a = 4.8\,\mathrm{m/s^2}}$$

2.4.19
GOAL: Time before bicyclist B slips.
GIVEN: Initial conditions, dimensions, maximum sustainable acceleration, constraint of constant speed.
DRAW:



FORMULATE EQUATIONS:

Acceleration of B
$$\vec{a}_B = a_t\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n \tag{1}$$

SOLVE:

$$20\,\text{mph} = 29.3\,\text{ft/s}$$

(1) and constant speed condition$\Rightarrow$
$$\vec{a}_B = \frac{v^2}{r_C}\vec{e}_n = 28\,\text{ft/s}^2\vec{e}_n \tag{2}$$

(2) $\Rightarrow$
$$r_C = \frac{(29.3\,\text{ft/s})^2}{28\,\text{ft/s}^2} = 30.73\,\text{ft} \tag{3}$$

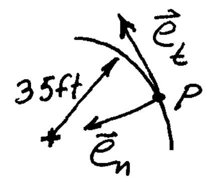
(3) $\Rightarrow$
$$r_C = 50 - 0.0025s^2 = 30.73\,\text{ft} \Rightarrow s = 87.8\,\text{ft}$$

$$\boxed{t = \frac{87.8\,\text{ft}}{29.3\,\text{ft/s}} = 3\,\text{s}}$$

2.4.20

GOAL: Find $|v_P|$ and $|a_P|$ of a point P on a ferris wheel.

GIVEN: Wheel's radius, $\dot{\theta}$ as a function of time and elapsed time.



FORMULATE EQUATIONS:

$$\vec{v}_P = v\vec{e}_t \tag{1}$$

$$\vec{a}_P = \dot{v}\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n \tag{2}$$

SOLVE:

$$\dot{\theta} = ct \Rightarrow \ddot{\theta} = c \tag{3}$$

Because P is moving in a circular fashion,

$$v = r_C\dot{\theta} = (35\,\text{ft})\dot{\theta} \tag{4}$$

(3), (4) → (1) ⇒ $\quad \vec{v}_P = (35\,\text{ft})[(0.05\,\text{rad/s}^2)(5\,\text{s})] = 8.75\,\text{ft/s}$

$$\boxed{v = |v_p| = 8.75\,\text{ft/s}}$$

(3), (4) ⇒ $$\dot{v} = r_C\ddot{\theta} = (35\,\text{ft})(0.05\,\text{rad/s}^2) = 1.75\,\text{ft/s}^2 \tag{5}$$

(4), (5) → (2) ⇒ $\quad \vec{a}_P = (1.75\,\text{ft/s}^2)\vec{e}_t + \dfrac{(8.75\,\text{ft/s})^2}{35\,\text{ft}}\vec{e}_n = (1.75\vec{e}_t + 2.19\vec{e}_n)\,\text{ft/s}^2$

$$\boxed{|\vec{a}_P| = 2.80\,\text{ft/s}^2}$$

2.4.21
GOAL: Find a track's radius of curvature.
DRAW:

FORMULATE EQUATIONS:
To solve this problem, start with the equation $ads = vdv$. The acceleration is constant, $a = 5\,\text{ft/s}^2$. So, integrating the equation

$$\int_{s_0}^{s_f} ads = \int_{v_0}^{v_f} vdv$$

gives

$$a(s_f - s_0) = \frac{1}{2}(v_f^2 - v_0^2)$$

SOLVE:
Using the given values gives

$$(5\,\text{ft/s}^2)(600\,\text{ft}) = \frac{1}{2}(v_f^2 - (80\,\text{ft/s})^2)$$

So the value for the final velocity is

$$v_f = 111\,\text{ft/s}$$

We know the magnitude of the car's total acceleration is $14.\bar{6}\,\text{ft/s}^2$ at C. The equation for the magnitude of the total acceleration is

$$||\vec{a}|| = \sqrt{a_n^2 + a_t^2}$$

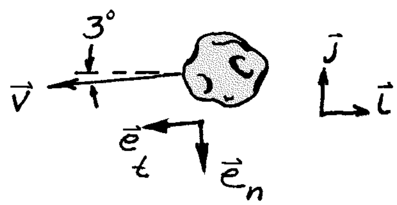
Since $a_n = \frac{v^2}{r_C}$ we can compute the radius of curvature

$$14.\bar{6}\,\text{ft/s}^2 = \sqrt{\left(\frac{(111\,\text{ft/s})^2}{r_C}\right)^2 + (5\,\text{ft/s}^2)^2}$$

and

$$\boxed{r_C = 899\,\text{ft}}$$

2.4.22
GOAL: Find the radius of curvature of a meteor's path at the given instant.
GIVEN: Speed, heading and acceleration information.
DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_t$	$-\cos 3°$	$-\sin 3°$
$\vec{e}_n$	$\sin 3°$	$-\cos 3°$

FORMULATE EQUATIONS:
We'll be using the formula for acceleration as expressed in path coordinates:

$$\vec{a} = \dot{v}\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n$$

SOLVE:
The first step is to convert the gravitational acceleration into path coordinates of the meteor and add that to the deceleration due to drag, so the total acceleration of the meteor is

$$\begin{aligned}\vec{a}_m &= -9.5\vec{\jmath}\,\mathrm{m/s^2} - 5.0\vec{e}_t\,\mathrm{m/s^2} \\ &= -9.5(-\cos 3°\vec{e}_n - \sin 3°\vec{e}_t)\,\mathrm{m/s^2} - 5\vec{e}_t\,\mathrm{m/s^2} \\ &= 9.49\vec{e}_n - 4.50\vec{e}_t \end{aligned} \tag{1}$$

Converting the velocity from kph to m/s

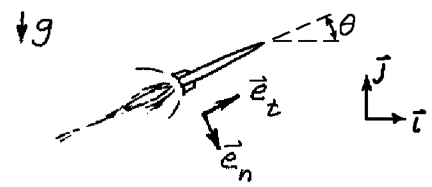
$$v = (25,000\,\mathrm{kph})\frac{1000\,\mathrm{m/km}}{3600\,\mathrm{s/hr}} = 6944\,\mathrm{m/s}$$

Using our expression for the normal acceleration component a_n gives us

$$\frac{(6944\,\mathrm{m/s})^2}{r_C} = 9.49\,\mathrm{m/s^2}$$

$$\boxed{r_C = 5.08\times10^3\,\mathrm{km}}$$

Do It 2.4.4
GOAL: Determine the acceleration of the rocket's engine, the total acceleration in path coordinates, and the radius of curvature of the path.
GIVEN: Rocket's acceleration and velocity.
DRAW:



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_t$	$\cos\theta$	$\sin\theta$
$\vec{e}_n$	$\sin\theta$	$-\cos\theta$

FORMULATE EQUATIONS:
We'll be using the formula for acceleration as expressed in path coordinates:

$$\vec{a} = \dot{v}\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n$$

SOLVE: The total acceleration of the rocket is

$$\vec{a}_R = \vec{a}_E + \vec{a}_G$$

where $\vec{a}_E$ is the acceleration due to the engine and $\vec{a}_G$ is the acceleration due to gravity. So

$$(5.657\vec{\imath} - 3.843\vec{\jmath})\,\text{m/s}^2 = (a_1\vec{\imath} + a_2\vec{\jmath})\,\text{m/s}^2 - 9.5\vec{\jmath}\,\text{m/s}^2$$

Equating coefficients gives $a_1 = 5.657\,\text{m/s}^2$ $a_2 = 5.657\,\text{m/s}^2$ The acceleration of the rocket engine is

$$\boxed{\vec{a}_E = (5.657\vec{\imath} + 5.657\vec{\jmath})\,\text{m/s}^2}$$

To convert the rocket's acceleration to path coordinates, you need to know the angle the rocket is flying with reference to the vectors $\vec{\imath}$ and $\vec{\jmath}$. The velocity vector,

$$\vec{v}_R = (5,000\vec{\imath} + 2,000\vec{\jmath})\,\text{m/s},$$

can be used to determine this angle.

$$\theta = \tan^{-1}\left(\frac{2,000}{5,000}\right) = 21.8^\circ$$

Converting the acceleration to path coordinates

$$\vec{a}_R = 5.657(\cos\theta\vec{e}_t + \sin\theta\vec{e}_n) - 3.843(\sin\theta\vec{e}_t - \cos\theta\vec{e}_n)$$

Evaluating this equation gives

$$\boxed{\vec{a}_R = (3.823\vec{e}_t + 5.669\vec{e}_n)}$$

To find the radius of curvature, use the equation $\frac{v^2}{r_C} = a_n$ with $v = \sqrt{(5000^2 + 2000^2)\,(\mathrm{m/s})^2} = 5,385\,\mathrm{m/s}$ and $a_n = 5.669\,\mathrm{m/s^2}$. Solving for r_C gives

$$\boxed{r_C = 5115\,\mathrm{km}}$$

2.4.23
GOAL: Determine the acceleration of a particle moving along the path $xy = 36\,\mathrm{m}^2$.
GIVEN: Horizontal speed of the particle.
FORMULATE EQUATIONS: We'll use the facts that displacement differentiated with respect to time gives us velocity and velocity differentiated with respect to time gives us acceleration.
SOLVE: $xy = 36\,\mathrm{m}^2$. If $x = 9$ m then $y = \frac{36\,\mathrm{m}^2}{9\,\mathrm{m}} = 4$ m.
Differentiating $xy = 36\,\mathrm{m}^2$ gives

$$\dot{x}y + x\dot{y} = 0 \Rightarrow \dot{y} = -\frac{y}{x}\dot{x} \tag{1}$$

$(1) \Rightarrow$

$$\vec{v} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} = \dot{x}\left(\vec{\imath} - \frac{y}{x}\vec{\jmath}\right) \tag{2}$$

$$\begin{aligned} \vec{a} = \dot{\vec{v}} &= \ddot{x}\left(\vec{\imath} - \frac{y}{x}\vec{\jmath}\right) + \dot{x}\left(-\frac{\dot{y}}{x}\vec{\jmath} + \frac{y\dot{x}}{x^2}\vec{\jmath}\right) \\ &= \ddot{x}\vec{\imath} + \quad -\frac{y\ddot{x}}{x} - \frac{\dot{x}\dot{y}}{x} + \frac{y\dot{x}^2}{x^2}\Bigg)\vec{\jmath} \end{aligned} \tag{3}$$

We're given that $v = 10$ m/s. Using (2) gives us

$$v = \dot{x} \quad 1 + \left(\frac{y}{x}\right)^2\Bigg)^{\frac{1}{2}} = 10\,\mathrm{m/s} \Rightarrow \dot{x} = 9.138\,\mathrm{m/s} \text{ (using } x = 9\,\mathrm{m},\ y = 4\,\mathrm{m})$$

From (1) we have $\dot{y} = -4.061$ m/s.
We're also given that v is constant. Thus $\dot{v} = 0$. From (2) this gives

$$\dot{v} = \frac{d}{dt}\left[\dot{x} \quad 1 + \left(\frac{y}{x}\right)^2\Bigg)^{\frac{1}{2}}\right] = 0$$

$$\ddot{x} \quad 1 + \left(\frac{y}{x}\right)^2\Bigg)^{\frac{1}{2}} + \dot{x} \quad 1 + \left(\frac{y}{x}\right)^2\Bigg)^{-\frac{1}{2}} \frac{y}{x}\left(\frac{\dot{y}}{x} - \frac{y\dot{x}}{x^2}\right) = 0$$

Using the values for x, y, $\dot{x}$ and $\dot{y}$ gives $\ddot{x} = 3.06\,\mathrm{m/s}^2$.
Using the values for x, y, $\dot{x}$, $\dot{y}$ and $\ddot{x}$ in (3) gives

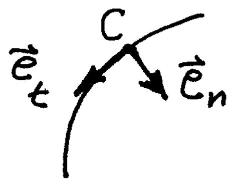
$$\boxed{\vec{a} = [3.06\vec{\imath} + 6.89\vec{\jmath}]\ \mathrm{m/s}^2}$$

2.4.24

GOAL: Find distance traveled and time elapsed when the magnitude of car's acceleration is 0.9 m/s^2.

GIVEN: Constraint of constant tangential acceleration, initial conditions, and dimensions.

DRAW:



FORMULATE EQUATIONS:

Speed
$$v = \int_0^t a_t dt = a_t t \tag{1}$$

Distance traveled
$$s = \int_0^t v dt = \frac{1}{2} a_t t^2 \tag{2}$$

Acceleration
$$\vec{a} = a_t \vec{e}_t + \frac{v^2}{r_C} \vec{e}_n \tag{3}$$

SOLVE:

(1) → (3) ⇒
$$\|\vec{a}\| = \sqrt{a_t^2 + \frac{a_t^4 t^4}{r_C^2}} = 0.9\,\mathrm{m/s^2} \tag{4}$$

(4) ⇒
$$t^4 = \left[(0.9\,\mathrm{m/s^2})^2 - (0.75\,\mathrm{m/s^2})^2\right] \frac{(300\,\mathrm{m})^2}{(0.75\,\mathrm{m/s^2})^4} \tag{5}$$

$$\boxed{t = 16.3\,\mathrm{s}} \tag{6}$$

(6) → (2) ⇒
$$s = \frac{1}{2}(0.75\,\mathrm{m/s^2})(16.3\,\mathrm{s})^2$$

$$\boxed{s = 99.5\,\mathrm{m}}$$

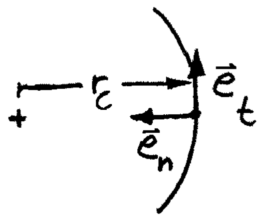
2.4.25
GOAL: Calculate the radius of curvature for a race car.
GIVEN: Track dimensional data, car's speed and car's overall acceleration level.
FORMULATE EQUATIONS: The acceleration vector written in the $(\vec{e}_t, \vec{e}_n)$ basis is

$$\vec{a} = \dot{v}\,\vec{e}_t + \frac{v^2}{r_C}\vec{e}_n$$

where r_C is the radius of curvature.
SOLVE: Given $a_n = 0.5g$ and $v = 200$ km/hr $= 55.5$ m/s, we can solve for r_C

$$r_C = \frac{v^2}{.5g} = \frac{(55.5\,\mathrm{m/s})^2}{0.5(9.81\,\mathrm{m/s^2})} = \boxed{629\,\mathrm{m}}$$

2.4.26
GOAL: Determine whether or not a car will lose traction during a turn.
GIVEN: The radius of curvature at entry and exit of the turn as well as the car's speed.
DRAW:



FORMULATE EQUATIONS:
All we need is the path coordinate form for acceleration:

$$\vec{a}_c = \dot{v}\vec{e}_t + \frac{v^2}{\rho_c}\vec{e}_n$$

SOLVE:
In this problem $\dot{v} = 0$ so

$$\vec{a}_c = \frac{v^2}{\rho_c}\vec{e}_n$$

Converting from mph to ft/s we have 100 mph=146.$\bar{6}$ ft/s
At the start of the curve we have

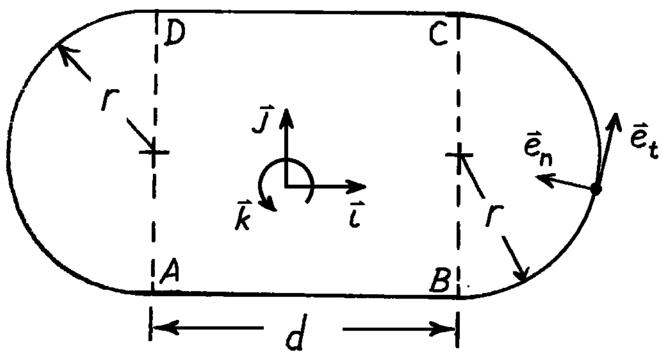
$$\frac{v^2}{r_c} = \frac{(146.\bar{6}\,\mathrm{ft/s})^2}{1000\,\mathrm{ft}} = 21.5\ \mathrm{ft/s^2}$$

At the end we have

$$\frac{v^2}{r_c} = \frac{(146.\bar{6})}{500} = 43\ \mathrm{ft/s^2}$$

43 ft/s^2 exceeds the tire's maximum sustainable acceleration of 35 ft/s^2 and so the car slips before exiting the turn.

2.4.27
GOAL: Determine v_B, a_{BC}, v_D, and the time to complete one circuit.
GIVEN: The straightaways have a length of $d = 50$ m, and the rounds have radius $r = 15$ m. The acceleration from A to B is $a_1 = 2$ m/s^2, and the acceleration from C to D is $a_2 = 1$ m/s^2. The chariot has a constant velocity in the rounds.
DRAW:



FORMULATE EQUATIONS:
In the straightaways, the chariot's displacement, velocity, acceleration, and travel time are related by

$$v_f^2 - v_i^2 = 2as \tag{1}$$

$$v_f - v_i = at \tag{2}$$

In the rounds, the chariot's displacement, velocity, acceleration, and travel time are related by

$$\vec{a} = \frac{v^2}{r}\vec{e}_n \tag{3}$$

$$\pi r = vt \tag{4}$$

SOLVE:
From A to B in the first straightaway,

(1) ⇒
$$v_B = \sqrt{2a_1 d}$$

$$v_B = \sqrt{2(2\,\text{m/s}^2)(50\,\text{m})}$$

$$\boxed{v_B = 14.14\,\text{m/s}}$$

(2) ⇒
$$t_{AB} = \frac{v_B}{a_1}$$

$$t_{AB} = \frac{14.14\,\mathrm{m/s}}{2\,\mathrm{m/s^2}}$$

$$t_{AB} = 7.07\,\mathrm{s} \tag{5}$$

From B to C in the first round,

(3) $\Rightarrow$

$$a_{BC} = \frac{v_B^2}{r}$$

$$a_{BC} = \frac{(14.14\,\mathrm{m/s})^2}{15\,\mathrm{m}}$$

$$\boxed{a_{BC} = 13.33\,\mathrm{m/s^2}}$$

(4) $\Rightarrow$

$$t_{BC} = \frac{\pi r}{v_B}$$

$$t_{BC} = \frac{\pi(15\,\mathrm{m})}{14.14\,\mathrm{m/s}}$$

$$t_{BC} = 3.33\,\mathrm{s} \tag{6}$$

From C to D in the second straightaway,

(1) $\Rightarrow$

$$v_D = \sqrt{v_C^2 + 2a_2 d} = \sqrt{v_B^2 + 2a_2 d}$$

$$v_D = \sqrt{(14.14\,\mathrm{m/s})^2 + 2(1\,\mathrm{m/s^2})(50\,\mathrm{m})}$$

$$\boxed{v_D = 17.32\,\mathrm{m/s}}$$

(2) $\Rightarrow$

$$t_{CD} = \frac{v_D - v_C}{a_2} = \frac{v_D - v_B}{a_2}$$

$$t_{CD} = \frac{17.32\,\mathrm{m/s} - 14.14\,\mathrm{m/s}}{1\,\mathrm{m/s^2}}$$

$$t_{CD} = 3.18\,\mathrm{s} \tag{7}$$

From D to A in the second round,

(4) $\Rightarrow$

$$t_{DA} = \frac{\pi r}{v_D}$$

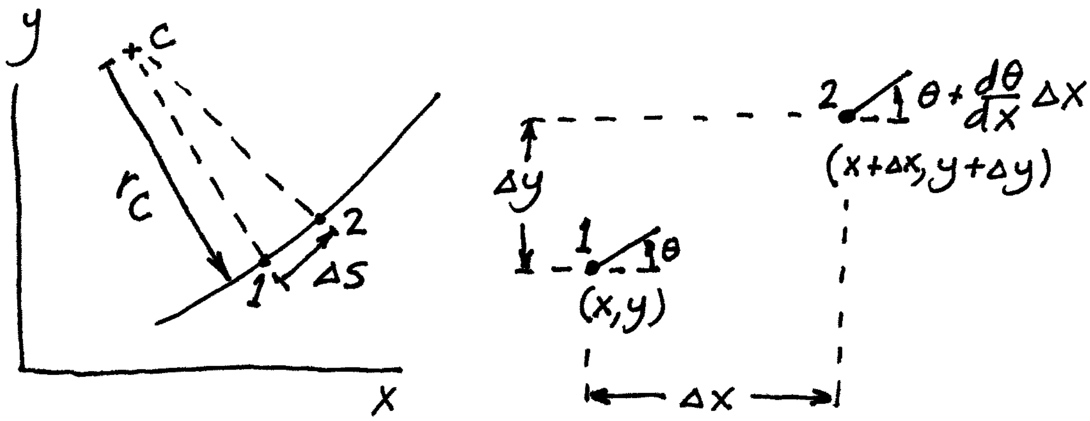
$$t_{DA} = \frac{\pi(15\,\mathrm{m})}{17.32\,\mathrm{m/s}}$$

$$t_{DA} = 2.72\,\mathrm{s} \tag{8}$$

(5) + (6) + (7) + (8) $\Rightarrow$

$$\boxed{t_{\text{total}} = 16.30\,\mathrm{s}}$$

2.4.28
GOAL: Derive the formula for the radius of curvature.
GIVEN: Coordinate geometry.
DRAW



FORMULATE EQUATIONS:
Two positions are drawn, 1 and 2 and we have the relationship

$$\Delta s = r_C \Delta\theta \tag{1}$$

between the two, i.e. the length of the curve traced out as θ changes slightly is equal to r_C times the angular change $\Delta\theta$. A further picture shows these two points along with their original and altered x, y coordinates.
SOLVE:
Start with $\Delta\theta$:

$$\Delta\theta = \frac{d\theta}{dx}\Delta x \tag{2}$$

At 1 we have $\theta = \arctan\frac{dy}{dx}$, the usual definition of slope.
To find $\frac{d\theta}{dx}$ we need to differentiate:

$$\tan\theta = \frac{dy}{dx}$$

$$\frac{d}{dx}(\tan\theta) = \frac{d^2y}{dx^2}$$

$$\frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\left(1+\tan^2\theta\right)\frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$1+\left(\frac{dy}{dx}\right)^2\Bigg)\frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2}\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{-1} \tag{3}$$

Δs is the Euclidean distance between points 1 and 2:

$$\Delta s = \sqrt{(x + \Delta x - x)^2 + (y + \Delta y - y)^2}$$

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{4}$$

Using $\Delta y = \frac{dy}{dx}\Delta x$ in (4) yields

$$\Delta s = \sqrt{(\Delta x)^2 + \left(\frac{dy}{dx}\right)^2 (\Delta x)^2} \tag{5}$$

$(1), (2), (3), (5) \Rightarrow$ $$\sqrt{(\Delta x)^2 + \left(\frac{dy}{dx}\right)^2 (\Delta x)^2} = r_C \left|\frac{d^2y}{d^2x}\right| \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{-1} \Delta x$$

Canceling Δx and rearranging yields

$$\boxed{r_C = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{d^2x}\right|}}$$

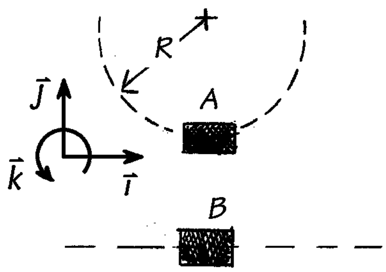
2.5 Relative Motion and Constraints

Do It 2.5.1

GOAL: Determine $\vec{v}_{A/B}$ and $\vec{a}_{A/B}$.

GIVEN: Jet A has velocity $\vec{v}_A = 660\vec{\imath}$ ft/s and acceleration $\vec{a}_{t,A} = 50\vec{\imath}$ ft/s^2, and it is at the bottom of a loop with radius $R = 100$ ft. Jet B has velocity $\vec{v}_B = -700\vec{\imath}$ ft/s and acceleration $\vec{a}_B = -40\vec{\imath}$ ft/s^2.

DRAW:



FORMULATE EQUATIONS:

The velocity of jet A as seen by jet B is

$$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B \tag{1}$$

The acceleration of jet A as seen by jet B is

$$\vec{a}_{A/B} = \vec{a}_A - \vec{a}_B = a_{t,A}\vec{\imath} + \frac{v_A^2}{R}\vec{\jmath} - \vec{a}_B \tag{2}$$

SOLVE:

(1) $\Rightarrow$
$$\vec{v}_{A/B} = (660\,\text{ft/s})\vec{\imath} - (-700\,\text{ft/s})\vec{\imath}$$

$$\boxed{\vec{v}_{A/B} = 1360\vec{\imath}\ \text{ft/s}}$$

(2) $\Rightarrow$
$$\vec{a}_{A/B} = (50\,\text{ft/s}^2)\vec{\imath} + \frac{(660\,\text{ft/s})^2}{100\,\text{ft}}\vec{\jmath} - (-40\,\text{ft/s}^2)\vec{\imath}$$

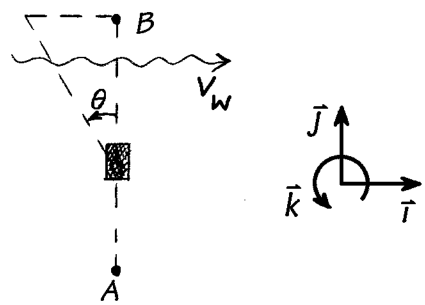
$$\boxed{\vec{a}_{A/B} = 90\vec{\imath} + 4356\vec{\jmath}\ \text{ft/s}^2}$$

2.5.1

GOAL: Determine how far over from B the plane will be at the anticipated time of arrival (i.e., with no wind) and at what speed and angle the plane should fly at relative to the wind to actually arrive at B in this time.

GIVEN: The plane initially has a velocity $\vec{v}_p = 200\vec{\jmath}$ mph and a strong wind blows at $\vec{v}_w = 80\vec{\imath}$ mph when the plane is $d = 100\,\text{mi}$ from B.

DRAW:



FORMULATE EQUATIONS:

The anticipated time of arrival t_B is based on when there is no wind, and so

$$t_B = \frac{d}{v_p} \tag{1}$$

The plane, if it doesn't course correct, will be pushed over a distance s to the right of B by the wind according to

$$s = v_w t_B \tag{2}$$

The actual velocity of the plane is given by

$$\vec{v}_{p,a} = \vec{v}_w + \vec{v}_{p/w} \tag{3}$$

SOLVE:

From (1), the plane's anticipated time of arrival is in

(1) ⇒
$$t_B = \frac{100\,\text{mi}}{200\,\text{mph}} = 0.5\,\text{hr}$$

By (2), the plane will actually arrive to the right of B at a distance

(2) ⇒
$$s = (80\,\text{mph})(0.5\,\text{hr})$$

$$\boxed{s = 40\,\text{mi}}$$

To arrive at B in the same amount of time as t_B, the actual velocity of the plane including the effects of the wind must be $\vec{v}_{p,a} = 200\vec{\jmath}$ mph, and so

$$\tan\theta = \frac{v_w}{v_{p,a}} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{80\,\text{mph}}{200\,\text{mph}}\right)$$

$$\boxed{\theta = 21.8^\circ}$$

(3) $\Rightarrow$ $$v_{p,a}\vec{\jmath} = v_w\vec{\imath} + v_{p/w}(-\sin\theta\vec{\imath} + \cos\theta\vec{\jmath})$$

$\vec{\imath}$: $$0 = v_w - v_{p/w}\sin\theta \quad \Rightarrow \quad v_{p/w} = \frac{80\,\text{mph}}{\sin(21.8^\circ)}$$

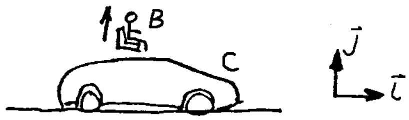
$$\boxed{v_{p/w} = 215\,\text{mph}}$$

2.5.2

GOAL: Solve for the velocity vector of the bad guy with respect to the ground as he leaves the car.

GIVEN: The car is moving to the right at 30 m/s and the ejector seat moves vertically at 10 m/s with respect to the car.

DRAW:



FORMULATE EQUATIONS:

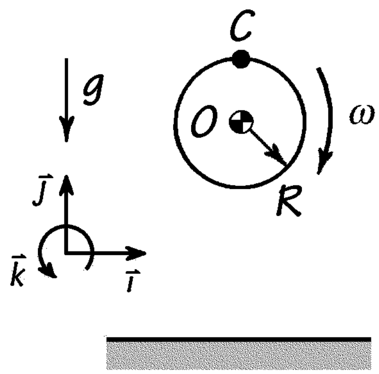
$$\vec{v}_B = \vec{v}_C + \vec{v}_{B/C}$$

SOLVE:

The bad guy (B) is traveling 30 m/s in the $\vec{\imath}$ direction when the ejector seat button is pushed. Immediately after the button is pushed, he also has a 10 m/s velocity in the $\vec{\jmath}$ direction. The overall velocity of the bad guy is therefore

$$\boxed{\vec{v}_B = (30\vec{\imath} + 10\vec{\jmath})\,\text{m/s}}$$

2.5.3
GOAL: Determine the speed and acceleration of the disk's top C.
GIVEN: The disk has a radius of $R = 7$ in. and spins at a constant angular speed of $\omega = 500$ rpm in the clockwise direction. The disk is released from rest far above the ground and allowed to fall $h = 3$ ft.
DRAW:



FORMULATE EQUATIONS:
The absolute velocity and acceleration of the disk's top C can be expressed as, respectively,

$$\vec{v}_C = \vec{v}_O + \vec{v}_{C/O} \quad (1)$$

$$\vec{a}_C = \vec{a}_O + \vec{a}_{C/O} \quad (2)$$

SOLVE:
Since the disk is released from rest, the speed of its mass center O after falling a distance h is given by $v_O = \sqrt{2gh}$, and thus the speed of the disk's top C is

(1) $\Rightarrow$
$$\vec{v}_C = -\sqrt{2gh}\,\vec{\jmath} + R\omega\vec{\imath}$$

$$\|\vec{v}_C\| = \sqrt{\vec{v}_C \cdot \vec{v}_C} = \sqrt{(R\omega)^2 + 2gh}$$

$$\|\vec{v}_C\| = \sqrt{\left[\left(\frac{7}{12}\,\text{ft}\right)\left(500\frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\,\text{s}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right)\right]^2 + 2(32.2\,\text{ft/s}^2)(3\,\text{ft})}$$

$$\boxed{\|\vec{v}_C\| = 33.6\,\text{ft/s}}$$

Noting that the disk is spinning at a constant angular speed and that its mass center is acted on by gravity alone, we have that the acceleration of the disk's top is

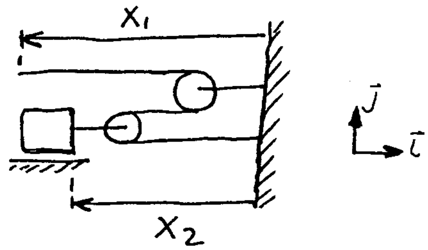
(2) $\Rightarrow$
$$\vec{a}_C = -g\vec{\jmath} - R\omega^2\vec{\jmath}$$

$$\|\vec{a}_C\| = \sqrt{\vec{a}_C \cdot \vec{a}_C} = g + R\omega^2$$

$$\|\vec{a}_C\| = 32.2\,\text{ft/s}^2 + \left(\frac{7}{12}\,\text{ft}\right)\left[\left(500\frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\,\text{s}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right)\right]^2$$

$$\boxed{\|\vec{a}_C\| = 1.63 \times 10^3\,\text{ft/s}^2}$$

2.5.4
GOAL: Solve for the velocity of the box.
GIVEN: The end of the rope is moving left at 2 m/s.
DRAW:



FORMULATE EQUATIONS:
The displacement of the worker is Δx_1 and the displacement of the box is Δx_2. So,

$$2\Delta x_2 + \Delta x_1 = 0$$

Now find the velocity relation by dividing by Δt and taking the limit as Δt goes to zero. This gives you

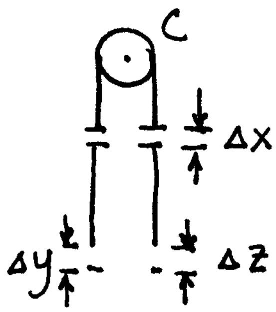
$$\dot{x}_1 = -2\dot{x}_2$$

SOLVE:

$$\dot{x}_2 = -\frac{1}{2}\dot{x}_1 = -\frac{1}{2}(2\,\text{m/s}) = -1\,\text{m/s}$$

$$\boxed{\vec{v}_{box} = 1\vec{\imath}\ \text{m/s}}$$

2.5.5
GOAL: Find the absolute velocity of B.
GIVEN: Pulley arrangement, $\dot{x}$ and $\dot{y}$.
DRAW



ASSUME: An overall constraint for this problem is the fact that both A and B are affected in the exact same manner by the central reel C's motion.
FORMULATE EQUATIONS:

$$-\Delta y - \Delta z + 2\Delta x = 0$$

SOLVE:
Differentiating gives us

$$-\dot{y} - \dot{z} + 2\dot{x} = 0$$

$$\dot{z} = 2\dot{x} - \dot{y}$$

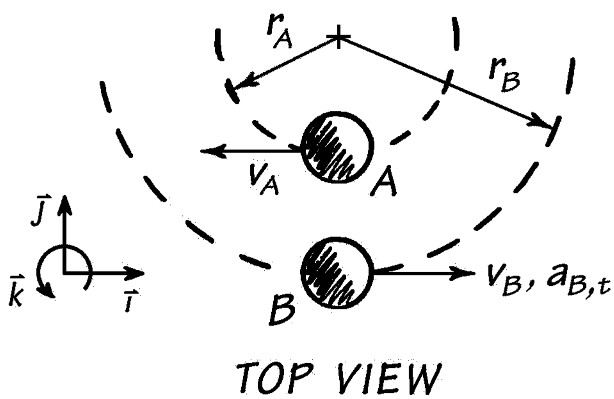
$$\boxed{\vec{v}_B = (2\dot{x} - \dot{y})\vec{\jmath}}$$

2.5.6

GOAL: Determine the velocity and acceleration of B as seen by A.

GIVEN: Two ball bearings (A and B) are constrained to move in the horizontal plane along concentric circular paths of radii $r_A = 0.4$ m and $r_B = 0.9$ m. At the given instant, the velocity of A is a constant $\vec{v}_A = -5\vec{\imath}$ m/s, whereas B is traveling at $\vec{v}_B = 8\vec{\imath}$ m/s with a tangential acceleration of $\vec{a}_{B,t} = 0.7\vec{\imath}$ m/s^2.

DRAW:



FORMULATE EQUATIONS:

The velocity and acceleration of B as seen by A are given by, respectively,

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A \tag{1}$$

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A \tag{2}$$

SOLVE:

The velocity of B relative to A is

(1) $\Rightarrow$

$$\vec{v}_{B/A} = v_B\vec{\imath} - (-v_A)\vec{\imath}$$

$$\vec{v}_{B/A} = (8\,\text{m/s} + 5\,\text{m/s})\,\vec{\imath}$$

$$\boxed{\vec{v}_{B/A} = 13\vec{\imath}\,\text{m/s}}$$

Since the velocity of A is constant at this instant, its acceleration has a normal component only, and therefore the acceleration of B relative to A is

(2) $\Rightarrow$

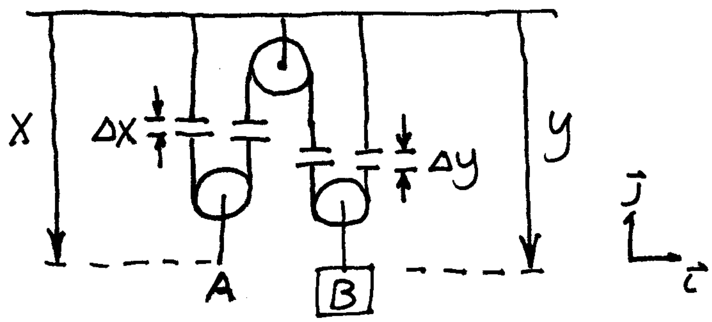
$$\vec{a}_{B/A} = (a_{B,t}\vec{\imath} + a_{B,n}\vec{\jmath}) - a_{A,n}\vec{\jmath}$$

$$\vec{a}_{B/A} = a_{B,t}\vec{\imath} + \left(\frac{v_B^2}{r_B} - \frac{v_A^2}{r_A}\right)\vec{\jmath}$$

$$\vec{a}_{B/A} = (0.7\,\mathrm{m/s^2})\,\vec{\imath} + \left[\frac{(8\,\mathrm{m/s})^2}{0.9\,\mathrm{m}} - \frac{(5\,\mathrm{m})^2}{0.4\,\mathrm{m}}\right]\vec{\jmath}$$

$$\boxed{\vec{a}_{B/A} = 0.7\,\vec{\imath} + 8.61\,\vec{\jmath}\ \mathrm{m/s^2}}$$

2.5.7
GOAL: Determine the velocity of block B given the velocity of the free end A.
GIVEN: A is moving down at $2\,\mathrm{m/s}$.
DRAW:



FORMULATE EQUATIONS:
Moving A (increasing x) will require us to add $2\Delta x$ units of rope. Moving B (increasing y) will require us to add $2\Delta y$ units of rope. Applying conservation of rope tells us that the net change must be zero:

$$2\Delta x + 2\Delta y = 0$$

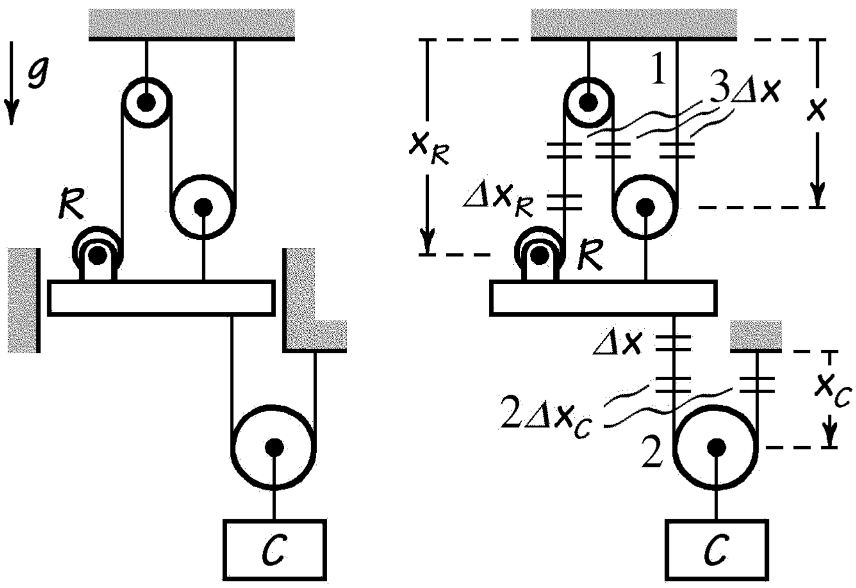
Dividing by time Δt and taking the limit as Δt goes to zero gives

$$2\dot{x} + 2\dot{y} = 0 \Rightarrow \dot{y} = -\dot{x}$$

SOLVE:
We're given $\dot{x} = 2\,\mathrm{m/s}$ and thus $\dot{y} = -2\,\mathrm{m/s}$

$$\boxed{\vec{v}_B = 2\vec{\jmath}\ \mathrm{m/s}}$$

2.5.8
GOAL: Determine v_C.
GIVEN: The motor at R reels in rope at $v_R = 3\,\text{m/s}$.
DRAW:



FORMULATE EQUATIONS:
By conservation of rope,

Rope 1 ⇒ $$3\Delta x + \Delta x_R = 0 \tag{1}$$

Rope 2 ⇒ $$2\Delta x_C - \Delta x = 0 \tag{2}$$

SOLVE:

Differentiate (1) ⇒ $$3\dot{x} + v_R = 0 \tag{3}$$

Differentiate (2) ⇒ $$2v_C - \dot{x} = 0 \tag{4}$$

(4) → (3) ⇒ $$6v_C + v_R = 0$$

$$v_C = -\frac{1}{6}v_R$$

$$v_C = -\frac{1}{6}(3\,\text{m/s})$$

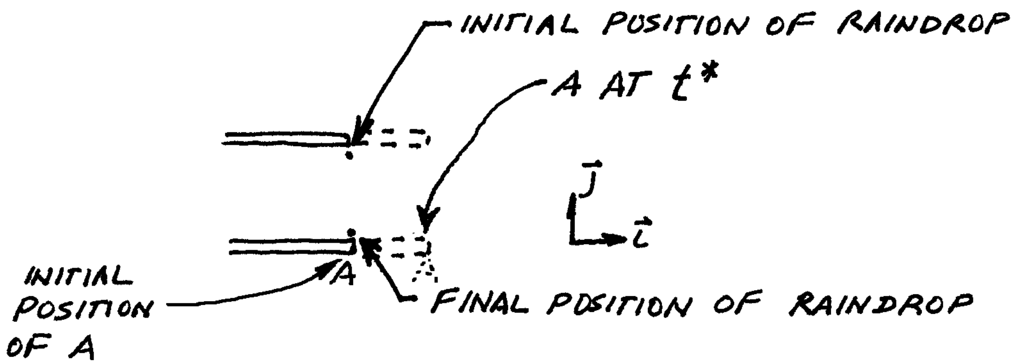
$$\boxed{v_C = -\frac{1}{2}\,\text{m/s}}$$

2.5.9

GOAL: Calculate the necessary constant acceleration for a raindrop to hit the floor of the train car eight feet from the point A and calculate the angle made by its velocity vector with the floor.

GIVEN: The initial speed of the train is $v_A = 80$ ft/s, the speed of the raindrop is $v_{rd} = 40$ ft/s and the height of the overhang is $h = 8$ ft.

DRAW



FORMULATE EQUATIONS:

Distance the point A travels in time t^*:

$$d = v_A t^* + \frac{a_A t^{*2}}{2} \tag{1}$$

Distance the raindrop falls in time t^*:

$$h = v_{rd} t^* \tag{2}$$

Relative velocity of the raindrop:

$$\vec{v}_{rd} = \vec{v}_A + \vec{v}_{rd/A} \tag{3}$$

SOLVE: First we'll determine the time at which the raindrop strikes the floor of the train car.

(2) $\Rightarrow$
$$8\,\text{ft} = (40\,\text{ft/s})t^* \;\Rightarrow\; t^* = 0.2\,\text{s}$$

We want to ensure that in 0.2 s the point A moves forward by 8 ft:

(1) $\Rightarrow$
$$8\,\text{ft} = (80\,\text{ft/s})(0.2\,\text{s}) + \frac{a_A(0.2\,\text{s})^2}{2}$$

$$\boxed{a_A = -400\,\text{ft/s}^2}$$

(Note that this deceleration is *very* nonrealistic for an actual train.)

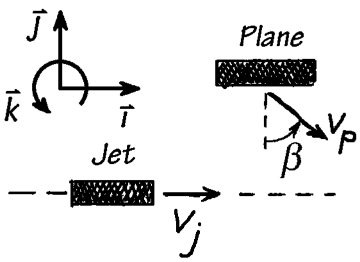
(3) $\Rightarrow$
$$\vec{v}_{rd} = \vec{v}_A + \vec{v}_{rd/A} \;\Rightarrow\; \vec{v}_{rd/A} = \vec{v}_{rd} - \vec{v}_A$$

$$\vec{v}_{rd/A} = -40\vec{\jmath}\ \text{ft/s} - 80\vec{\imath}\ \text{ft/s} + (400\,\text{ft/s}^2)(0.2\,\text{s})\vec{\imath} = -40\vec{\jmath}\ \text{ft/s}$$

The deceleration is such that at 0.2 s the horizontal speed of the raindrop with respect to the train has gone to zero and the velocity is purely vertical.

$$\boxed{\text{angle with respect to the floor is } 90^\circ}$$

2.5.10
GOAL: Determine the approach speed $\|\vec{v}_a\|$ of the fuel nozzle with respect to the jet, and find what speed the jet should slow down to if it wants to maintain the same $\|\vec{v}_a\|$ when the refueling plane reduces its speed.
GIVEN: The refueling plane comes in at an angle $\beta = 85°$ to the vertical with an initial speed $v_p = 350\,\text{mph}$, while the jet flies at $\vec{v}_j = 360\vec{\imath}\,\text{mph}$. The plane then reduces its speed to $v_p^* = 340\,\text{mph}$.
DRAW:



FORMULATE EQUATIONS:
The approach velocity of the fuel nozzle as seen by the jet is

$$\vec{v}_a = \vec{v}_p - \vec{v}_j \tag{1}$$

SOLVE:

(1) $\Rightarrow$
$$\vec{v}_a = v_p(\sin\beta\,\vec{\imath} - \cos\beta\,\vec{\jmath}) - v_j\,\vec{\imath}$$

$$\|\vec{v}_a\| = \sqrt{\vec{v}_a \cdot \vec{v}_a} = \sqrt{(v_p\sin\beta - v_j)^2 + (v_p\cos\beta)^2} \tag{2}$$

$$\|\vec{v}_a\| = \sqrt{\{(350\,\text{mph})\sin(85°) - 360\,\text{mph}\}^2 + \{(350\,\text{mph})\cos(85°)\}^2}$$

$$\boxed{\|\vec{v}_a\| = 32.5\,\text{mph}}$$

(2) $\Rightarrow$
$$\|\vec{v}_a\| = v_a = \sqrt{(v_p^*\sin\beta - v_j^*)^2 + (v_p^*\cos\beta)^2}$$

$$\begin{aligned}
v_a^2 &= (v_p^*\sin\beta - v_j^*)^2 + (v_p^*\cos\beta)^2 \\
0 &= (v_j^*)^2 - (2v_p^*\sin\beta)v_j^* + (v_p^*\sin\beta)^2 + (v_p^*\cos\beta)^2 - v_a^2 \\
0 &= (v_j^*)^2 - (2v_p^*\sin\beta)v_j^* + \{(v_p^*)^2 - v_a^2\}
\end{aligned}$$

$$\begin{aligned}
v_j^* &= \frac{2v_p^*\sin\beta \pm \sqrt{(2v_p^*\sin\beta)^2 - 4\{(v_p^*)^2 - v_a^2\}}}{2} \\
v_j^* &= \frac{2(340\,\text{mph})\sin(85°) \pm \sqrt{\{2(340\,\text{mph})\sin(85°)\}^2 - 4\{(340\,\text{mph})^2 - (32.5\,\text{mph})^2\}}}{2} \\
v_j^* &= 325.3\,\text{mph},\ 352.2\,\text{mph}
\end{aligned}$$

The slower speed corresponds to the jet and plane moving *away* from each other, so the jet speed we want is

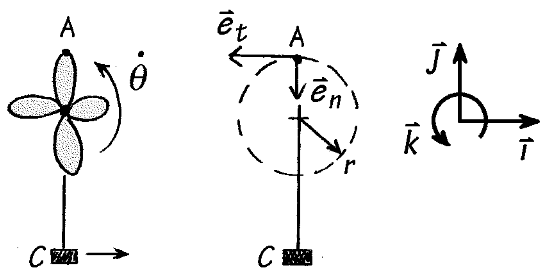
$$\boxed{v_j^* = 352.2\,\text{mph}}$$

2.5.11

GOAL: Determine $\|\vec{v}_A\|$ and $\|\vec{a}_A\|$ at the given instant.

GIVEN: The car travels with $\vec{v}_c = 30\vec{\imath}$ mph and $\vec{a}_c = 5\vec{\imath}$ ft/s^2. The windmill has a radius of $r = 4$ in. and spins at $\dot{\theta} = 200$ rpm counterclockwise. At the given instant, $\vec{e}_t = -\vec{\imath}$ and $\vec{e}_n = -\vec{\jmath}$ at A.

DRAW:



FORMULATE EQUATIONS:

The velocity of point A on the windmill is given by

$$\vec{v}_A = \vec{v}_c + \vec{v}_{A/c} \tag{1}$$

The acceleration of A is given by

$$\vec{a}_A = \vec{a}_c + \vec{a}_{A/c} \tag{2}$$

SOLVE:

(1) $\Rightarrow$

$$\vec{v}_A = v_c\vec{\imath} + r\dot{\theta}\vec{e}_t = (v_c - r\dot{\theta})\vec{\imath}$$

$$\|\vec{v}_A\| = \sqrt{\vec{v}_A \cdot \vec{v}_A} = v_c - r\dot{\theta}$$

$$\|\vec{v}_A\| = 30\,\text{mph} - (4\,\text{in.})\left(\frac{\text{ft}}{12\,\text{in.}}\right)\left(\frac{\text{mi}}{5280\,\text{ft}}\right)(200\,\text{rpm})\left(\frac{60\,\text{min}}{\text{hr}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right)$$

$$\boxed{\|\vec{v}_A\| = 25.2\,\text{mph}}$$

(2) $\Rightarrow$

$$\vec{a}_A = a_c\vec{\imath} + r\dot{\theta}^2\vec{e}_n = a_c\vec{\imath} - r\dot{\theta}^2\vec{\jmath}$$

$$\|\vec{a}_A\| = \sqrt{\vec{a}_A \cdot \vec{a}_A} = \sqrt{a_c^2 + (r\dot{\theta}^2)^2}$$

$$\|\vec{a}_A\| = \sqrt{(5\,\text{ft/s}^2)^2 + \left[\left(\frac{4}{12}\,\text{ft}\right)\left(\frac{200\cdot 2\pi}{60}\,\text{rad/s}\right)^2\right]^2}$$

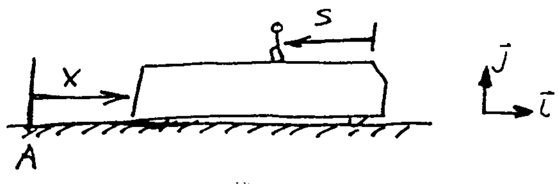
$$\boxed{\|\vec{a}_A\| = 146.3\,\text{ft/s}^2}$$

2.5.12

GOAL: Find the agent's acceleration so that he will be moving at zero velocity with respect to the ground. Also, find the point on the ground where he will land.

GIVEN: Agent is 30 ft from the back of the bus and moving with a constant acceleration with respect to the bus. Bus is moving at 10 mph with respect to ground.

DRAW:



FORMULATE EQUATIONS:

To move at zero velocity with respect to the ground the agent must be moving at $-10\,\text{mph}$ with respect to the bus. You can solve for the acceleration with respect to the bus using

$$\ddot{s}ds = \dot{s}\,d\dot{s} \tag{1}$$

Converting the speed from mph to ft/sec gives

$$10\,\text{mph} = 14.\overline{6}\,\text{ft/sec}$$

SOLVE: Integrating (1) yields

$$\int_0^{30} \ddot{s}\,ds = \int_0^{14.\overline{6}} \dot{s}\,d\dot{s}$$

$$\ddot{s}(30) = \frac{1}{2}\left(14.\overline{6}^{\,2}\right) \Rightarrow \ddot{s} = 3.59\,\text{ft/s}^2$$

When the agent leaves the bus, he is traveling with zero velocity with respect to the ground in the $\vec{\imath}$ direction and thus he'll drop straight down from the back of the bus. To determine the location of the bus when the agent jumps, we look at the equation for change of position under constant acceleration

$$s = \frac{1}{2}\ddot{s}t^2 + \dot{s}_0 t + s_0.$$

where $\dot{s}_0$ and s_0 are the speed and position of the agent with respect to the bus at $t = 0$. Since $\dot{s}_0$ and s_0 equal zero, the equation simplifies to

$$30\,\text{ft} = \frac{1}{2}(3.59\,\text{ft/s}^2)t^2$$

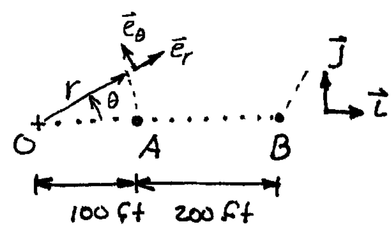
Solving for t gives $t = 4.08\,\text{s}$. Using this fact and the fact that the bus is moving at a constant speed gives us

$$x = \dot{x}t$$

$$x = (14.\overline{6}\,\text{ft/s})(4.08\,\text{s}) = 60\,\text{ft}$$

The position of the agent with respect to A is $\boxed{x = 60\,\text{ft}}$

2.5.13
GOAL: Find the velocity and acceleration of car A relative to car B.
GIVEN: Absolute positions and velocities of cars A and B.
DRAW



FORMULATE EQUATIONS:

Velocity of car A relative to car B:
$$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B \tag{1}$$

Acceleration of car A relative to car B:
$$\vec{a}_{A/B} = \vec{a}_A - \vec{a}_B \tag{2}$$

Absolute velocity of car A:
$$\vec{v}_A = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{3}$$

Absolute acceleration of car A:
$$\vec{a}_A = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{e}_\theta \tag{4}$$

SOLVE:

(1) $\Rightarrow$
$$\vec{v}_{A/B} = \left[30\vec{\jmath} - 60 \quad 0.5\vec{\imath} + \frac{\sqrt{3}}{2}\vec{\jmath}\Big)\right] \text{mph} \tag{5}$$

$$\boxed{\vec{v}_{A/B} = (-30\vec{\imath} - 21.96\vec{\jmath})\,\text{mph}}$$

Because car A is moving at a constant speed, in a circle of constant radius, at angular position $\theta = 0$ we have

(3) $\Rightarrow$
$$\vec{v}_A = r\dot{\theta}\vec{e}_\theta = r\dot{\theta}\vec{\jmath} \tag{6}$$

(4) $\Rightarrow$
$$\vec{a}_A = -r\dot{\theta}^2\vec{e}_r = -r\dot{\theta}^2\vec{\imath} \tag{7}$$

We can get $\dot{\theta}$ from (6) $\Rightarrow$
$$30\,\text{mph}\,\vec{\jmath} = (100\,\text{ft})\dot{\theta}\vec{\jmath}$$
$$\dot{\theta} = \left(\frac{30\,\text{mph}}{100\,\text{ft}}\right)\left(1.4667\frac{\text{ft/s}}{\text{mph}}\right) = 0.44\,\text{rad/s} \tag{8}$$

(8) $\rightarrow$ (7) $\Rightarrow$
$$\vec{a}_A = -(100\,\text{ft})(0.44\,\text{rad/s})^2\vec{\imath} = -19.36\ \text{ft/s}^2\,\vec{\imath} \tag{9}$$

Because car B is moving in a straight line:
$$\vec{a}_B = 0 \tag{10}$$

(9), (10) $\rightarrow$ (2) $\Rightarrow$
$$\vec{a}_{A/B} = -19.36\ \text{ft/s}^2\,\vec{\imath}$$

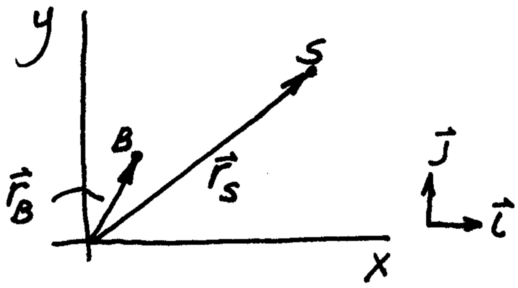
$$\boxed{\vec{a}_{A/B} = -19.36\ \text{ft/s}^2\,\vec{\imath}}$$

2.5.14

GOAL: Find how long it will take for a fisherman's reel to fully unwind and the velocity and acceleration of a swordfish with respect to the fisherman's boat.

GIVEN: Relative positions of the swordfish and boat and both boat and swordfish's velocity.

DRAW:



FORMULATE EQUATIONS:

$$\vec{r}_S(t) = \vec{v}_S(0)t + \vec{r}_S(0) \quad (1)$$
$$\vec{r}_B(t) = \vec{v}_B(0)t + \vec{r}_B(0) \quad (2)$$

ASSUME:

$$\max|\vec{r}_S(t) - \vec{r}_B(t)| = 500\,\mathrm{m} \quad (3)$$

$$\begin{aligned} \vec{r}_S(0) &= 50\vec{\imath}\,\mathrm{m} & \vec{v}_S(0) &= 10\vec{\imath}\,\mathrm{m/s} \\ \vec{r}_B(0) &= -40\vec{\jmath}\,\mathrm{m} & \vec{v}_B(0) &= -3\vec{\jmath}\,\mathrm{m/s} \end{aligned} \quad (4)$$

SOLVE:

(1),(2),(4)→(3)⇒

$$\begin{aligned} |((10\,\mathrm{m/s})t_f + 50\,\mathrm{m})\vec{\imath} + (-(3\,\mathrm{m/s})t_f - 40\,\mathrm{m/s})\vec{\jmath}| &= 500\,\mathrm{m} \\ \sqrt{((10\,\mathrm{m/s})t_f + (50\,\mathrm{m}))^2 + ((3\,\mathrm{m/s})t_f + (40\,\mathrm{m}))^2} &= 500 \\ (109\,\mathrm{s}^{-2})t_f^2 + (1240\,\mathrm{s}^{-1})t_f - 245900 &= 0 \\ t_f &= -53.5\,\mathrm{s},\ \boxed{42.1\ \mathrm{s}} \end{aligned}$$

Note that only the positive solution makes sense for this problem.

(1),(2)⇒

$$\begin{aligned} \vec{v}_{S/B} &= \vec{v}_S - \vec{v}_B = 10\vec{\imath}\,\mathrm{m/s} - (-3\vec{\jmath}\,\mathrm{m/s}) \\ \vec{v}_{S/B} &= \boxed{(10\vec{\imath} + 3\vec{\jmath})\,\mathrm{m/s}} \end{aligned}$$

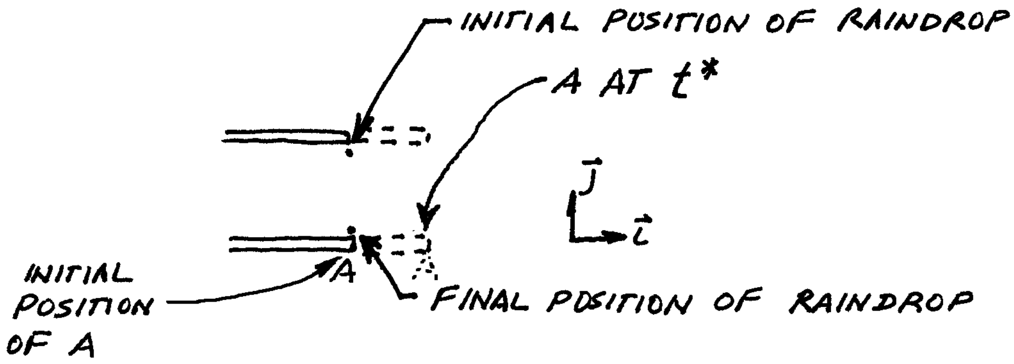
$$\vec{a}_{S/B} = \frac{d}{dt}\vec{v}_{S/B} = \boxed{0} \quad (5)$$

2.5.15

GOAL: Find the distance raindrops travel into the train, and the velocity of raindrops relative to the train.

GIVEN: The speed of point A, v_A, the speed of the raindrops, v_{rd}, and the height of the overhang, h.

DRAW



FORMULATE EQUATIONS:

Distance the train travels in time t^*:

$$d = v_A t^* \tag{1}$$

Distance the raindrops fall in time t^*:

$$h = v_{rd} t^* \tag{2}$$

Relative velocity of the raindrop:

$$\vec{v}_{rd} = \vec{v}_A + \vec{v}_{rd/A} \tag{3}$$

SOLVE:

A raindrop will fall by a height h in the same time as the train moves forward by a distance d.

$(1), (2) \Rightarrow$

$$d = \left(\frac{v_A}{v_{rd}}\right) h = \left(\frac{30 \text{ mph}}{25 \text{ mph}}\right)(7 \text{ ft}) = 8.4\,\text{ft}$$

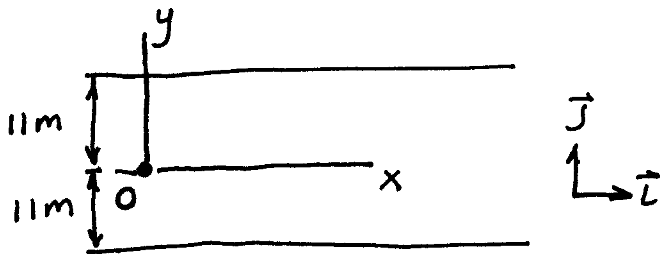
$$\boxed{d = 8.4\,\text{ft}}$$

Solving (3) for the velocity of the raindrop relative to point A gives us:

$$\vec{v}_{rd/A} = \vec{v}_{rd} - \vec{v}_A = (-25\vec{\jmath})\,\text{mph} - (30\vec{\imath})\,\text{mph}$$

$$\boxed{\vec{v}_{rd/A} = (-30\vec{\imath} - 25\vec{\jmath})\,\text{mph}}$$

Do It 2.5.2
GOAL: Find the distance you will travel in the $\vec{\imath}$ direction before landing on shore.
GIVEN: Absolute velocity of the river, $\vec{v}_r$, velocity of the boat relative to the river, $\vec{v}_{B/r}$, distance to shore in the $\vec{\jmath}$ direction.
DRAW



FORMULATE EQUATIONS:
The position you will land on the shore relative to your starting position:

$$\vec{r}_{S/O} = x\vec{\imath} + 11\,\text{m}\vec{\jmath} \tag{1}$$

The absolute velocity of the boat:

$$\vec{v}_B = \vec{v}_R + \vec{v}_{B/R} \tag{2}$$

The velocity of the boat relative to the river:

$$\vec{v}_{B/R} = (4\cos 45^\circ\,\vec{\imath} + 4\sin 45^\circ\,\vec{\jmath})\,\text{m/s} \tag{3}$$

The distance travelled in t^* seconds:

$$\vec{r}_{S/O} = \vec{v}_B t^* \tag{4}$$

SOLVE:

(3) → (2) ⇒
$$\vec{v}_B = [(3 + 4\cos 45^\circ)\,\vec{\imath}\ \text{m/s} + 4\sin 45^\circ\,\vec{\jmath}\ \text{m/s}] \tag{5}$$

(1),(5) → (4) ⇒
$$x\vec{\imath} + 11\,\text{m}\vec{\jmath} = [(3 + 4\cos 45^\circ)\,\vec{\imath}\ \text{m/s} + 4\sin 45^\circ)\,\vec{\jmath}\ \text{m/s}]\,t^* \tag{6}$$

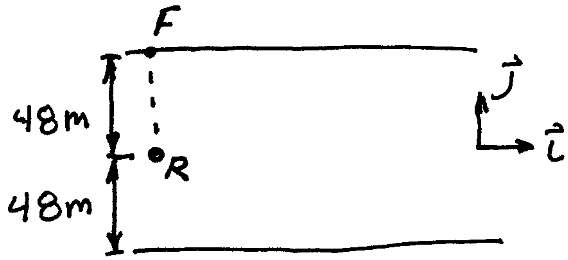
$\vec{\imath}$:
$$x = [(3 + 4\cos 45^\circ)\,\text{m/s}]\,t^* \tag{7}$$

$\vec{\jmath}$:
$$11\,\text{m} = [(4\cos 45^\circ)\,\text{m/s}]\,t^* \tag{8}$$

(7), (8) ⇒
$$x = \left[\frac{(3 + 4\cos 45^\circ)\,\text{m/s}}{(4\cos 45^\circ)\,\text{m/s}}\right](11\,\text{m}) = 22.7\,\text{m}$$

$$\boxed{x = 22.7\,\text{m}}$$

2.5.16
GOAL: Find direction to paddle.
GIVEN: Absolute velocity of river, $\vec{v}_R$, position vector of friend with respect to raft's initial position $\vec{r}_0$, time to reach friend, t^*.
DRAW



ASSUME:

Velocity of raft relative to the river: $$\vec{v}_{Ra/R} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath}$$

FORMULATE EQUATIONS:

Direction to paddle: $$\theta = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) \quad (1)$$

Absolute velocity of the raft: $$\vec{v}_{Ra} = \vec{v}_R + \vec{v}_{Ra/R} = (1\,\text{m/s} + \dot{x})\vec{\imath} + \dot{y}\vec{\jmath} \quad (2)$$

Distance travelled in time t^*: $$\vec{r}_0 = \vec{v}_{Ra}t^* \quad (3)$$

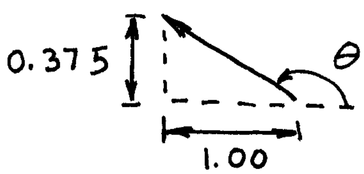
SOLVE:

(2) → (3) ⇒ $$48\,\text{m}\vec{\jmath} = [(1\,\text{m/s} + \dot{x})\vec{\imath} + \dot{y}\vec{\jmath}]\,(120\,\text{s}) \quad (4)$$

$\vec{\imath}$: $$\dot{x} = -1\,\text{m/s}$$

$\vec{\jmath}$: $$\dot{y} = \frac{48\,\text{m}}{120\,\text{s}} = 0.375\,\text{m/s}$$

Plugging these into (1) ⇒ $$\theta = \tan^{-1}\left(\frac{0.375\,\text{m/s}}{-1\,\text{m/s}}\right) = \boxed{159^\circ}$$

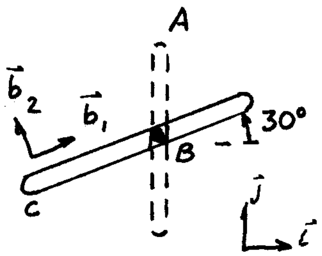


2.5.17

GOAL: A rod is constrained to move within two independent, overlapping slots. Find the rod's velocity with respect to one of the constraining boards.

GIVEN: Speed of board C and orientation of slot.

DRAW:



	$\vec{i}$	$\vec{j}$
$\vec{b}_1$	$\sqrt{3}/2$	$1/2$
$\vec{b}_2$	$-1/2$	$\sqrt{3}/2$

FORMULATE EQUATIONS: We'll use the relative velocity formula

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

SOLVE:

$$\vec{v}_{Body\ A} = 0 \tag{1}$$

$$\vec{v}_{Body\ C} = 1\vec{\imath}\ \mathrm{m/s} \tag{2}$$

$$\vec{v}_{B/Body\ A} = \dot{y}\vec{\jmath} \tag{3}$$

$$\vec{v}_{B/Body\ C} = \dot{s}\vec{b}_1 = \dot{s}\left(\frac{\sqrt{3}}{2}\vec{\imath} + \frac{1}{2}\vec{\jmath}\right) \tag{4}$$

$$\begin{aligned}\vec{v}_B &= \vec{v}_{Body\ C} + \vec{v}_{B/BodyC} = \vec{\imath} + \dot{s}\left(\frac{\sqrt{3}}{2}\vec{\imath} + \frac{1}{2}\vec{\jmath}\right)\\ &= \vec{\imath}\left(1\,\mathrm{m/s} + \frac{\sqrt{3}}{2}\dot{s}\right) + \frac{1}{2}\dot{s}\vec{\jmath}\end{aligned} \tag{5}$$

$$\vec{v}_B = \vec{v}_{Body\ A} + \vec{v}_{B/Body\ A} = \dot{y}\vec{\jmath} \tag{6}$$

$\vec{v}_B$ must be the same in (5) and (6). Equating these two expressions gives

$$\vec{\imath}\left(1\,\mathrm{m/s} + \frac{\sqrt{3}}{2}\dot{s}\right) + \vec{\jmath}\left(\frac{\dot{s}}{2}\right) = 0\vec{\imath} + \dot{y}\vec{\jmath}$$

Equating Coefficients:

$\vec{\imath}$:
$$1\,\mathrm{m/s} + \frac{\sqrt{3}}{2}\dot{s} = 0 \tag{7}$$

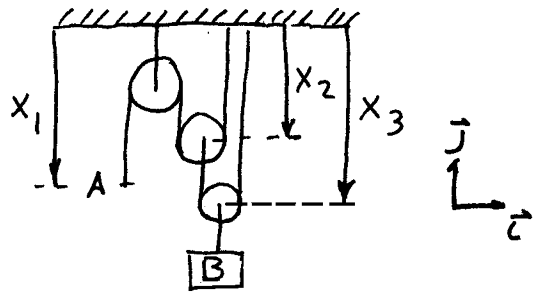
$\vec{\jmath}$:
$$\frac{\dot{s}}{2} = \dot{y} \tag{8}$$

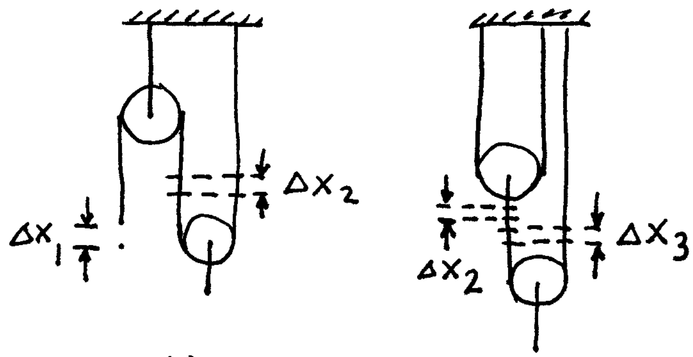
(7) $\Rightarrow$
$$\dot{s} = -\frac{2}{\sqrt{3}}\,\mathrm{m/s} \tag{9}$$

(9) $\rightarrow$ (4) $\Rightarrow$

$$\boxed{\vec{v}_{B/BodyC} = \dot{s}\left(\frac{\sqrt{3}}{2}\vec{\imath} + \frac{1}{2}\vec{\jmath}\right) = \left(-\vec{\imath} - \frac{1}{\sqrt{3}}\vec{\jmath}\right)\,\mathrm{m/s}}$$

Do It 2.5.3
GOAL: Find velocity of block B.
GIVEN: Velocity of free end A.
DRAW:





(a) (b)

ASSUME: Both ropes have a fixed length (conservation of rope).
FORMULATE EQUATIONS:
First consider the left pulley (Figure (a)):

Increase x_1 and x_2:

$$2\Delta x_2 + \Delta x_1 = 0$$

$$\dot{x}_2 = -\frac{1}{2}\dot{x}_1 \tag{1}$$

Next consider the right pulley (Figure (b)):

Increase x_2 and x_3:

$$2\Delta x_3 - \Delta x_2 = 0$$

$$\dot{x}_3 = \frac{1}{2}\dot{x}_2 \tag{2}$$

SOLVE:

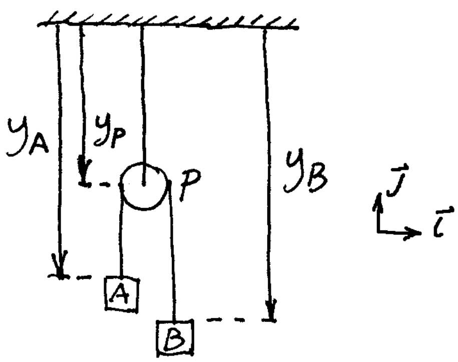
$(1), (2) \Rightarrow$

$$\dot{x}_3 = \frac{1}{2}\left(-\frac{1}{2}\dot{x}_1\right) = -\frac{1}{4}\dot{x}_1$$

We're given $\dot{x}_1 = 3\,\mathrm{m/s}$ and so have

$$\boxed{\vec{v}_B = \frac{3}{4}\vec{\jmath}\ \mathrm{m/s}}$$

2.5.18
GOAL: Find the absolute velocity of A.
GIVEN: Pulley arrangement, speed of B and rate at which reel is pulling in rope.
DRAW



ASSUME: An overall constraint for this problem is the fact that the speed with which the pulley P drops is equal to the rate at which rope is fed from the reel.
FORMULATE EQUATIONS:
Ignoring motion of the pulley P we see by inspection that the velocities of A and B must be equal in magnitude and opposite in sign. We also have the relative velocity relationships

$$\vec{v}_A = \vec{v}_P + \vec{v}_{A/P}$$

$$\vec{v}_B = \vec{v}_P + \vec{v}_{B/P}$$

SOLVE:
We're given $\vec{v}_P = 14\vec{\jmath}$ in/s and $\vec{v}_B = -2\vec{\jmath}$ in/s. Using our relative velocity relationships gives us

$$\vec{v}_{B/P} = \vec{v}_B - \vec{v}_P$$

$$\vec{v}_{B/P} = -2\vec{\jmath}\text{ in/s} - 14\vec{\jmath}\text{ in/s} = -16\vec{\jmath}\text{ in/s}$$

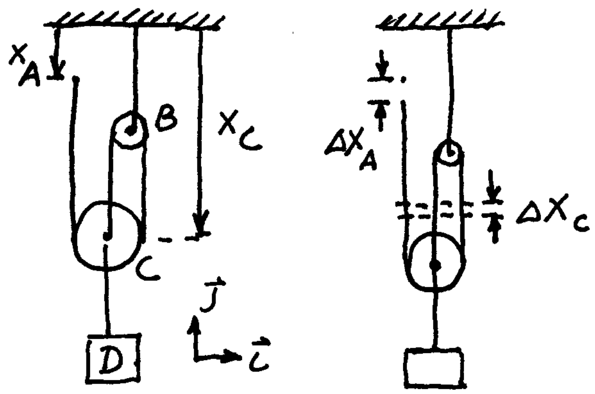
We've already observed that $\vec{v}_{A/P} = -\vec{v}_{B/P}$ and so have

$$\vec{v}_{A/P} = 16\vec{\jmath}\text{ in/s}$$

We can now solve for $\vec{v}_A$.

$$\vec{v}_A = \vec{v}_P + \vec{v}_{A/P} = 14\vec{\jmath}\text{ in/s} + 16\vec{\jmath}\text{ in/s} = \boxed{30\vec{\jmath}\text{ in/s}}$$

2.5.19
GOAL: Find the absolute velocity of Block D.
GIVEN: Pulley arrangement and speed of the rope.
DRAW



FORMULATE EQUATIONS: Let's use the "conservation-of-rope" principle. If we increase x_A by a small amount Δx_A, then we will need to add THREE sections of rope at pulley C, which is $3\Delta x_C$. But it is very important to note that when you increase x_A, you're actually taking rope away, and not adding it to the overall length of the rope. Thus, our condition is

$$-\Delta x_A + 3\Delta x_C = 0 \tag{1}$$

Differentiating gives us
$$-v_A + 3v_C = 0 \tag{2}$$

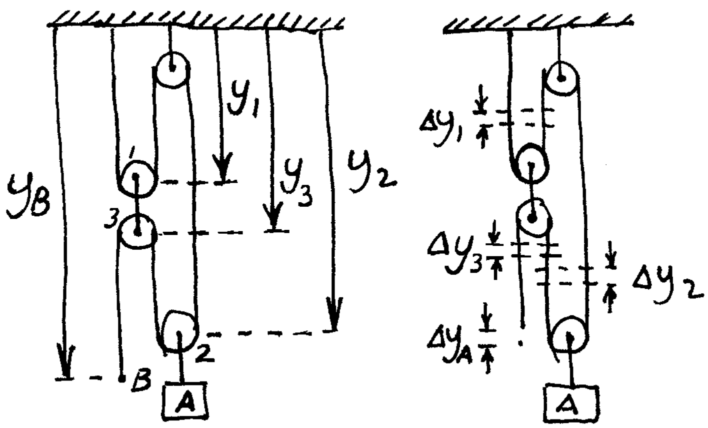
SOLVE:

Solving (2) for $v_C \Rightarrow$
$$v_C = \frac{1}{3}v_A = -\frac{0.4\,\text{m/s}}{3}$$

Because the velocity of Block D is the same as that of Block C, we get

$$\boxed{\vec{v}_D = 0.133\vec{\jmath}\ \text{m/s}}$$

2.5.20
GOAL: Find the absolute velocity of A.
GIVEN: Pulley arrangement and speed of the rope.
DRAW



FORMULATE EQUATIONS: In addition to the constraint that there is a fixed amount of rope, there is a constraint that the distance between pulley 1 and pulley 3 is fixed. This means that if you change the position of one, the position of the other will change by the same amount. Using the notation in the figure, this can be expressed as

$$\Delta y_3 = \Delta y_1 \tag{1}$$

Using the "conservation-of-rope" principle, the other constraint is

$$\Delta y_B - 2\Delta y_3 + 2\Delta y_2 + 2\Delta y_1 = 0 \tag{2}$$

SOLVE:

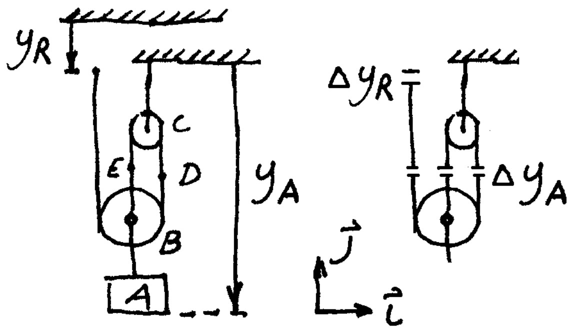
(1) → (2) ⇒
$$\Delta y_B + 2\Delta y_2 = 0 \tag{3}$$

Differentiating (3) and rearranging ⇒
$$v_2 = -\frac{1}{2} v_B = -\frac{4\,\text{ft/s}}{2} = -2\,\text{ft/s}$$

$$\boxed{\vec{v}_A = 2\vec{\jmath}\ \text{ft/s}}$$

2.5.21

GOAL: Determine the velocity of a point on the rightmost rope of a pulley.
GIVEN: Pulley geometry and rate at which the reel is taking in pulley rope.
DRAW



ASSUME: To account for the reel we introduce a coordinate y_R that shows the vertical position of the leftmost pulley rope. A positive value for $\dot{y}_R$ indicates that the rope is unreeling and a negative value means the reel is retracting rope.
Conservation of rope gives us

$$-\Delta y_R + 3\Delta y_A = 0 \tag{1}$$

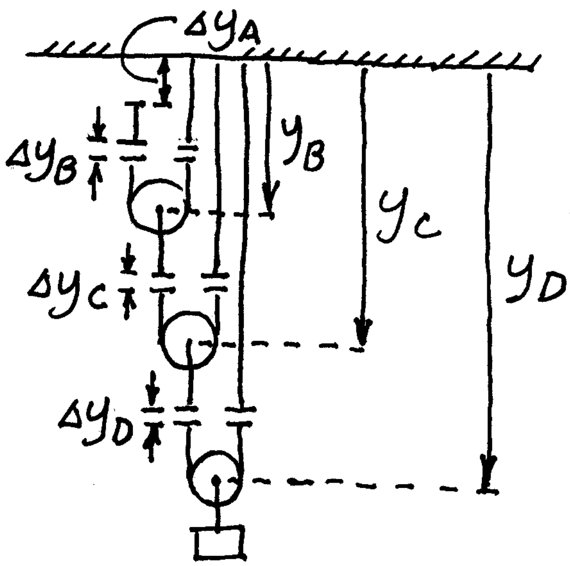
FORMULATE EQUATIONS:

(1) ⇒
$$\dot{y}_A = \frac{1}{3}\dot{y}_R = \frac{10}{3}\text{ in/s} \tag{2}$$

SOLVE:
We can proceed simply using observation at this point. We've already seen that the pulley B is moving up at $\frac{10}{3}$ in/s. The rope that D's a part of goes up over the top pulley (C) and then down to pulley B. Hence, if B is rising at $\frac{10}{3}$ in/s, the rope at D must be going down at the same rate.

$$\boxed{\vec{v}_D = -\tfrac{10}{3}\vec{\jmath}\text{ in/s}}$$

2.5.22
GOAL: Find velocity of weight B.
GIVEN: Speed with which rope is drawn into reel (v_0).
DRAW:



ASSUME: All three ropes have a constant length. Positive Δy_A indicates that the rope is being unreeled.
FORMULATE EQUATIONS:
Conservation of rope gives us

$$-\Delta y_A + 2\Delta y_B = 0 \tag{1}$$

$$-\Delta y_B + 2\Delta y_C = 0 \tag{2}$$

$$-\Delta y_C + 2\Delta y_D = 0 \tag{3}$$

SOLVE:

(1) $\Rightarrow$ $$\dot{y}_A = 2\dot{y}_B \tag{4}$$

(2) $\Rightarrow$ $$\dot{y}_B = 2\dot{y}_C \tag{5}$$

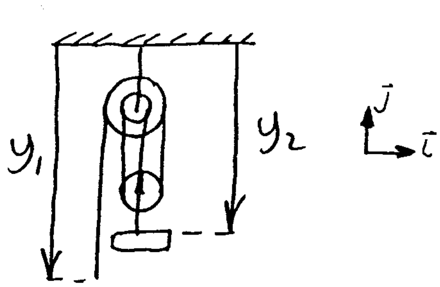
(3) $\Rightarrow$ $$\dot{y}_C = 2\dot{y}_D \tag{6}$$

$(4), (5), (6) \Rightarrow$ $$\dot{y}_A = 8\dot{y}_D$$

Realizing that the reel is taking in rope ($\dot{y}_A = -v_0$) and that a positive $\dot{y}$ implies that B is rising ($\dot{y}_D = -\dot{y}$) we have

$$\boxed{\vec{v}_B = \frac{v_0}{8}\vec{\jmath}}$$

2.5.23
GOAL: Determine the acceleration of the mass given the acceleration of the free end of rope.
GIVEN: Free end of the rope is accelerating downward at $4\,\text{ft/s}^2$.
DRAW:



FORMULATE EQUATIONS:
The displacement of the free end is Δy_1. The displacement of the mass is Δy_2. So the displacement relation is

$$3\Delta y_2 + \Delta y_1 = 0$$

Taking the limit as Δt goes to zero gives

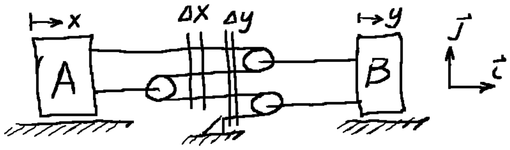
$$\dot{y}_2 = -\frac{1}{3}\dot{y}_1$$

Differentiating again to find acceleration gives

$$\ddot{y}_2 = -\frac{1}{3}\ddot{y}_1$$

SOLVE:
Letting $\ddot{y}_1 = 4\,\text{ft/s}^2$, we get $\boxed{\vec{a}_{mass} = \frac{4}{3}\vec{j}\ \text{ft/s}^2}$.

2.5.24
GOAL: Determine the velocity of block B given the velocity of block A.
GIVEN: Block A is moving to the right at 10 m/s.
DRAW:



FORMULATE EQUATIONS:
Moving A (increasing x) will require us to lose $3\Delta x$ units of rope. Moving B (increasing y) will require us to add $4\Delta y$ units of rope. Applying conservation of rope tells us that the net change must be zero:

$$-3\Delta x + 4\Delta y = 0$$

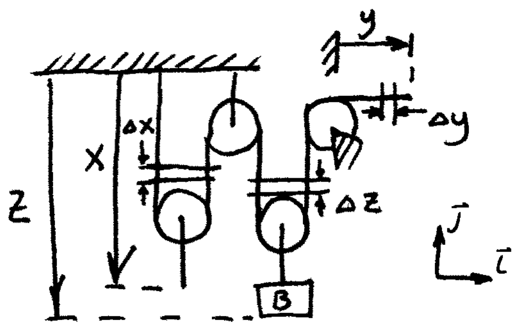
Dividing by time Δt and taking the limit as Δt goes to zero gives

$$-3\dot{x} + 4\dot{y} = 0 \Rightarrow \dot{y} = \frac{3}{4}\dot{x}$$

SOLVE:
We're given $\dot{x} = 10\,\text{m/s}$ and thus

$$\boxed{\vec{v}_B = \tfrac{3}{4}(10\,\vec{\imath}\text{ m/s}) = 7.5\,\vec{\imath}\text{ m/s}}$$

2.5.25
GOAL: Find Block B's velocity.
GIVEN: Pulley geometry.
DRAW:



FORMULATE EQUATIONS:

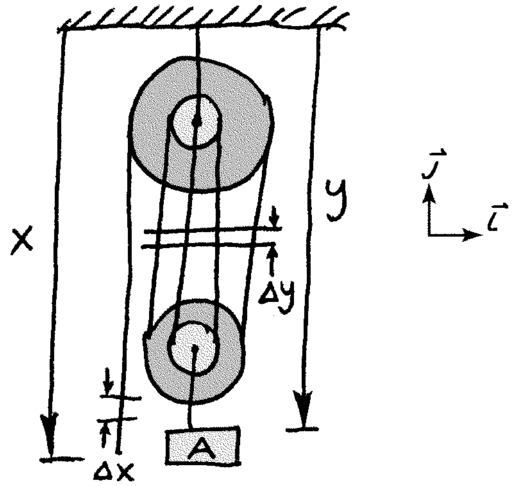
$$2\Delta x + 2\Delta z + \Delta y = 0$$
$$2\dot{x} + 2\dot{z} + \dot{y} = 0 \tag{1}$$

SOLVE:
We're given $\dot{x} = 20\,\text{in./s}$ and $\dot{y} = 10\,\text{in./s}$. Using this information in (1) yields

$$\dot{z} = \frac{-\dot{y} - 2\dot{x}}{2} = -\frac{1}{2}(10\,\text{in./s} + 2(20\,\text{in./s})) = -25\,\text{in/s}$$

$$\boxed{\vec{v}_B = 25\vec{\jmath}\ \text{in./s}}$$

2.5.26
GOAL: Find $\vec{v}_A$ of a pulley system.
GIVEN: Pulley geometry.
DRAW:



ASSUME: The ropes connecting the pulleys are vertical.
FORMULATE EQUATIONS:

$$\Delta x + 4\Delta y = 0 \Rightarrow \ \dot{y} = -\frac{1}{4}\dot{x}$$

SOLVE:
We're given $\dot{x} = 10$ in/s. Thus we have

$$\dot{y} = -\frac{1}{4}\dot{x} = -\frac{5}{2}\text{in/s}$$

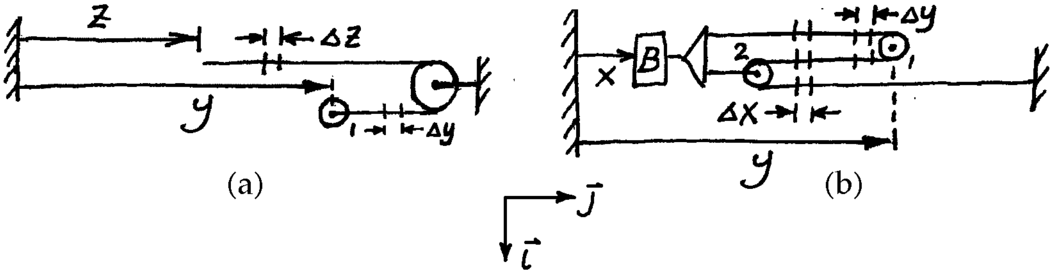
$$\boxed{\vec{v}_A = \tfrac{5}{2}\vec{\jmath}\ \text{in/s}}$$

2.5.27

GOAL: GOAL: Find the velocity of block B.

GIVEN: The velocity of the free end A.

DRAW:



ASSUME: Both ropes have a fixed length (conservation of rope).

FORMULATE EQUATIONS: We'll examine the two ropes separately. Note that y and Δy is shown for both.

SOLVE:

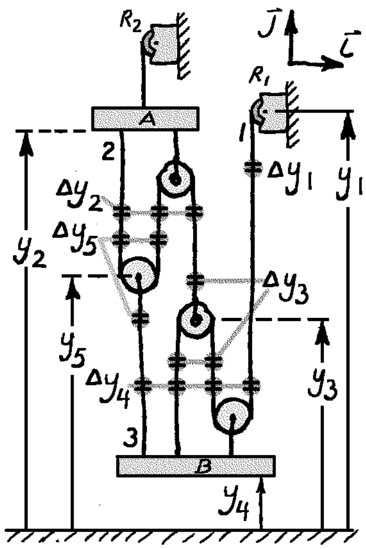
From (a) we see that if the free end of the rope is being pulled down z decreases. Thus we have $\dot{z} = -v_A$.

Rope 1:
$$-\Delta z - \Delta y = 0 \Rightarrow -\dot{z} = \dot{y} \Rightarrow \dot{y} = v_A$$

Rope 2:
$$2\Delta y - 3\Delta x = 0 \Rightarrow \dot{x} = \frac{2}{3}\dot{y}$$

$$\boxed{\vec{v}_B = \dot{x}\vec{\jmath} = \tfrac{2}{3}v_A\vec{\jmath}}$$

Do It 2.5.4
GOAL: Determine $\dot{y}_3$ and $\dot{y}_4$.
GIVEN: The reels R_1 and R_2 are taking in rope at speeds v_1 and v_2, respectively.
DRAW:



ASSUME: All three ropes have a fixed length (conservation of rope).
FORMULATE EQUATIONS: We'll examine the three ropes separately. Note that the Δy_i represent small changes to the position of the coordinates y_i and hence small changes in the length of rope.
SOLVE:

Rope 1:
$$\Delta y_1 - 3\Delta y_4 + 2\Delta y_3 = 0 \Rightarrow \dot{y}_1 - 3\dot{y}_4 + 2\dot{y}_3 = 0 \tag{1}$$

Rope 2:
$$-\Delta y_3 + 3\Delta y_2 - 2\Delta y_5 = 0 \Rightarrow -\dot{y}_3 + 3\dot{y}_2 - 2\dot{y}_5 = 0 \tag{2}$$

Rope 3:
$$\Delta y_5 - \Delta y_4 = 0 \Rightarrow \dot{y}_5 = \dot{y}_4 \tag{3}$$

$(1), (2), (3) \Rightarrow$
$$\dot{y}_1 - 3\left(\frac{-\dot{y}_3 + 3\dot{y}_2}{2}\right) + 2\dot{y}_3 = 0$$

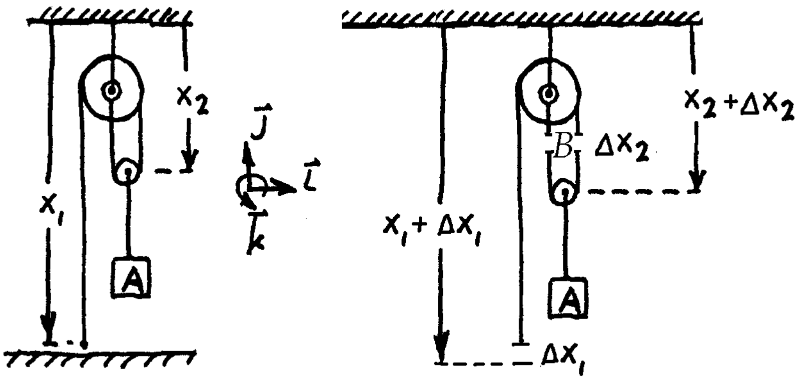
$$\boxed{\dot{y}_3 = -\tfrac{2}{7}v_1 + \tfrac{9}{7}v_2}$$

Using (2) and (3), along with our solution for $\dot{y}_3$ gives us
$$\dot{y}_4 = \frac{1}{2}\left(\frac{2}{7}v_1 - \frac{9}{7}v_2 + 3v_2\right)$$

$$\boxed{\dot{y}_4 = \tfrac{1}{7}v_1 + \tfrac{6}{7}v_2}$$

2.5.28

GOAL: Determine the velocity of the pulley rope's free end.
GIVEN: Block A is observed to be dropping down at a steady 0.9 ft/s.
DRAW



FORMULATE EQUATIONS: We'll use the "conservation of rope" principle. To lower Block A we'll have to lower the pulley directly above it. To allow the pulley to drop we have to "break" the rope. If we increase x_2 by a small amount (to $x_2 + \Delta x_2$), then we'll have two gaps (shown in the figure at B) that will need to be filled in by additional rope. Similarly, bringing the free end of the pulley rope down means adding rope (bringing it from x_1 to $x_1 + \Delta x_1$). Thus, our conservation of rope condition is

$$\Delta x_1 + 2\Delta x_2 = 0 \tag{1}$$

Differentiating gives us
$$v_1 + 2 v_2 = 0 \tag{2}$$

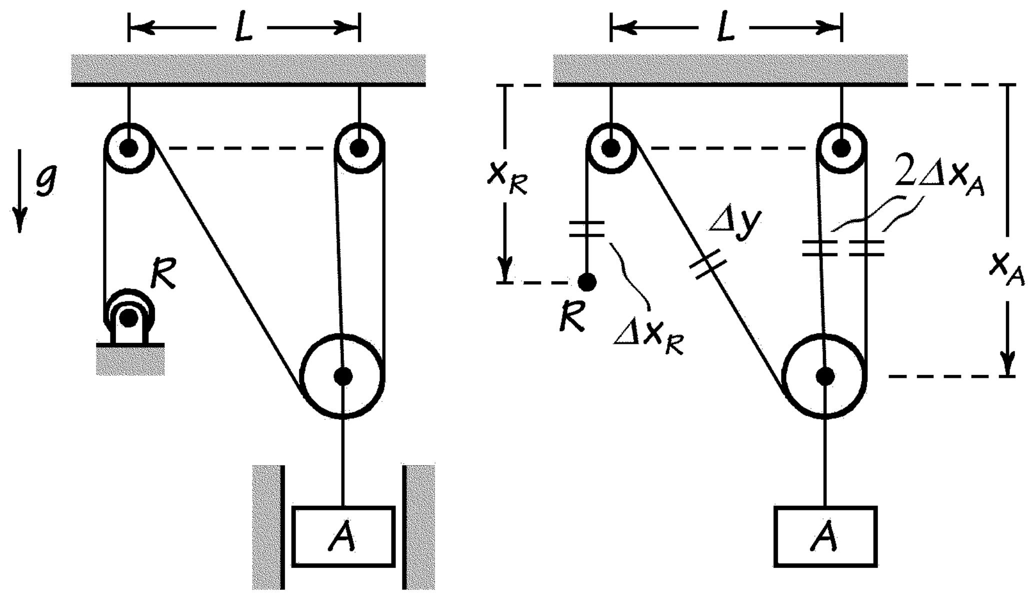
SOLVE:

(2) ⇒
$$v_1 = -2v_2 = -2(0.9\,\text{ft/s}) = -1.8\,\text{ft/s}$$

x_1 is oriented downward and thus the negative value for v_1 indicates that the free end is moving upward and has a velocity of

$$\boxed{\vec{v}_1 = 1.8\vec{\jmath}\ \text{ft/s}}$$

2.5.29
GOAL: Derive an expression for v_A.
GIVEN: A motor at R reels in rope at a rate of v_R.
DRAW:



FORMULATE EQUATIONS:
If we define y as the diagonal length of rope between the pulley supporting block A and the pulley above the motor at R, then conservation of rope gives us

$$\Delta x_R + 2\Delta x_A + \Delta y = 0 \tag{1}$$

The diagonal length of rope y is related to the block's vertical position x_A by

$$y = \sqrt{L^2 + x_A^2} \tag{2}$$

SOLVE:

Differentiate (1) $\Rightarrow$
$$v_R + 2v_A + \dot{y} = 0 \tag{3}$$

Differentiate (2) $\Rightarrow$
$$\dot{y} = \frac{x_A v_A}{\sqrt{L^2 + x_A^2}} \tag{4}$$

(4) $\rightarrow$ (3) $\Rightarrow$
$$v_R + 2v_A + \frac{x_A v_A}{\sqrt{L^2 + x_A^2}} = 0$$

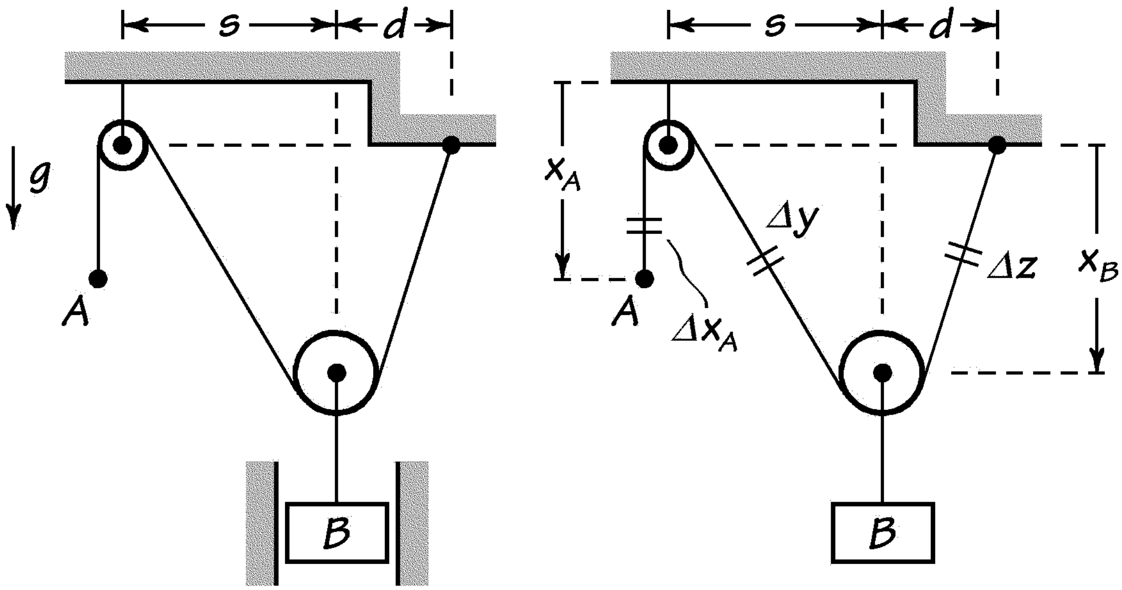
$$\boxed{v_A = \frac{-v_R\sqrt{L^2 + x_A^2}}{x_A + 2\sqrt{L^2 + x_A^2}}}$$

2.5.30

GOAL: Derive an expression for v_B.

GIVEN: The rope is pulled down at A at a rate of v_A.

DRAW:



FORMULATE EQUATIONS:

Let's first define y as the diagonal length of rope between the pulley supporting block B and the pulley above A. Likewise, we'll let z be the diagonal length of rope between the pulley supporting block B and the wall to its right. Thus, by conservation of rope, we have that

$$\Delta x_A + \Delta y + \Delta z = 0 \tag{1}$$

The diagonal lengths y and z are related to the block's vertical position x_B by, respectively,

$$y = \sqrt{s^2 + x_B^2} \tag{2}$$

$$z = \sqrt{d^2 + x_B^2} \tag{3}$$

SOLVE:

Differentiate (1) $\Rightarrow$
$$v_A + \dot{y} + \dot{z} = 0 \tag{4}$$

Differentiate (2) $\Rightarrow$
$$\dot{y} = \frac{x_B v_B}{\sqrt{s^2 + x_B^2}} \tag{5}$$

Differentiate (3) $\Rightarrow$
$$\dot{z} = \frac{x_B v_B}{\sqrt{d^2 + x_B^2}} \tag{6}$$

(5), (6) → (4) ⇒ $$v_A + x_B v_B \left(\frac{1}{\sqrt{s^2 + x_B^2}} + \frac{1}{\sqrt{d^2 + x_B^2}} \right) = 0$$

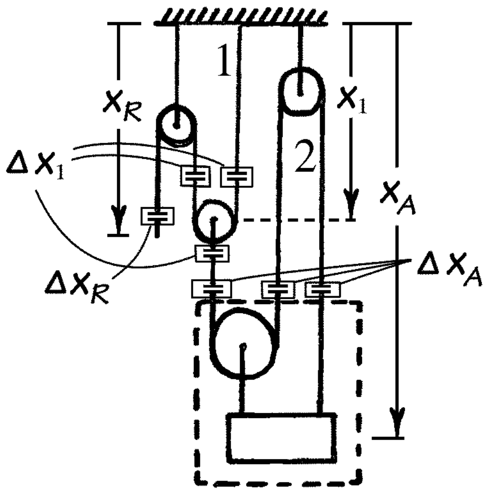
$$\boxed{v_B = \frac{-v_A}{x_B \left(\frac{1}{\sqrt{s^2 + x_B^2}} + \frac{1}{\sqrt{d^2 + x_B^2}} \right)}}$$

2.5.31

GOAL: Determine v_A.

GIVEN: Rope is reeled in at R at the rate $v_R = 3\,\text{ft/s}$.

DRAW:



FORMULATE EQUATIONS:

By conservation of rope,

Rope 1 $\Rightarrow$
$$\Delta x_R + 2\Delta x_1 = 0 \tag{1}$$

Rope 2 $\Rightarrow$
$$-\Delta x_1 + 3\Delta x_A = 0 \tag{2}$$

SOLVE:

Differentiate (1) $\Rightarrow$
$$v_R + 2\dot{x}_1 = 0 \tag{3}$$

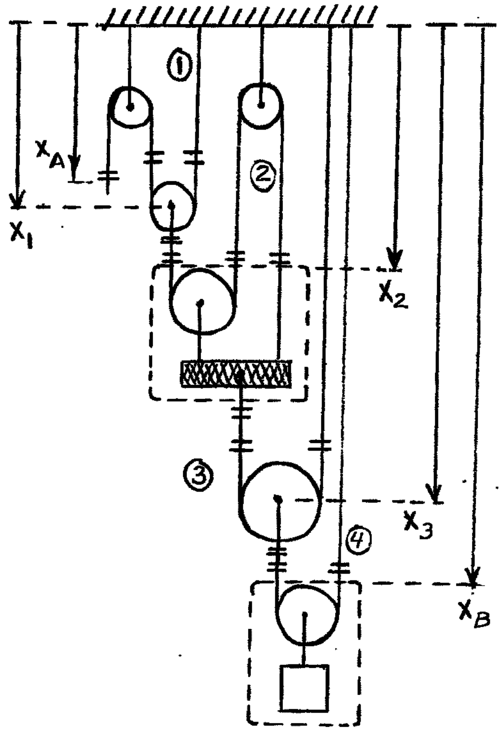
Differentiate (2) $\Rightarrow$
$$-\dot{x}_1 + 3v_A = 0 \tag{4}$$

(4) $\rightarrow$ (3) $\Rightarrow$
$$v_R + 6v_A = 0$$

$$v_A = -\frac{1}{6}v_R$$
$$v_A = -\frac{1}{6}(3\,\text{ft/s})$$

$$\boxed{v_A = -\frac{1}{2}\,\text{ft/s}}$$

2.5.32
GOAL: Determine v_B.
GIVEN: The person pulls down at A with a speed $v_A = 2\,\text{ft/s}$.
DRAW:



FORMULATE EQUATIONS:
By conservation of rope,

Rope 1 ⇒ $$\Delta x_A + 2\Delta x_1 = 0 \quad (1)$$

Rope 2 ⇒ $$-\Delta x_1 + 3\Delta x_2 = 0 \quad (2)$$

Rope 3 ⇒ $$-\Delta x_2 + 2\Delta x_3 = 0 \quad (3)$$

Rope 4 ⇒ $$-\Delta x_3 + 2\Delta x_B = 0 \quad (4)$$

SOLVE:

Differentiate (1) ⇒ $$v_A + 2\dot{x}_1 = 0 \quad (5)$$

Differentiate (2) ⇒ $$-\dot{x}_1 + 3\dot{x}_2 = 0 \quad (6)$$

Differentiate (3) ⇒ $$-\dot{x}_2 + 2\dot{x}_3 = 0 \quad (7)$$

Differentiate (4) ⇒ $$-\dot{x}_3 + 2v_B = 0 \quad (8)$$

(6), (7), (8) → (5) ⇒

$$v_B = -\frac{1}{24} v_A$$

$$v_B = -\frac{1}{24}(2\,\text{ft/s})$$

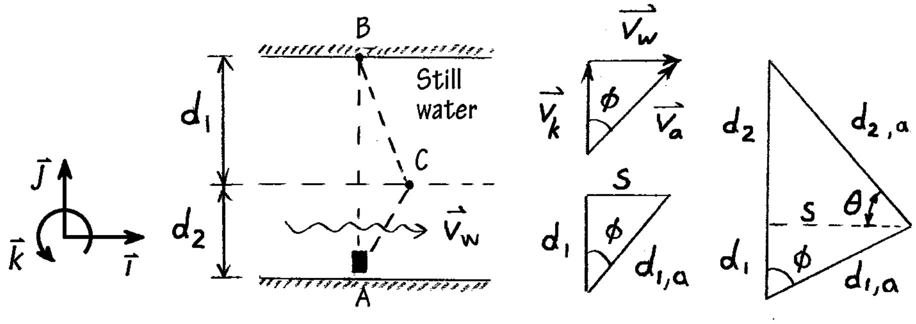
$$\boxed{v_B = -\frac{1}{12}\,\text{ft/s}}$$

2.5.33

GOAL: Determine the kayaker's actual speed in the current, how far over the kayak is at C, the angle the kayaker needs to travel at to get to B, and how much longer it takes than kayaking in completely still water.

GIVEN: The kayaker paddles from A to B with a maximum velocity of $\vec{v}_k = 2\vec{\jmath}$ m/s. The water has a current of $\vec{v}_w = 3\vec{\imath}$ m/s for $d_1 = 20$ m into the pond. The water is still to the other side of the pond, $d_2 = 50$ m.

DRAW:



FORMULATE EQUATIONS:

The actual velocity of the kayaker is given by

$$\vec{v}_a = \vec{v}_k + \vec{v}_w \tag{1}$$

The kayaker's speed, distance traveled, and travel time are related by

$$d = vt \tag{2}$$

SOLVE:

The kayaker's actual speed while in the current is

(1) $\Rightarrow$
$$\vec{v}_a = v_k\vec{\jmath} + v_w\vec{\imath}$$

$$v_a = \|\vec{v}_a\| = \sqrt{\vec{v}_A \cdot \vec{v}_A} = \sqrt{v_k^2 + v_w^2}$$

$$v_a = \sqrt{(2\,\mathrm{m/s})^2 + (3\,\mathrm{m/s})^2}$$

$$\boxed{v_a = 3.61\,\mathrm{m/s}}$$

Using similar triangles to find how far the kayaker has moved over,

$$\frac{v_k}{d_1} = \frac{v_w}{s}$$

$$s = d_1\frac{v_w}{v_k}$$

$$s = (20\,\mathrm{m})\left(\frac{3\,\mathrm{m/s}}{2\,\mathrm{m/s}}\right)$$

$$\boxed{s = 30\,\mathrm{m}}$$

The actual distance traveled while in the current is

$$\begin{aligned} d_{1,a} &= \sqrt{d_1^2 + s^2} \\ d_{1,a} &= \sqrt{(20\,\mathrm{m})^2 + (30\,\mathrm{m})^2} \\ d_{1,a} &= 36.1\,\mathrm{m} \end{aligned}$$

The travel time in the current is

(2) ⇒
$$t_1 = \frac{d_{1,a}}{v_a}$$

$$\begin{aligned} t_1 &= \frac{36.1\,\mathrm{m}}{3.61\,\mathrm{m/s}} \\ t_1 &= 10.0\,\mathrm{s} \end{aligned}$$

The angle the kayaker needs to paddle at to get to B is

$$\begin{aligned} \theta &= \tan^{-1}\frac{d_2}{s} \\ \theta &= \tan^{-1}\frac{50\,\mathrm{m}}{30\,\mathrm{m}} \end{aligned}$$

$$\boxed{\theta = 59.0°}$$

The actual distance traveled while in the still water portion is

$$\begin{aligned} d_{2,a} &= \sqrt{d_2^2 + s^2} \\ d_{2,a} &= \sqrt{(50\,\mathrm{m})^2 + (30\,\mathrm{m})^2} \\ d_{2,a} &= 58.3\,\mathrm{m} \end{aligned}$$

The travel time in the still water is

(2) ⇒
$$t_2 = \frac{d_{2,a}}{v_k}$$

$$
\begin{aligned}
t_2 &= \frac{58.3\,\mathrm{m}}{2\,\mathrm{m/s}} \\
t_2 &= 29.15\,\mathrm{s}
\end{aligned}
$$

The total travel time is

$$
\begin{aligned}
t_{\text{total}} &= t_1 + t_2 \\
t_{\text{total}} &= 10.0\,\mathrm{s} + 29.15\,\mathrm{s} \\
t_{\text{total}} &= 39.15\,\mathrm{s}
\end{aligned}
$$

When the entire pond is still water, the total travel time is

(2) $\Rightarrow$ $$t^* = \frac{d_1 + d_2}{v_k}$$

$$
\begin{aligned}
t^* &= \frac{20\,\mathrm{m} + 50\,\mathrm{m}}{2\,\mathrm{m/s}} \\
t^* &= 35.0\,\mathrm{s}
\end{aligned}
$$

The extra time it takes to paddle in the pond with a current is then

$$
\begin{aligned}
\Delta t &= t_{\text{total}} - t^* \\
\Delta t &= 39.15\,\mathrm{s} - 35.0\,\mathrm{s}
\end{aligned}
$$

$$\boxed{\Delta t = 4.15\,\mathrm{s}}$$

2.5.34

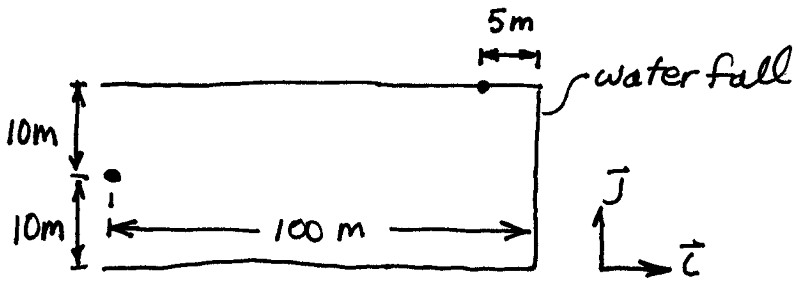
GOAL: Find the direction in which the student should paddle.
GIVEN: Distances to shore, velocity of the river, $\vec{v}_r$, and speed of the canoe v_c.
ASSUME: The absolute velocity vector of the canoe is given by

$$\vec{v}_c = \dot{x}_c\vec{\imath} + \dot{y}_c\vec{\jmath} \tag{1}$$

The velocity vector of the canoe relative to the river is given by

$$\vec{v}_{c/r} = \dot{x}_{c/r}\,\vec{\imath} + \dot{y}_{c/r}\,\vec{\jmath} \tag{2}$$

DRAW



FORMULATE EQUATIONS:
The direction the student should paddle, measured with respect to $\vec{\imath}$, is

$$\theta = \tan^{-1}\frac{\dot{y}_{c/r}}{\dot{x}_{c/r}} \tag{3}$$

The absolute velocity of the canoe is

$$\vec{v}_c = \vec{v}_r + \vec{v}_{c/r} \tag{4}$$

Because the absolute velocity vector of the canoe should point directly at the point on the shore where the student wishes to land, we get the relation

$$\frac{\dot{y}_c}{\dot{x}_c} = \frac{10}{95} \tag{5}$$

Lastly, the speed at which the student paddles is related to the two components of the relative velocity by

$$v_c = \sqrt{\dot{x}^2_{c/r} + \dot{y}^2_{c/r}} = 3\,\mathrm{m/s} \tag{6}$$

SOLVE:

(2) → (4) ⇒

$$\vec{v}_c = (4 + \dot{x}_{c/r})\vec{\imath} + \dot{y}_{c/r}\vec{\jmath} \tag{7}$$

By comparing (7) to (1), we can see that

$$\dot{x}_c = 4 + \dot{x}_{c/r} \tag{8}$$

$$\dot{y}_c = \dot{y}_{c/r} \tag{9}$$

(8), (9) → (5) ⇒
$$\frac{\dot{y}_{c/r}}{4+\dot{x}_{c/r}} = \frac{10}{95} \tag{10}$$

Rearranging (6) ⇒
$$\dot{x}_{c/r} = \sqrt{v_c^2 - \dot{y}_{c/r}^2} \tag{11}$$

(11) → (10) ⇒
$$\left[1+\left(\frac{10}{95}\right)^2\right]\dot{y}_{c/r}^2 - \frac{2(10)(4\,\mathrm{m/s})}{95}\dot{y}_{c/r} + \left[\left(\frac{10(4\,\mathrm{m/s})}{95}\right)^2 - \left(\frac{10}{95}\right)^2 v_c^2\right] = 0$$

$$1.011\dot{y}_{c/r}^2 - 0.842\dot{y}_{c/r} + 0.0776 = 0 \tag{12}$$

Solving (12) yields two possible solutions ⇒

$$\dot{y}_{c/r} = 0.727\,\mathrm{m/s},\ \text{or}\ 0.105\,\mathrm{m/s}$$

Plugging these into (11) ⇒
$$\dot{x}_{c/r} = 2.91\,\mathrm{m/s},\ \text{or}\ -3.00\,\mathrm{m/s}$$

Thus the two possible solution pairs are

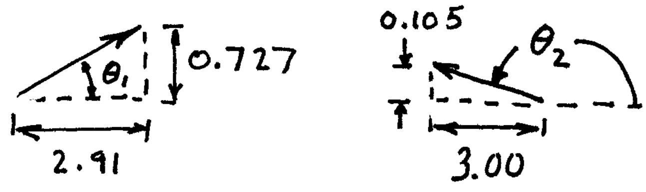
$$\dot{x}_{c/r} = 2.911\,\mathrm{m/s},\ \dot{y}_{c/r} = 0.727\,\mathrm{m/s},$$

and
$$\dot{x}_{c/r} = -3.00\,\mathrm{m/s},\ \dot{y}_{c/r} = 0.105\,\mathrm{m/s},$$

Plugging these into (3) ⇒

$$\theta_1 = \tan^{-1}\left(\frac{0.727\,\mathrm{m/s}}{2.911\,\mathrm{m/s}}\right) = \boxed{0.245\,\mathrm{rad}}$$

$$\theta_2 = \pi\,\mathrm{rad} - \tan^{-1}\left(\frac{0.105\,\mathrm{m/s}}{3.00\,\mathrm{m/s}}\right) = \boxed{3.11\,\mathrm{rad}}$$

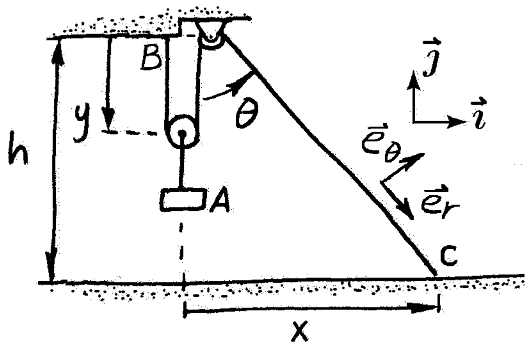


2.5.35

GOAL: Find the speed and acceleration of the pulley's load as a function of x, $\dot{x}$ and $\ddot{x}$.

GIVEN: System configuration.

DRAW



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\sin\theta$	$-\cos\theta$
$\vec{e}_\theta$	$\cos\theta$	$\sin\theta$

FORMULATE EQUATIONS:

Well use the general polar expressions for velocity and acceleration:

$$\vec{v}_C = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

$$\vec{a}_C = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

ASSUME: The motion of the point C is constrained such that $\vec{v}_C = \dot{x}\vec{\imath}$ and $\vec{a}_C = \ddot{x}\vec{\imath}$.

SOLVE:

Equating the two expressions for $\vec{v}_C$ gives us

$$\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta = \dot{x}\vec{\imath} = \dot{x}(\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta)$$

$\vec{e}_r$: $$\dot{r} = \dot{x}\sin\theta \qquad (1)$$

$\vec{e}_\theta$: $$r\dot{\theta} = \dot{x}\cos\theta \qquad (2)$$

(1) $\Rightarrow$ $$\dot{r} = x\dot{x}(x^2 + h^2)^{-0.5} \qquad (3)$$

(2) $\Rightarrow$ $$\dot{\theta} = \dot{x}h(x^2 + h^2)^{-1} \qquad (4)$$

We know from the kinematics of pulleys that the speed of the load will be one half the speed at which rope pulls through the pulley. Hence we have

$$\boxed{\vec{v}_A = \tfrac{1}{2}x\dot{x}(x^2 + h^2)^{-0.5}\vec{\jmath}}$$

Moving on to acceleration, we have

$$\vec{a}_C = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta = \ddot{x}\vec{\imath} = \ddot{x}(\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta)$$

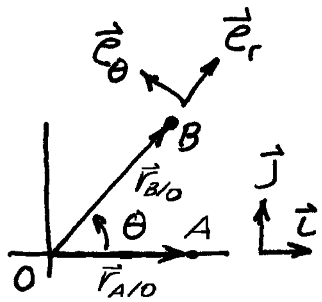
$\vec{e}_r$: $$\ddot{r} = r\dot{\theta}^2 + \ddot{x}\sin\theta \qquad (5)$$

$(4),(5) \Rightarrow$ $$\ddot{r} = \dot{x}^2h^2(x^2+h^2)^{-1.5} + \ddot{x}x(x^2+h^2)^{-0.5}$$

We know the acceleration of the load is one half the acceleration with which rope pulls through the pulley and thus have:

$$\boxed{\vec{a}_A = \frac{1}{2}\left[\dot{x}^2h^2(x^2+h^2)^{-1.5} + \ddot{x}x(x^2+h^2)^{-0.5}\right]\vec{\jmath}}$$

2.5.36
GOAL: Find time for B to reach $\frac{\pi}{2}$ rad and evaluate $\vec{r}_{B/A}$, $\vec{v}_{B/A}$, $\vec{a}_{B/A}$
GIVEN: Velocity of A and path of B
DRAW



	$\vec{\imath}$	$\vec{\jmath}$
$\vec{e}_r$	$\cos\theta$	$\sin\theta$
$\vec{e}_\theta$	$-\sin\theta$	$\cos\theta$

FORMULATE EQUATIONS:

$$\vec{r}_{B/O} = bt^2\vec{e}_r \tag{1}$$

$$\vec{v}_B = 2bt\vec{e}_r + bt^2\frac{d\vec{e}_r}{dt} = 2bt\vec{e}_r + bt^2\dot{\theta}\vec{e}_\theta \tag{2}$$

$$\vec{a}_B = 2b\vec{e}_r + 2bt\frac{d\vec{e}_r}{dt} + 2bt\dot{\theta}\vec{e}_\theta + bt^2\ddot{\theta}\vec{e}_\theta + bt^2\dot{\theta}\frac{d\vec{e}_\theta}{dt}$$
$$= 2b\vec{e}_r + 4bt\dot{\theta}\vec{e}_\theta + bt^2\ddot{\theta}\vec{e}_\theta - bt^2\dot{\theta}^2\vec{e}_r$$

$$= (2b - bt^2\dot{\theta}^2)\vec{e}_r + (4bt\dot{\theta} + bt^2\ddot{\theta})\vec{e}_\theta \tag{3}$$

$$\theta = at^2 \tag{4}$$

$$\dot{\theta} = 2at \tag{5}$$

$$\ddot{\theta} = 2a \tag{6}$$

$(2), (5) \Rightarrow$
$$\vec{v}_B = 2bt\vec{e}_r + 2abt^3\vec{e}_\theta \tag{7}$$

$(3), (5), (6) \Rightarrow$
$$\vec{a}_B = (2b - 4ba^2t^4)\vec{e}_r + 10abt^2\vec{e}_\theta \tag{8}$$

SOLVE:
(a) Find t such that $\theta = \frac{\pi}{2}$:

$$\frac{\pi}{2} = at^2 \Rightarrow \boxed{t = \sqrt{\frac{\pi}{2a}}} \tag{9}$$

(b)

$$\vec{v}_A = v_A \vec{i} \Rightarrow \vec{r}_{A/O} = v_A t \vec{i} \text{ and } \vec{a}_A = 0 \tag{10}$$

$$\vec{r}_{B/A} = \vec{r}_{B/O} - \vec{r}_{A/O}$$

(1), (10) $\Rightarrow$

$$\begin{aligned} \vec{r}_{B/A} &= bt^2 \vec{e}_r - v_A t \vec{i} \\ &= bt^2(\cos\theta \vec{i} + \sin\theta \vec{j}) - v_A t \vec{i} \\ &= (bt^2 \cos\theta - v_A t)\vec{i} + bt^2 \sin\theta \vec{j} \end{aligned}$$

At $t = \sqrt{\frac{\pi}{2a}}$ and $\theta = \frac{\pi}{2}$ we have

$$\boxed{\vec{r}_{B/A} = -v_A \sqrt{\frac{\pi}{2a}} \vec{i} + \frac{b\pi}{2a} \vec{j}}$$

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$$

(5), (10) $\Rightarrow$

$$\begin{aligned} \vec{v}_{B/A} &= 2bt\vec{e}_r + 2abt^3 \vec{e}_\theta - v_A \vec{i} \\ &= (2bt\cos\theta - 2abt^3 \sin\theta - v_A)\vec{i} + (2bt\sin\theta + 2abt^3\cos\theta)\vec{j} \end{aligned}$$

Evaluating at $t = \sqrt{\frac{\pi}{2a}}$, $\theta = \frac{\pi}{2}$ yields

$$\boxed{\vec{v}_{B/A} = (-\pi b \sqrt{\frac{\pi}{2a}} - v_A)\vec{i} + 2b\sqrt{\frac{\pi}{2a}} \vec{j}}$$

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A$$

(5), (8), (10) $\Rightarrow$

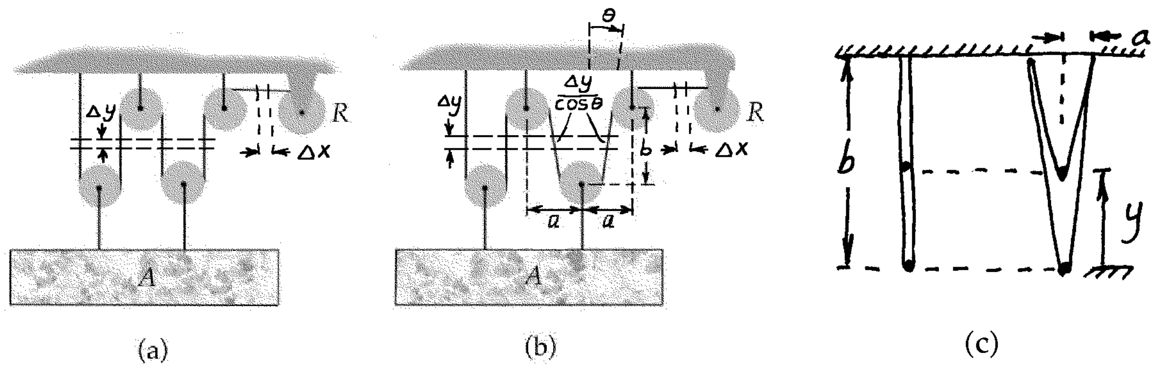
$$\begin{aligned} \vec{a}_{B/A} &= (2b - 4ba^2t^4)\vec{e}_r + 10abt^2 \vec{e}_\theta \\ &= [(2b - 4ba^2t^4)\cos\theta - 10abt^2 \sin\theta]\vec{i} + [(2b - 4ba^2t^4)\sin\theta + 10abt^2\cos\theta]\vec{j} \end{aligned}$$

At $t = \sqrt{\frac{\pi}{2a}}$, $\theta = \frac{\pi}{2}$ we have

$$\boxed{\vec{a}_{B/A} = -5\pi b \vec{i} + b(2 - \pi^2)\vec{j}}$$

SYSTEM ANALYSIS: Chapter 2

SA2.1
GOAL:
Compare speed of a block being raised by a pulley for two different cases.
GIVEN: System geometry for the two configurations.
DRAW:



Question (a):
FORMULATE EQUATIONS:
(Case a): From Figure (a) we have

$$4\Delta y + \Delta x = 0 \;\Rightarrow\; \Delta y = -\frac{1}{4}\Delta x$$

SOLVE:
Block A rises at $\frac{1}{4}\dot{x}$.

$$\dot{y} = -\frac{1}{4}\dot{x} = -\frac{1}{4}(1.0\,\mathrm{m/s}) = -0.25\,\mathrm{m/s}$$

Block A is raised at $\boxed{0.25\,\mathrm{m/s}}$
(Case b):
FORMULATE EQUATIONS:
From Figure (b) we see that the amount of rope used on the right pulley is more than in the left. y is defined from the Block A's initial position, as shown in Figure (c).
The original length of hanging rope is $2b + 2\sqrt{a^2+b^2}$ and after the block moves up y the length of rope is

$$2(b-y) + 2\sqrt{(b-y)^2 + a^2}$$

After the reel has retracted an amount x of rope we know the retracted rope and the rope currently in the pulley is equal to the original amount of rope, giving us

$$x + 2(b-y) + 2\sqrt{(b-y)^2+a^2} = 2b + 2\sqrt{a^2+b^2}$$

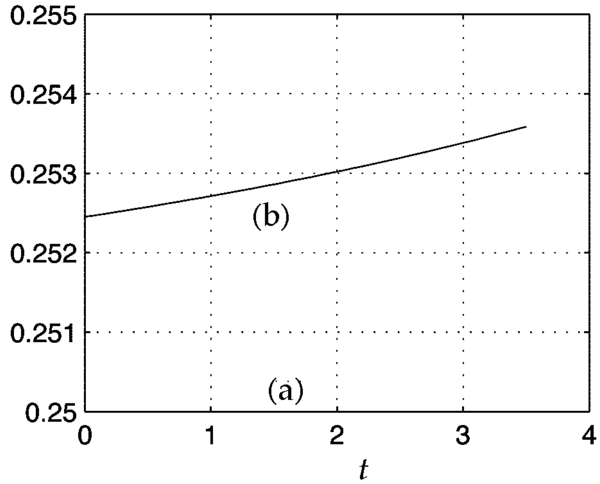
SOLVE:

$$x = 2y + 2\sqrt{a^2+b^2} - 2\sqrt{(b-y)^2+a^2}$$

$$\dot{x} = 2\dot{y} - 2\left(\frac{1}{2}\right)\left((b-y)^2+a^2\right)^{-\frac{1}{2}}(-2b\dot{y} + 2y\dot{y})$$

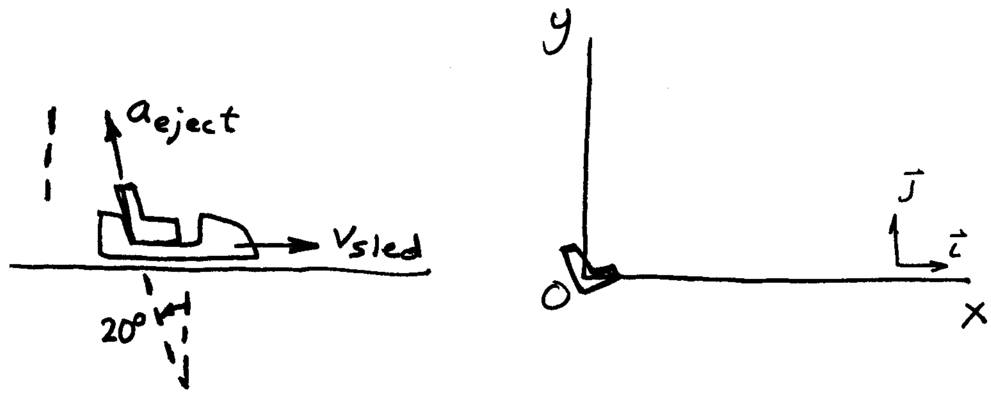
$$\boxed{\dot{y} = \frac{\dot{x}}{2}\frac{((b-y)^2+a^2)^{\frac{1}{2}}}{((b-y)^2+a^2)^{\frac{1}{2}} + b - y}}$$

To plot this we'll need to numerically integrate from 0 to 3.5 seconds, $a = 1\,\text{m}$, $b = 5\,\text{m}$, using an initial condition of $y(0) = 0$ The plot (obtained from a MATLAB integration of our equation for $\dot{y}$) shows the elevation rate for the two cases (given in m/s). Note how the elevation rate is higher from the start for the case of non-vertical supporting ropes and that the rate continues to increase as the mass is lifted higher.



Question (b): Left as a design exercise.

SA2.2
GOAL: Analyze the behavior of an ejector system.
GIVEN: System configuration and vehicle launch kinematics.
DRAW:



Ejection phase:
FORMULATE EQUATIONS: Motion in the x and y directions are independent. Gravity induces a negative vertical acceleration to the body and the ejection rocket provides an acceleration directed along the long axis of the seat. The seat ejects at an angle of 20° and so we have x and y components of the acceleration given by

$$\begin{aligned} a_x &= -[16 - (9\,\mathrm{s}^{-2})t^2](\sin 20°)(32.2\,\mathrm{ft/s}^2) \\ a_y &= [(16 - (9\,\mathrm{s}^{-2})t^2)\cos 20° - 1](32.2\,\mathrm{ft/s}^2) \end{aligned}$$

SOLVE: The speed at launch is 600 KEAS$= 600(1.15\,\mathrm{mph}) = 1012\,\mathrm{ft/s}$.
The acceleration components can be evaluated to give

$$\begin{aligned} a_x &= -176\,\mathrm{ft/s}^2 + (99.1\,\mathrm{ft/s}^4)t^2 \\ a_y &= 452\,\mathrm{ft/s}^2 - (272\,\mathrm{ft/s}^4)t^2 \end{aligned}$$

Integration gives us

$$\begin{aligned} v_x &= -(176\,\mathrm{ft/s}^2)t + (33.0\,\mathrm{ft/s}^4)t^3 + 1012\,\mathrm{ft/s} \\ v_y &= (452\,\mathrm{ft/s}^2)t - (90.8\,\mathrm{ft/s}^4)t^3 \\ x &= -(88.1\,\mathrm{ft/s}^2)t^2 + (8.26\,\mathrm{ft/s}^4)t^4 + (1012\,\mathrm{ft/s})t \\ y &= (226\,\mathrm{ft/s}^2)t^2 - (22.7\,\mathrm{ft/s}^4)t^4 \end{aligned}$$

Evaluating at $t = 0.8\,\mathrm{s}$ yields

$$\begin{aligned} v_x &= 888\,\mathrm{ft/s} \\ v_y &= 315\,\mathrm{ft/s} \\ x &= \boxed{757\,\mathrm{ft}} \\ y &= \boxed{135\,\mathrm{ft}} \end{aligned}$$

Drag phase
FORMULATE EQUATIONS:
We're given that the acceleration follows

$$a_x = -0.003v_x^2$$

We can determine the change in distance as a function of the initial and final speed through evaluating

$$a\,dx = v\,dv \tag{1}$$

and find the elapsed time as a function of initial and final speed through

$$\frac{dv}{dt} = a \tag{2}$$

SOLVE: The drag phase ends when the seat reaches 100 KEAS= 100(1.15) mph = 168.7 ft/s.

(1) $\Rightarrow$

$$\begin{aligned} -(0.003\,\text{ft}^{-1})v_x^2\,ds_x &= v_x\,dv_x \\ -(0.003\,\text{ft}^{-1})\,ds_x &= \frac{dv_x}{v_x} \\ -(0.003\,\text{ft}^{-1})\Delta s_x &= \ln[v_x]\Big|_f - \ln[v_x]\Big|_i \end{aligned}$$

Using $\ln[v_x]\Big|_f = 168.7\,\text{ft/s}$ and $\ln[v_x]\Big|_i = 888\,\text{ft/s}$ gives us

$$\Delta x = 554.\,\text{ft}$$

(2) $\Rightarrow$

$$\begin{aligned} \frac{dv_x}{dt} &= -(0.003\,\text{ft}^{-1})v_x^2 \\ \frac{dv_x}{v_x^2} &= -(0.003\,\text{ft}^{-1})dt \\ -\left[\frac{1}{v_x}\Big|_f - \frac{1}{v_x}\Big|_i\right] &= -(0.003\,\text{ft}^{-1})\Delta t \end{aligned}$$

Solving gives us

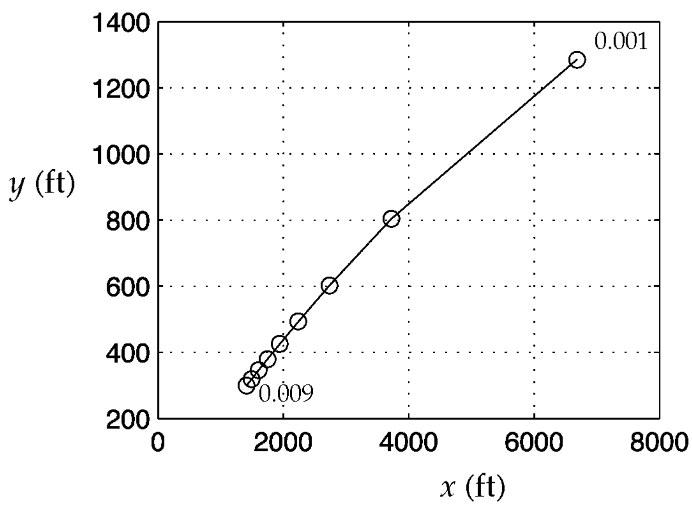
$$\Delta t = 1.60\,\text{s}$$

Knowing how long the drag phase lasts and how far the seat travels due to the drag deceleration lets us calculate the final horizontal and vertical positions of the seat.

$$x = 757\,\text{ft} + (888\,\text{ft/s})(1.60\,\text{s}) + 554\,\text{ft} = \boxed{2732\,\text{ft}}$$

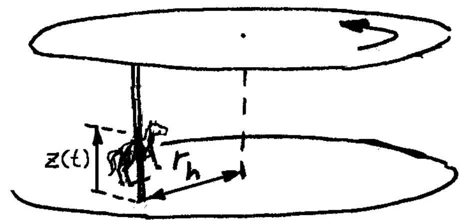
$$y = 135\,\text{ft} + (315\,\text{ft/s})(1.60\,\text{s}) - \frac{1}{2}(32.2\,\text{ft/s}^2)(1.60\,\text{s})^2 = \boxed{598\,\text{ft}}$$

Calculating the range variation for c going from $0.001\,\text{ft}^{-1}$ to $0.009\,\text{ft}^{-1}$ yields the following plot, which shows the x,y positions for increments of $0.001\,\text{ft}^{-1}$.



1400
1200
1000
800
600
400
200
0
2000
4000
6000
8000
y (ft)
x (ft)
0.001
0.009

SA2.3
GOAL:
Determine maximal acceleration levels of a carousel horse.
GIVEN: Position as a function of time for the horse.
DRAW:



FORMULATE EQUATIONS:
We're given

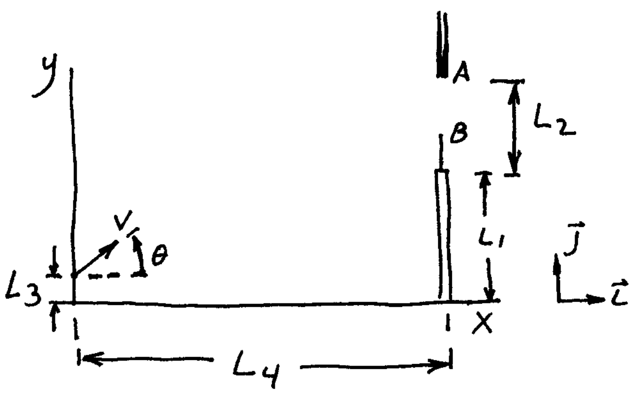
$$z(t) = z_0 + z_1 \cos(\omega_h t)$$

SOLVE:
Differentiating twice yields

$$\boxed{\ddot{z}(t) = -z_1 \omega_h^2 \cos(\omega_h t)}$$

Parametric plots left as an open-ended exercise.

SA2.4
GOAL:
a) Construct coordinate system
b) Calculate time for ball (b) to reach the building
c) Calculate height of ball when it hits the building.
d) Calculate height of cover when ball hits the building.
e) Graphically find the permissible launch angles so the ball goes through the opening.
GIVEN:
$\vec{r}_b(0) = L_3\vec{\jmath}, \vec{v}_b(0) = v(\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}), \vec{r}_B(0) = L_4\,\vec{\imath} + L_1\,\vec{\jmath}, \vec{v}_B(0) = v_B\vec{\jmath}\, v = 40\,\mathrm{m/s}, v_B = 6.0\,\mathrm{m/s}, L_1 = 14\,\mathrm{m}, L_2 = 25\,\mathrm{m}, L_3 = 2\,\mathrm{m}, L_4 = 100\,\mathrm{m}$
DRAW:



FORMULATE EQUATIONS:

$$\vec{r}_b(t) = \left(-\frac{1}{2}gt^2\right)\vec{\jmath} + \vec{v}_b(0)t + \vec{r}_b(0) \quad (1)$$
$$\vec{r}_B(t) = \vec{v}_B(0)t + \vec{r}_B(0) \quad (2)$$

SOLVE:

(1)⇒
$$\left(-\frac{1}{2}gt_f^2\right)\vec{\jmath} + \vec{v}_b(0)t_f + \vec{r}_b(0) = L_4\vec{\imath} + h\vec{\jmath}$$

$$\boxed{t_f = \frac{L_4}{v\cos\theta}} \quad (3)$$

$\vec{\jmath}$:
$$h = -\frac{g}{2}t_f^2 + v\,t_f\sin\theta + L_3$$
$$= -\frac{g}{2}\left(\frac{L_4}{v\cos\theta}\right)^2 + v\sin\theta\left(\frac{L_4}{v\cos\theta}\right) + L_3$$

$$\boxed{h = L_4\tan\theta - \frac{gL_4^2}{2v^2\cos^2\theta} + L_3} \quad (4)$$

$$
\begin{aligned}
v_B\vec{\jmath}\,t_f + L_4\vec{\imath} + L_1\vec{\jmath} &= L_4\vec{\imath} + h_B\vec{\jmath} \\
\vec{\jmath}:\Rightarrow \quad v_B t_f + L_1 &= h_B \\
h_B &= \boxed{v_B\left(\frac{L_4}{v\cos\theta}\right) + L_1}
\end{aligned}
$$

(3)→(2)⇒

Plotting the height of the cover, the height of the ball and the maximum height of the opening produces the following graph, where (a) indicates the height of the ball, (b) indicates the height of the moving cover and (c) indicates the height of the top edge of the opening. From the plot it can be seen that the ball is above the cover at a launch angle of slightly less than 0.7 rad and moves beyond the top of the opening at around 0.77 rad.