Two Phase Flow Boiling and Condensation In Conventional and Miniature Systems 2nd edition Ghiaasiaan Solutions Manual

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Problem 2.1

We have

$$dP/dx = \mu_L \frac{\partial^2 U}{\partial x^2} \qquad (1)$$

Also

$$dP/dy = -\rho_L g \qquad (2)$$

Assuming incompressible flow and steady-state, mass conservation requires that:

$$\int_0^H U(y)\,dy = 0 \qquad (3)$$

The velocity boundary conditions are:

$$U = 0 \quad \text{at } y = 0 \qquad (4)$$

$$\mu_L \frac{\partial U}{\partial y} = \frac{d\sigma}{dx} \quad \text{at } y = H \qquad (5)$$

Eq. (1) can now be integrated, with boundary conditions (4) & (5), to get:

$$\mu_L U = \left(\frac{d\sigma}{dx} - H \frac{\partial P}{\partial x} \right) y + \frac{1}{2} \frac{\partial P}{\partial x} y^2 \qquad (6)$$

Given that at any location $P = P_\infty + \rho_L g (H - y)$, then

$$\frac{\partial P}{\partial x} = \rho_L g \, dH/dx \qquad (7)$$

We can now substitute for U in Eq. (3) from Eq. (6), and perform the integration to get

$$\frac{\partial \sigma}{\partial x} = \frac{2}{3} H \frac{\partial P}{\partial x} \qquad (8)$$

Elimination of $\frac{\partial P}{\partial x}$ between Eqs. (7) and (8) then gives

$$\frac{d\sigma}{dx} = \frac{2}{3} \rho_L g H \, dH/dx \qquad (9)$$

Substitution in Eq. (6), from (7) and (9) eventually leads to the desired result.

Reference:
R. F. Probstein, Physicochemical Hydrodynamics, Butterworth, 1989.

Problem 2.3

Refering to Fig 1 below, at an arbitrary point Q we must have

$$r_1 + r_c = \frac{\sigma}{P_B - P_\infty} \qquad (1)$$

where r_1 & r_c are the principle radii of curvature. Taking r_1 as the radius in the plane of the page, we can write

$$r_1 = \left[\frac{(1+y'^2)^{3/2}}{|y''|} \right]_Q \qquad (2)$$

where $y' = dy/dx$ on the bubble surface.
The radius of curvature in the plane perpendicular to the page and passing through the point Q is r_c as shown in the figure (note that the bubble is axi-symmetric), and it can be easily shown from geometry that

$$r_c = \left\{ |x| \left[1 + \frac{1}{y'^2} \right]^{1/2} \right\} \qquad (3)$$

Combining Eqs. (1) ~ (3), the equation

defining the shape of the bubble's profile is derived as

$$\frac{(1+y'^2)^{\frac{3}{2}}}{|y''|} + |x|\left[1+\frac{1}{y'^2}\right]^{1/2} = \frac{\sigma}{P_B - P_\infty} \qquad (4)$$

To find $P_B - P_\infty$, we note that it is a constant, and must apply to point D as well. At point D, we can assume the bubble's surface is spherical with a radius of curvature $r_{c0}/\sin\theta_0$. Then

$$P_B - P_\infty = \frac{2\sigma_w}{r_{c0}/\sin\theta_0} \qquad (5)$$

The right side of Eq. (4) then becomes

$$\frac{\sigma\, r_{c0}/\sin\theta_0}{2\sigma_w} \qquad (6)$$

Substitution of Eq.(6) into Eq.(4) then gives a second order ordinary differential equation. The boundary (initial) conditions for the equation will be:

At $x = r_{co}$

$$y = 0$$

$$\frac{dy}{dx} \hat{=} y' = \tan\theta_0$$

Now, non-dimensionalization using the definitions of X and Y, will give the desired dimensionless second-order ordinary differential equation. The initial conditions will then be:

at $X = 0$

$$Y = 0$$

$$\frac{dY}{dX} = \tan\theta_0$$

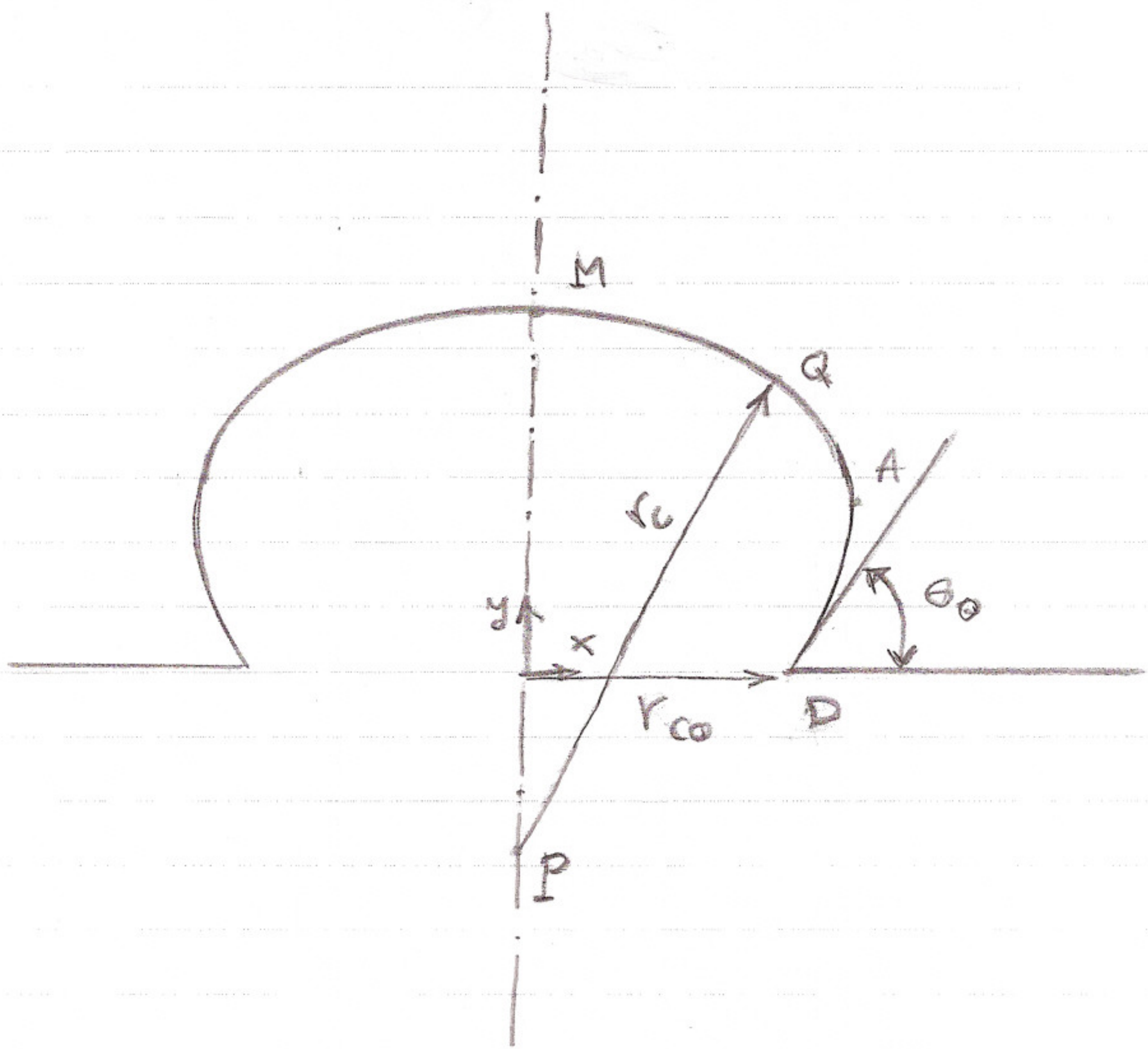

Figure 1

(Problem 2.3)

Problem 2.4

a)

$X_G = 3.774 \times 10^{-4}$ (CO_2 mole fraction in air)

$C_{He,290} = 1.28 \times 10^8$ Pa (C_{He} at 290 K)

$C_{He,300} = 1.71 \times 10^8$ Pa (C_{He} at 300K)

Apply Henry's law to the interphase

$P_0 = 101330$ Pa

$X_{L,290} = P_0 X_G / C_{He,290} = 2.988 \times 10^{-7}$

$X_{L,300} = P_0 X_G / C_{He,300} = 2.236 \times 10^{-7}$

$$m_{L,290} = X_{L,290} \frac{M_{CO_2}}{M_{H_2O}} = 7.30 \times 10^{-7}$$

$$m_{L,300} = X_{L,300} \frac{M_{CO_2}}{M_{H_2O}} = 5.47 \times 10^{-7}$$

b) Interpolation gives, at 287.5°C

$C_{He} = 1.17 \times 10^8$ Pa

$$X_L = P_0 \frac{X_G}{C_{He}} = 3.264 \times 10^{-7}$$

$m_L = X_L M_{CO_2} / M_{H_2O} = 7.98 \times 10^{-7}$

c) $\forall_W = 1.35 \times 10^{18}$ m^3

$Mass_W = \rho_L \forall_W = 1.35 \times 10^{21}$ kg (Water mass)

$Mass_{CO_2} = Mass_W \, m_L = 1.077 \times 10^{15}$ kg

2.4-2

$X_{G,1} = 3.756 \times 10^{-4}$

$X_{L,1} = P_0 X_{G,1} / C_{He} = 3.248 \times 10^{-7}$

$m_{L,1} = X_{L,1} M_{CO2} / M_{H2O} = 7.94 \times 10^{-7}$

$mass_{CO2,1} = \forall_w \, m_{L,1} = 1.072 \times 10^{15}$

$\Delta mass_{CO2} = mass_{CO2} - mass_{CO2,1} = 5.137 \times 10^{12}$ kg

This is change in the mass of dissolved CO2, if the oceans remain at equilibrium with the atmosphere.

$$Ratio = \frac{\Delta mass_{CO2}}{7.25 \times 10^{12}} = 0.7086$$

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{Example 2.5}
{Based on assumptions, T_s=25 C, and concentration of CO2 at s surface is zero.}
MM_v=18.
MM_air=29.
MM_CO2=44
D_d=0.55e-3
U_d=2.
T_G=25.+273.
T_L=25.+273.
P_G=1.0133e5
phi=0.6
P_sat=P_sat(Water, T=T_G)
P_v=phi*P_sat
X_v=P_v/P_G
m_v=X_v*MM_v/(X_v*MM_v+(1.-X_v)*MM_air)
X_vs=P_sat/P_G
m_vs=X_vs*MM_v/(X_vs*MM_v+(1.-X_vs)*MM_air)
mu_G=viscosity(AIR,T=T_G)
rho_G=density(AIR,T=T_G,P=P_G)
k_G=conductivity(AIR,T=T_G)
nu_G=mu_G/rho_G
C_PG=cp(STEAM,T=T_G,x=1)
D_12=(1.97e-5)*(101330./P_G)*((T_G/256.)^(1.685))
Sc_G=nu_G/D_12
Pr_G=prandtl(AIR,T=T_G)
Sh_G=2.
Sh_G=K_massG*D_d/(rho_G*D_12)
Nus_G=2.
Nus_G=H_G*D_d/k_G
{Part a:  First neglect the blowing effect}
m_dd_va1=K_massG*(m_vs-m_v)
{Now include blowing effect}
B_mG=(m_v-m_vs)/(m_vs-1.)
m_dd_va=K_massG*ln(1.+B_mg)
{Part b: Now we have heat transfer as well.  However, by fixing T_s at 50 C, evaporation rate will also be fixed, and will be decoupled from HT}
T_sb=273+50
P_vsb=P_sat(water, T=T_sb)
X_vsb=P_vsb/P_G
m_vsb=X_vsb*MM_v/(X_vsb*MM_v+(1.-X_vsb)*MM_air)
B_mGb=(m_v-m_vsb)/(m_vsb-1.)
m_dd_vb=K_massG*ln(1.+B_mgb)
{c:  This time we deal with CO2 mass transfer rate. Let us solve for case (a)}
m_CO2s=0.0
X_CO2G=500.e-6
m_CO2G=X_CO2G*MM_CO2/(X_CO2G*MM_CO2+X_v*MM_v+(1-X_v-X_CO2G)*MM_air)
D_aCO2=0.16e-4
Sh_G=K_massCO2*D_d/(rho_G*D_aCO2)
{First without the blowing effect:}
m_ddCO2=K_massCO2*(m_CO2s-m_CO2G)
{Now with blowing}
m_ddtot=m_dd_va+m_ddCO2b
B_mGCO2=(m_CO2G-m_CO2s)/(m_CO2s-(m_ddCO2b/m_ddtot))
m_ddCO2b=m_ddtot*m_CO2s+K_massCO2*(ln(1.+B_mGCO2)/B_mGCO2)*(m_CO2s-m_CO2G)
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$MM_v = 18$

$MM_{air} = 29$

$MM_{CO2} = 44$

$D_d = 0.00055$

$U_d = 2$

$T_G = 25 + 273$

$T_L = 25 + 273$

$P_G = 101330$

$\phi = 0.6$

$P_{sat} = \mathbf{P}_{sat}\,[\text{'Water'}, T = T_G]$

$P_v = \phi \cdot P_{sat}$

$$X_v = \frac{P_v}{P_G}$$

$$m_v = X_v \cdot \left[\frac{MM_v}{X_v \cdot MM_v + (1 - X_v) \cdot MM_{air}}\right]$$

$$X_{vs} = \frac{P_{sat}}{P_G}$$

$$m_{vs} = X_{vs} \cdot \left[\frac{MM_v}{X_{vs} \cdot MM_v + (1 - X_{vs}) \cdot MM_{air}}\right]$$

$\mu_G = \mathbf{Visc}\,[\text{'Air'}, T = T_G]$

$\rho_G = \rho\,[\text{'Air'}, T = T_G, P = P_G]$

$k_G = \mathbf{k}\,[\text{'Air'}, T = T_G]$

$$\nu_G = \frac{\mu_G}{\rho_G}$$

$C_{PG} = \mathbf{Cp}\,[\text{'Steam'}, T = T_G, x = 1]$

$$D_{12} = 0.0000197 \cdot \frac{101330}{P_G} \cdot \left[\frac{T_G}{256}\right]^{1.685}$$

$$Sc_G = \frac{\nu_G}{D_{12}}$$

$Pr_G = \mathbf{Pr}\,[\text{'Air'}, T = T_G]$

$Sh_G = 2$

$$Sh_G = K_{massG} \cdot \frac{D_d}{\rho_G \cdot D_{12}}$$

$$Nus_G = 2$$

$$Nus_G = H_G \cdot \frac{D_d}{k_G}$$

$$m_{dd,va1} = K_{massG} \cdot [m_{vs} - m_v]$$

$$B_{mG} = \frac{m_v - m_{vs}}{m_{vs} - 1}$$

$$m_{dd,va} = K_{massG} \cdot \mathbf{ln}[1 + B_{mG}]$$

$$T_{sb} = 273 + 50$$

$$P_{vsb} = \mathbf{P}_{sat}['Water', T = T_{sb}]$$

$$X_{vsb} = \frac{P_{vsb}}{P_G}$$

$$m_{vsb} = X_{vsb} \cdot \left[\frac{MM_v}{X_{vsb} \cdot MM_v + (1 - X_{vsb}) \cdot MM_{air}}\right]$$

$$B_{mGb} = \frac{m_v - m_{vsb}}{m_{vsb} - 1}$$

$$m_{dd,vb} = K_{massG} \cdot \mathbf{ln}[1 + B_{mGb}]$$

$$m_{CO2s} = 0$$

$$X_{CO2G} = 0.0005$$

$$m_{CO2G} = X_{CO2G} \cdot \left[\frac{MM_{CO2}}{X_{CO2G} \cdot MM_{CO2} + X_v \cdot MM_v + (1 - X_v - X_{CO2G}) \cdot MM_{air}}\right]$$

$$D_{aCO2} = 0.000016$$

$$Sh_G = K_{massCO2} \cdot \frac{D_d}{\rho_G \cdot D_{aCO2}}$$

$$m_{ddCO2} = K_{massCO2} \cdot [m_{CO2s} - m_{CO2G}]$$

$$m_{ddtot} = m_{dd,va} + m_{ddCO2b}$$

$$B_{mGCO2} = \frac{m_{CO2G} - m_{CO2s}}{m_{CO2s} - \frac{m_{ddCO2b}}{m_{ddtot}}}$$

$$m_{ddCO2b} = m_{ddtot} \cdot m_{CO2s} + K_{massCO2} \cdot \frac{\mathbf{ln}[1 + B_{mGCO2}]}{B_{mGCO2}} \cdot [m_{CO2s} - m_{CO2G}]$$

SOLUTION

Unit Settings: [J]/[K]/[Pa]/[kg]/[degrees]

$B_{mG} = 0.007998$ $B_{mGb} = 0.07276$ $B_{mGCO2} = 0.01198$

C_{PG} = 1887
D_{12} = 0.00002545
D_{aCO2} = 0.000016
D_d = 0.00055
H_G = 92.72
k_G = 0.0255
$K_{massCO2}$ = 0.06893
K_{massG} = 0.1096
MM_{air} = 29
MM_{CO2} = 44
MM_v = 18
μ_G = 0.00001848
m_{CO2G} = 0.0007638
m_{CO2s} = 0
m_{ddCO2} = -0.00005265
m_{ddCO2b} = -0.00005233
m_{ddtot} = 0.000821
$m_{dd,va}$ = 0.0008733
$m_{dd,va1}$ = 0.0008597
$m_{dd,vb}$ = 0.007699
m_v = 0.01163
m_{vs} = 0.01947
m_{vsb} = 0.07866
Nus_G = 2
ν_G = 0.0000156
ϕ = 0.6
Pr_G = 0.7281
P_G = 101330
P_{sat} = 3141
P_v = 1884
P_{vsb} = 12253
ρ_G = 1.185
Sc_G = 0.613
Sh_G = 2
T_G = 298
T_L = 298
T_{sb} = 323
U_d = 2
X_{CO2G} = 0.0005
X_v = 0.0186
X_{vs} = 0.031
X_{vsb} = 0.1209

6 potential unit problems were detected.

Problem 2.5

The assumptions mentioned regarding the droplet are met with $T_s = 25°C$ and $m_{CO_2,s} = 0$.

Define

① → vapor

② → air

③ → CO_2

a) $T_L = T_G = T_s = 25°C = 298\,K$

$\phi = 0.6$

$P_1 = \phi\, P_{sat} = (0.6)(3141\,Pa) = 1884\,Pa$

$P_0 = 101330\,Pa$

$X_{1,G} = P_1 / P_0 = 0.0186$

$$m_{1,G} = \frac{X_1 M_1}{X_1 M_1 + (1 - X_1) M_2} = 0.01163$$

$X_{1s} = P_{sat}|_{T_s} / P = 3141/101330 = 0.031$

$$m_{1s} = \frac{X_{1s} M_1}{X_{1s} M_1 + (1 - X_{1s}) M_2} = 0.0195$$

From Appendix E

$$D_{12} = (1.97 \times 10^{-5}\,m^2/s)\left(\frac{101330}{P_0}\right)\left(\frac{T_G}{256}\right)^{1.685}$$

$$= 2.545 \times 10^{-5}\,m^2/s$$

$$Sh_G = \frac{K_G D}{\rho_G D_{12}} = 2$$

As an approximation use

$$\rho_G = \rho_{air} = 1.185 \ kg/m^3$$

Also, $D = 5.5 \times 10^{-4} \ m$

$$\Rightarrow K_G \approx 0.116593 \ kg/m^2 s$$

Without consideration for the blowing effect, we will have

$$m_1'' = K_G (m_{1,s} - m_{1,G}) = 0.00085 \ kg/m^2 s$$

With the blowing effect included, then

$$B_{m,G} = \frac{m_{1,G} - m_{1,s}}{m_{1,s} - 1} \approx 0.0086$$

$$m_1'' = K_G \ln(1 + B_{m,G}) = 0.000873 \ kg/m^2 s$$

b) $T_{s,b} = 323 \ K$

$$P_{1,sb} = P_{sat}\big|_{T_{s,b}} = 1.225 \times 10^4 \ Pa$$

$$X_{1,sb} = P_{1,sb} / P_0 = 0.1209$$

$$\Rightarrow m_{1,sb} = 0.0787$$

$$B_{mG,b} = \frac{m_{1,G} - m_{1,sb}}{m_{1,sb} - 1} = 0.07276$$

$$m_{1,b}'' = K_G \ln(1 + B_{mG,b}) = 0.0077 \ kg/m^2 s$$

c) We will do this for Part a only.

$m_{3,s} = 0$

$X_{3,G} = 500\times10^{-6}$

$$m_{3,G} = \frac{X_{3G} M_3}{X_{3G} M_3 + X_{1G} M_1 + (1 - X_{1G} - X_{3G}) X_2} = 7.64\times10^{-4}$$

For diffusion coefficient, let us assume diffusivity of CO2 with respect to the mixture is equal to D_{13} (for simplicity).

$D_{13} = 1.6\times10^{-5}\ m^2/s$

$$Sh_{G,CO2} = \frac{K_{G,CO2}\, D}{\rho_G\, D_{13}} \Rightarrow K_{G,CO2} = 0.0689\ kg/m^2s$$

$m_{3,s} = 0$

$$B_{mG} = \frac{m_{3,G} - m_{3,s}}{m_{3,s} - \dfrac{m''_{CO2}}{m''_{tot}}} \quad (1)$$

$$m''_3 = m''_{tot} m_{3,s} + K_{G,CO2} \frac{\ln(1+B_{mG})}{B_{mG}} (m_{3,s} - m_{3,G}) \quad (2)$$

$$m''_{tot} = m''_3 + m''_1 \quad (3)$$

Solution of (1) ~ (3) gives

$m''_3 = -0.0000523\ kg/m^2s.$

Problem 2.6

The threshold diameter, according to Suo and Griffith (1964) is

$$D = 0.3 \left[\frac{\sigma}{g \Delta \rho} \right]^{1/2}$$

Define

① air-water at room temperature

② saturated water-steam at 10 MPa

③ saturated R-134a mixture at 1.02 MPa

	ρ_L (kg/m^3)	ρ_G (kg/m^3)	σ (N/m)	$0.3\sqrt{\frac{\sigma}{g\Delta\rho}}$ (m)
①	997	1.185	0.072	8.15×10^{-4}
②	688.6	55.48	0.012	4.19×10^{-4}
③	1,146	50.23	0.0061	2.26×10^{-4}

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{P 2.7}
P_inf=101330.
T_sat_inf=T_sat(Water, P=P_inf)
T_cr_w=647.15
sigma=0.238*(1-T_sat_inf/T_cr_w)^1.25*(1.-0.639*(1.-T_sat_inf/T_cr_w))
rho_f=density (water, P=P_inf, x=0.0)
rho_g=density(water, P=P_inf, x=1.)
h_f=enthalpy(Water, P=P_inf, x=0.)
h_g=enthalpy(Water, P=P_inf, x=1.)
h_fg=h_g-h_f
{R=1.e-6}
DelT_sup=2.*sigma*T_sat_inf/(R*h_fg*rho_g)
DelT_subc=2.*sigma*T_sat_inf/(R*h_fg*rho_f)
```

$$P_{inf} = 101330$$

$$T_{sat,inf} = \mathbf{T}_{sat}\left[water, P = P_{inf}\right]$$

$$T_{cr,w} = 647.15$$

$$\sigma = 0.238 \cdot \left[1 - \frac{T_{sat,inf}}{T_{cr,w}}\right]^{1.25} \cdot \left[1 - 0.639 \cdot \left(1 - \frac{T_{sat,inf}}{T_{cr,w}}\right)\right]$$

$$\rho_f = \rho\left[water, P = P_{inf}, x = 0\right]$$

$$\rho_g = \rho\left[water, P = P_{inf}, x = 1\right]$$

$$h_f = \mathbf{h}\left[water, P = P_{inf}, x = 0\right]$$

$$h_g = \mathbf{h}\left[water, P = P_{inf}, x = 1\right]$$

$$h_{fg} = h_g - h_f$$

$$DelT_{sup} = 2 \cdot \sigma \cdot \frac{T_{sat,inf}}{R \cdot h_{fg} \cdot \rho_g}$$

$$DelT_{subc} = 2 \cdot \sigma \cdot \frac{T_{sat,inf}}{R \cdot h_{fg} \cdot \rho_f}$$

SOLUTION

Unit Settings: SI K Pa J mass deg

(Table 1, Run 14)

$DelT_{subc}$ = 0.0004092	$DelT_{sup}$ = 0.6563
h_f = 419073	h_{fg} = 2.257E+06
h_g = 2.676E+06	P_{inf} = 101330
R = 0.00005	ρ_f = 958.4
ρ_g = 0.5975	σ = 0.05929
$T_{cr,w}$ = 647.2	$T_{sat,inf}$ = 373.2

5 potential unit problems were detected.

Parametric Table: Table 1

	R	$DelT_{sup}$
Run 1	1.000E-06	*32.82*
Run 2	0.000002	*16.41*
Run 3	0.000002	*16.41*
Run 4	0.000003	*10.94*
Run 5	0.000004	*8.204*
Run 6	0.000005	*6.563*
Run 7	0.000006	*5.469*
Run 8	0.000007	*4.688*
Run 9	0.000008	*4.102*
Run 10	0.000009	*3.646*
Run 11	0.00001	*3.282*
Run 12	0.00002	*1.641*
Run 13	0.00004	*0.8204*
Run 14	0.00005	*0.6563*

Problem 2.8

We have

$\lambda = 100\ m$

$\zeta_0 = 0.5\ m$

$$k = \frac{2\pi}{\lambda} = 0.0628\ m^{-1}$$

$\sigma = 0.071\ N/m$

$\rho_L \approx 1000\ kg/m^3$

From Eq. (2.105)

$$\omega = \left[\sigma k^3/\rho_L + g k\right]^{1/2} = 0.785\ Rad/s$$

Eq. (2.104) gives

$$A = \zeta_0 \frac{\omega}{k} = 6.248\ m^2/s$$

The velocity in z direction is $\partial\phi/\partial z$,

and from Eq. (2.98)

$$v_{z,max}\Big|_{z=0} = Ak = 0.3924\ m/s$$

At location $-z^*$ (i.e., at a depth z^*),

again from Eq. (2.98)

$$v_{z,max}\Big|_{z=-z^*} = Ake^{-kz^*}$$

We thus get

$$v_{z,max}\Big|_{z=-10m} = 1.78\times10^{-5}\ m/s$$

$$v_{z,max}\Big|_{z=-50m} \approx 0.0$$

$$\rho_L\left(\frac{\partial \phi_L'}{\partial t} + U_L\frac{\partial \phi_L'}{\partial x} + U_L\frac{\partial \phi_L'}{\partial y} - g\zeta\right)$$

$$\rho_G\left(\frac{\partial \phi_G'}{\partial t} + U_G\frac{\partial \phi_G'}{\partial x} + U_G\frac{\partial \phi_G'}{\partial y} - g\zeta\right) = \sigma\left(-\frac{\partial^2 \zeta}{\partial x^2} - \frac{\partial^2 \zeta}{\partial y^2}\right) \quad \text{— (1)}$$

$$\frac{\partial \zeta}{\partial t} + U_L\frac{\partial \zeta}{\partial x} + U_L\frac{\partial \zeta}{\partial y} \simeq \left.\frac{\partial \phi_L'}{\partial z}\right|_{z=0} \quad \text{— (2)}$$

$$\frac{\partial \zeta}{\partial t} + U_G\frac{\partial \zeta}{\partial x} + U_G\frac{\partial \zeta}{\partial y} \simeq \left.\frac{\partial \phi_G'}{\partial z}\right|_{z=0} \quad \text{— (3)}$$

$$\zeta = \zeta_0 \exp\left[i(\omega t - k_x x - k_y y)\right] \quad \text{— (4)}$$

Employing (4) into (2)

$$-bk = \zeta_0\left[i\omega + U_L(-ik_x) + U_L(-ik_y)\right] = i\zeta_0\left[\omega - U_L k_x - U_L k_y\right]$$

Employing (4) into (3)

$$b'k = \zeta_0\left[i\omega - U_G i k_x - i U_G k_y\right] = i\zeta_0\left[\omega - U_G k_x - U_G k_y\right]$$

$$\rho_L\left[i\omega b - ik_x U_L b - ik_y U_L b - g\zeta_0\right]$$

$$-\rho_G\left[i\omega b' - ik_x U_G b' - ik_y U_G b' - g\zeta_0\right] = \sigma\zeta_0\left(k_x^2 + k_y^2\right)$$

$\checkmark$ $\nabla^2\phi_L = 0$

$$(-ik_x)^2 + (-ik_y)^2 + (-k)^2 = -k_x^2 - k_y^2 + k^2 = 0$$

$$\Rightarrow k^2 = k_x^2 + k_y^2 \; \checkmark$$

$$\rho_L \left\{ \frac{\zeta_0 \omega [\omega - U_L(k_x+k_y)]}{-k^2} - U_L (k_x+k_y) \frac{\zeta_0 [\omega - U_L(k_x+k_y)]}{k^2} \right.$$

$$\left. - \frac{g\zeta_0}{k} \right\} - \rho_G \left\{ - \frac{\zeta_0 \omega [\omega - U_G(k_x+k_y)]}{k^2} + U_G(k_x+k_y) \frac{\zeta_0 [\omega - U_G(k_x+k_y)]}{k^2} \right.$$

$$\left. - \frac{g\zeta_0}{k} \right\} = \sigma \zeta_0 k$$

$$(\rho_L+\rho_G)\left(\frac{\omega}{k}\right)^2 - \frac{2U_L(k_x+k_y)}{k}\rho_L\left(\frac{\omega}{k}\right) - \frac{2\rho_G U_G (k_x+k_y)}{k}\left(\frac{\omega}{k}\right)$$

$$+ \frac{\rho_L U_L^2 (k_x+k_y)^2}{k^2} + \frac{\rho_G U_G^2 (k_x+k_y)^2}{k^2} - \left(\frac{\rho_L g - \rho_G g}{k} + \sigma k\right) = 0$$

$$\Rightarrow (\rho_L+\rho_G)\left(\frac{\omega}{k}\right)^2 - \frac{2(U_L\rho_L + U_G\rho_G)(k_x+k_y)}{k}\left(\frac{\omega}{k}\right) + \frac{(\rho_L U_L^2 + \rho_G U_G^2)(k_x+k_y)^2}{k^2}$$

$$- \frac{g(\rho_L-\rho_G)}{k} - \sigma k = 0$$

$$\frac{\omega}{k} = \frac{(U_L\rho_L + U_G\rho_G)}{(\rho_L+\rho_G)}\frac{(k_x+k_y)}{k} \pm \frac{1}{2(\rho_L+\rho_G)}\left\{ 4\frac{(U_L\rho_L+U_G\rho_G)^2(k_x+k_y)^2}{k^2} \right.$$

$$\left. - 4\left[\frac{(\rho_L U_L^2 + \rho_G U_G^2)(k_x+k_y)^2}{k^2} - g\frac{(\rho_L-\rho_G)}{k} - \sigma k\right](\rho_L+\rho_G) \right\}^{\frac{1}{2}}$$

$$\frac{w}{k} = \frac{(U_L\rho_L + U_G\rho_G)(k_x+k_y)}{(\rho_L+\rho_G)\quad k} \pm \frac{1}{(\rho_L+\rho_G)}\Big\{ (k_x^2+k_y^2)\left(2\rho_L\rho_G U_L U_G - \rho_L\rho_G U_G^2 - \rho_L\rho_G U_L^2\right) - kg(\rho_L^2-\rho_G^2) + \sigma k^3(\rho_L+\rho_G)\Big\}^{1/2}$$

Critical happens when

$$\frac{g}{k}\left(\frac{\rho_L-\rho_G}{\rho_L+\rho_G}\right) + \frac{(k_x+k_y)^2}{k^2}\frac{\rho_L\rho_G}{(\rho_L+\rho_G)^2}(U_L-U_G)^2 - \frac{\sigma k}{\rho_L+\rho_G} = 0$$

Taylor Instability happens when $U_L = U_G = 0$

$$\frac{g}{k}(\rho_L-\rho_G) - \sigma k = 0$$

$$k_c = \sqrt{\frac{g\Delta\rho}{\sigma}} = \frac{1}{L}$$

$$\lambda_c = \frac{2\pi}{k_c} = 2\pi L \qquad \lambda_{c_2} = \sqrt{2}\,\lambda_c = 2\pi\sqrt{2}\,L$$

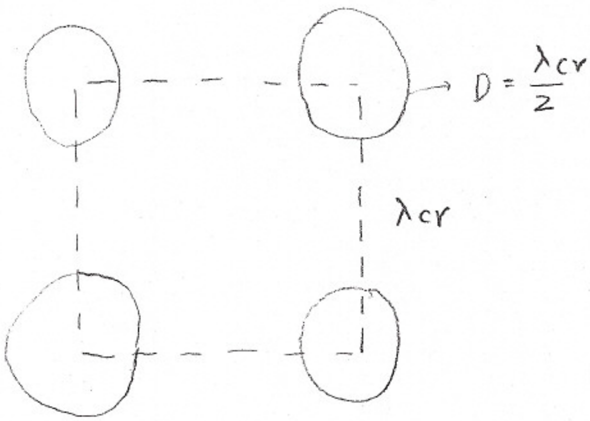
To find the fastest growing wave length

$$\frac{dw}{dk} = 0 \quad , \quad U_L = U_G = 0$$

$$w = \left(g\left(\frac{\rho_L-\rho_G}{\rho_L+\rho_G}\right)k - \frac{\sigma k^3}{\rho_L+\rho_G}\right)^{\frac{1}{2}}$$

$$\frac{dw}{dk} = \frac{g\left(\frac{\rho_L-\rho_G}{\rho_L+\rho_G}\right) - 3\frac{\sigma}{\rho_L+\rho_G}k^2}{\sqrt{gk\left(\frac{\rho_L-\rho_G}{\rho_L+\rho_G}\right) - \frac{\sigma k^3}{\rho_L+\rho_G}}} = 0 \Rightarrow k = \sqrt{\frac{g\Delta\rho}{3\sigma}} = \frac{1}{\sqrt{3}\,L}$$

$$\lambda_{d_2} = \sqrt{2}\,\lambda_d = 2\pi\sqrt{6}\,L$$



$\lambda_H = \lambda_c = 2\pi R_j$, $U_{G,cr} = \sqrt{\frac{2\pi\sigma}{\rho_G \lambda_H}}$

mass flow rate

$$\dot{m} = \rho_G A_G U_G$$

$$= \rho_G \frac{\pi}{4}\left(\frac{\lambda_{cr}}{2}\right)^2 \sqrt{\frac{2\pi\sigma}{\rho_G \lambda_H}}$$

$\lambda_H = 2\pi\left(\frac{\lambda_c}{2}\right) = \pi\lambda_c$

$$= \rho_G \frac{\pi}{4}\frac{1}{4}\lambda_c^2 \sqrt{\frac{2\pi\sigma}{\rho_G \lambda_c}}$$

$$= \frac{1}{16}\rho_G \pi \lambda_c^2 \sqrt{\frac{2\pi\sigma}{\rho_G \lambda_c}}$$

$$= \frac{1}{16}\rho_G \pi\, 4\pi^2 R_j^2 \sqrt{\frac{\sigma}{\rho_G R_j}}$$

ok

$$= \frac{1}{4}\rho_G \pi^3 R_j^2 \sqrt{\frac{\sigma}{\rho_G R_j}}$$

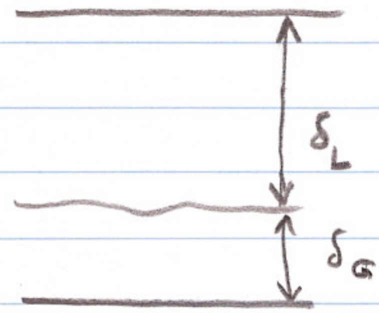
mass flow rate per unit surface area:

$$\frac{\rho_G \frac{1}{16}\pi \cdot \lambda_c^2 \sqrt{\frac{2\pi\sigma}{\rho_G \lambda_c}}}{\lambda_c^2} = \frac{1}{16}\rho_G \pi \sqrt{\frac{2\pi\sigma}{\rho_G \lambda_c}} = \frac{1}{16}\sqrt{\frac{2\pi^3\sigma\rho_G}{2\pi R_j}}$$

$$= \frac{1}{16}\sqrt{\frac{\pi\sigma\rho_G}{R_j}}$$

Numbers can now be plugged in

Problem 2.11 and 2.12



Starting from Eqs. (2.117) and (2.118) we must express ϕ_L' and ϕ_G' such that

$$\frac{d\phi_G'}{dz} = 0 \quad \text{at} \quad z = -\delta_G$$

$$\frac{d\phi_L'}{dz} = 0 \quad \text{at} \quad z = +\delta_L$$

Thus, instead of Eq. (2.117) use

$$\phi_L' = b \cosh\left[-k(z-\delta_L)\right] e^{i(\omega t - kx)}$$

and instead of Eq. (2.118) use

$$\phi_G' = b' \cosh\left[k(z+\delta_G)\right] e^{i(\omega t - kx)}$$

The result of the analysis will be similar to Eq. (2.129) except that ρ_L and ρ_G are replaced with ρ_L' and ρ_G' where

$$\rho_L' = \rho_L / \tanh(k\delta_L)$$

$$\rho_G' = \rho_G / \tanh(k\delta_G)$$

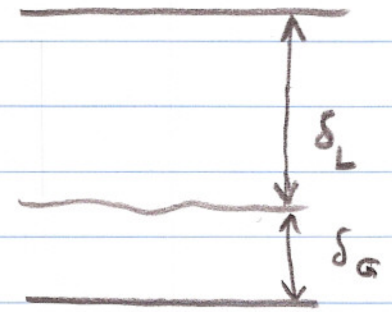
The neutral and the fastest growing wavelengths are found to be, when $U_L \to 0$ and $U_G \to 0$:

$$\lambda_{cr}' = 2\pi \lambda_L'$$

$$\lambda_d = 2\pi\sqrt{3}\ \lambda_L'$$

$$\lambda_L' = \sqrt{\frac{\sigma}{g(\rho_L' - \rho_G')}}$$

Problem 2.11



Starting from Eqs. (2.117) and (2.118) we must express ϕ'_L and ϕ'_G such that

$$\frac{d\phi'_G}{dz} = 0 \quad \text{at} \quad z = -\delta_G$$

$$\frac{d\phi'_L}{dz} = 0 \quad \text{at} \quad z = +\delta_L$$

Thus, instead of Eq. (2.117) use

$$\phi'_L = b \cosh\left[-k(z-\delta_L)\right] e^{i(\omega t - kx)}$$

and instead of Eq. (2.118) use

$$\phi'_G = b' \cosh\left[k(z+\delta_G)\right] e^{i(\omega t - kx)}$$

The result of the analysis will be similar to Eq. (2.129) except that ρ_L and ρ_G are replaced with ρ_L' and ρ_G' where

$$\rho_L' = \rho_L / \tanh(k\delta_L)$$

$$\rho_G' = \rho_G / \tanh(k\delta_G)$$

Solution

$$\nabla^2 \phi = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = 0 \qquad (a)$$

$$\frac{\partial \phi}{\partial r} = \dot{R} \quad \text{at} \quad r = R$$

$$\frac{\partial \phi}{\partial r} \to 0 \quad \text{at} \quad r \to \infty$$

The solution for these equations is

$$\phi = R\dot{R}\ln r$$

Therefore

$$\frac{\partial \phi}{\partial t} = \ln r\left[\dot{R}^2 + R\ddot{R}\right]$$

$$U = u_r = \frac{\partial \phi}{\partial r} = \frac{R\dot{R}}{r}$$

Insert in Euler's equation:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}U^2 + \frac{P}{\rho} = f(t)$$

$$\Rightarrow \left[\dot{R}^2 + R\ddot{R}\right]\ln r + \frac{1}{2}\left(\frac{R\dot{R}}{r}\right)^2 + \frac{P}{\rho} = f(t) \qquad (b)$$

Equation (b) must be applicable to a point far away, ~~were~~ where

$$\left[\dot{R}^2 + R\ddot{R}\right]\ln R_\infty + \frac{P_\infty}{\rho} = f(t) \qquad (c)$$

At the surface of the cylinder:

$$\left[\dot{R}^2 + R\ddot{R}\right]\ln R + \frac{1}{2}\dot{R}^2 + \frac{P_L}{\rho} = f(t) \qquad (d)$$

Subtracting (c) from (d) gives

$$\left[\dot{R}^2 + R\ddot{R}\right]\ln\frac{R}{R_\infty} + \frac{1}{2}\dot{R}^2 + \frac{P_L - P_\infty}{\rho} = 0$$

Since $\lim_{R_\infty \to \infty} \ln(R/R_\infty) \to -\infty$,

the following equations must both apply in order for the above expression to make sense:

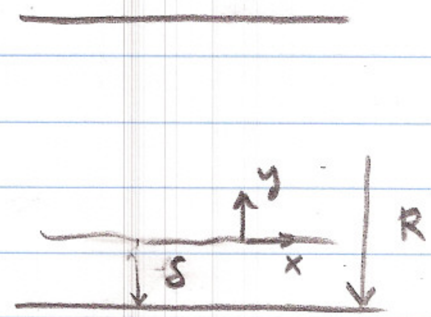
$$\dot{R}^2 + R\ddot{R} = 0$$

$$\frac{1}{2}\dot{R}^2 + \frac{P_L - P_\infty}{\rho} = 0$$

These imply:

$$R\ddot{R} - \frac{2(P_L - P_\infty)}{\rho_L} = 0$$

Problem 2.14



Assume

1. $\delta/R \ll 1$; therefore liquid can be infinitely thick
2. Both phases are incompressible & inviscid
3. No gravitational effect
4. Axi-symmetric conditions

In view of assumption 1, the waves can be treated as 2-D, in (x, y) coordinates. The solution proceeds as:

$$\phi_L = U_L x + \phi_L' \qquad (1)$$

$$\phi_G = U_G x + \phi_G' \qquad (2)$$

Integration of Euler's equation for the two phases (similar to Eq. (2.119)) gives:

$$\rho_L \left(\frac{\partial \phi_L'}{\partial t} + U_L \frac{\partial \phi_L'}{\partial x} \right) = -P_L + P_{oL} \quad (3)$$

$$\rho_G \left(\frac{\partial \phi_G'}{\partial t} + U_G \frac{\partial \phi_G'}{\partial x} \right) = -P_G + P_{oG} \quad (4)$$

At equilibrium we would have

$$P_{oL} - P_{oG} = \frac{\sigma}{R-h} \quad (5)$$

Also

$$P_L - P_G = \frac{\sigma}{R-h-\zeta} + \sigma \frac{\partial^2 \zeta}{\partial x^2} \quad (6)$$

Now assume

$$\phi_L' = A_L \, e^{-ky} \, e^{i(\omega t - kx)} \quad (7)$$

$$\phi_G' = A_G \cosh\left[k(y+\delta)\right] e^{i(\omega t - kx)} \quad (8)$$

These equations satisfy the following

$$\nabla^2 \phi_L' = \nabla^2 \phi_G' = 0 \quad (9)$$

$$\nabla \phi_L' \Big|_{y \to \infty} = 0 \quad (10)$$

$$\frac{d\phi_G'}{dy} \Big|_{y=-\delta} = 0 \quad (11)$$

Assume a disturbance of the form

$$\zeta = \zeta_0 \, e^{i(\omega t - kx)} \quad (12)$$

We now must satisfy

$$v_L\big|_{y=\zeta} \approx \frac{\partial \phi_L}{\partial y}\Big|_{y=0} = \frac{\partial \zeta}{\partial t} + U_L \frac{\partial \zeta}{\partial x} \quad (13)$$

$$v_G\big|_{y=\zeta} \approx \frac{\partial \phi_G}{\partial y}\Big|_{y=0} = \frac{\partial \zeta}{\partial t} + U_G \frac{\partial \zeta}{\partial x} \quad (14)$$

$$v_L\big|_{y=\delta} = v_G\big|_{y=\delta} \quad (15)$$

We also note that

$$-(P_L - P_G) + (P_{0L} + P_{0G}) = \frac{\sigma}{R-\delta} - \frac{\sigma}{(R-\delta)-\zeta} - \sigma \frac{\partial^2 \zeta}{\partial x^2} \quad (16\text{-a})$$

$$\frac{\sigma}{(R-\delta)-\zeta} \simeq \frac{1}{R-\delta}\sigma \left[1 - \frac{\zeta}{R-\delta}\right] \quad (16\text{-b})$$

Eqs. (13), (14), (15), then lead to, respectively

$$k A_L = -i\zeta_0 (\omega - U_L k) \quad (17)$$

$$k A_G \sinh(k\delta) = i\zeta_0 (\omega - U_G k) \quad (18)$$

$$\rho_L \left[i\omega A_L + U_L A_L(-ik)\right] - \rho_G \left[i\omega A_G \cosh(k\delta) + U_G A_G \cosh(k\delta)(-ik)\right] = \sigma \zeta_0 k^2 - \frac{\sigma \zeta_0^2}{(R-\delta)^2}$$

(19)

$$(17) \Rightarrow \quad A_L = -i \frac{J_0 (\omega - U_L k)}{k} \qquad (20)$$

$$(18) \Rightarrow \quad A_G = i \frac{J_0 (\omega - U_G k)}{k \sinh (k\delta)} \qquad (21)$$

substitution from Eqs. (20) and (21) in (19) gives the desired result.

Problem 2.16

Assume:

1. steady - state
2. $\delta / R << 1$, so that a planar, 2-D stability analysis at the interphase is justified
3. $\mu_G << \mu_L$, so that the gas can be treated as inviscid
4. Axi-symmetric flow
5. Negligible gravitational effect

2.16.2 The conservation equations for the liquid phase can then be represented as:

$$\frac{\partial u_L}{\partial x}+\frac{\partial v_L}{\partial y}=0 \tag{5}$$

$$\rho_L \frac{\partial u_L}{\partial t}=-\frac{\partial P_L}{\partial x}+\mu_L\left(\frac{\partial^2 u_L}{\partial x^2}+\frac{\partial^2 u_L}{\partial y^2}\right) \tag{6}$$

$$\rho_L \frac{\partial v_L}{\partial t}=-\frac{\partial P_L}{\partial y}+\mu_L\left(\frac{\partial^2 v_L}{\partial x^2}+\frac{\partial^2 v_L}{\partial y^2}\right) \tag{7}$$

The gas phase conservation equation are similar, except that they do not include viscosity.

Periodic motion of incompressible fluid surfaces obeying the Navier-Stokes equations can be represented using a potential and a stream function [6, 17]. Each velocity component of the liquid is therefore assumed to consist of an inviscid term and a perturbation representing the effect of viscosity [17].

$$u_L=u_L^0+U_L \tag{8}$$

$$v_L=v_L^0+V_L \tag{9}$$

$$P=P^0 \tag{10}$$

where parameters with superscript *0* represent ideal fluid conditions, and *U* and *V* represent the velocities in *x* and *y* directions due to viscosity, respectively. The gas phase is assumed to be inviscid; the gas velocity components therefore do not include terms representing viscosity. The above ideal-fluid velocities are represented by a potential function of the form:

$$\Phi_L=A_L\, e^{-ky+\omega t}\cos k\, x \tag{11}$$

$$\Phi_G = A_G \cosh\left[k\left(y+\delta\right)\right]e^{\omega t} \cos k\, x \tag{12}$$

where for both phases:

$$\left(u^o, v^o\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\Phi \tag{13}$$

These potential functions fully represent the irrotational, ideal liquid and gas fields. The liquid velocity terms that account for the viscosity-induced rotational motion can be represented by a stream function, ψ_L, that must satisfy [17]:

$$\frac{\partial \psi_L}{\partial t} = \nu_L \nabla^2 \psi_L \tag{14}$$

where:

$$\left(U_L, V_L\right) = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)\psi_L \tag{15}$$

Equation (14) and all the necessary boundary conditions can be satisfied using:

$$\psi_L = B_L e^{-my+\omega t} \sin kx \tag{16}$$

Note that the above equation approximates the liquid core as infinitely-deep. Using Eq. (14), it can be shown that:

$$m^2 = k^2 + \frac{\omega}{\nu_L} \tag{17}$$

The above potential and stream functions are now utilized in a linear stability analysis, using the following boundary conditions at the interphase:

$$v_L = v_G \tag{18}$$

$$-P_L + 2\mu_L\left(\frac{\partial V_L}{\partial y}\right) = -P_G - \sigma\frac{\partial^2 \eta}{\partial x^2} - \Delta P_\sigma \tag{19}$$

$$\mu_L\left(\frac{\partial u_L}{\partial y}+\frac{\partial v_L}{\partial x}\right)=0 \tag{20}$$

where:

$$\eta=\frac{k}{\omega}(B_L-A_L)e^{\omega t}\cos kx \tag{21}$$

$$P_L-P_G=-\rho_L\left.\frac{\partial\Phi_L}{\partial t}\right|_{y=0}+\rho_G\left.\frac{\partial\Phi_G}{\partial t}\right|_{y=0} \tag{22}$$

The term ΔP_σ in Eqn. (19), which is de-stabilizing, is the surface tension force caused by the curvature of the channel, and can be represented as:

$$\Delta P_\sigma=\frac{\sigma}{R-\delta}-\frac{\sigma}{R-\delta-\eta}\approx\frac{\sigma}{R-\delta}\left(1+\frac{\eta}{R-\delta}\right) \tag{23}$$

Equations (19-21) following substitution for velocities using the aforementioned potential and stream functions, lead to:

$$-A_L+B_L-A_G\sinh(kh)=0 \tag{24}$$

$$2k^2A_L-\left(m^2+k^2\right)B_L=0 \tag{25}$$

$$\begin{gathered}\left[2\mu_Lk^2+\frac{k^3\sigma}{\omega}-\frac{\sigma}{(R-\delta)^2}\frac{k}{\omega}+\rho_L\omega\right]A_L+\\ \left[-2\mu_Lkm-\frac{k^3\sigma}{\omega}+\frac{\sigma}{(R-\delta)^2}\frac{k}{\omega}\right]B_L-\left[\rho_G\alpha\cosh(k\delta)\right]A_G=0\end{gathered} \tag{26}$$

A non-trivial solution is possible if the determinant of the coefficient matrix of Eqns. (24-26) vanishes, and that leads to:

2.16.5

$$\left[\rho^* \coth\left(R^* KH\right)+1\right]\Omega^2 - \frac{K}{R^{*2}(1-H)^2} + K^3$$
$$+4K^4\left[1-\left(1+\frac{\Omega}{K^2}\right)^{1/2}\right] + 4\Omega K^2 = 0 \qquad (27)$$

where $\rho^* = \rho_G/\rho_L$; $H = \delta/R$; $R^* = R\frac{\sigma}{\rho_L v_L^2}$; $K = k\left(\frac{\sigma}{\rho_L v_L^2}\right)^{-1}$; and $\Omega = \omega\left(\frac{\rho_L^2 v_L^3}{\sigma^2}\right)$. The interface is evidently unstable when $\Omega > 0$, and the neutral condition ($\Omega = 0$) occurs when

$$\left(KR^*\right)_{cr} = \frac{1}{1-H} \qquad (28)$$

Disturbances with wavelengths longer than λ_{cr} thus cause instability, where:

$$\lambda_{cr} = 2\pi\left(R-\delta\right) \qquad (29)$$

The neutral wavelength thus is similar to the predictions of the Kelvin-Helmholtz stability; and does not depend on liquid viscosity, in agreement with the Taylor stability analysis for film boiling of viscous liquids. The fastest growing wavelength, Ω_d, occurs when $d\Omega/dK = 0$, whereby:

$$\left(\frac{\rho^* R^* H}{\sinh^2\left(R^* KH\right)}\right)\Omega_d^2 - 4K\left[2+\left(1+\frac{\Omega_d}{K_d^2}\right)^{-1/2}\right]\Omega_d$$
$$-16K_d^3\left[1-\left(1+\frac{\Omega_d}{K_d^2}\right)^{1/2}\right] - 3K_d^2 = 0 \qquad (30)$$

Problem 2.17

We have

$P_L = 1.0133 \times 10^5$ Pa

$T_L = 298$

$\sigma = 0.071$ N/m

$\rho_L \approx 1000$ kg/m^3

$\gamma = C_{PG}/C_{vG} = 1.4$

$\rho_G = 1.18$ kg/m^3

The volumetric and second mode oscillation frequencies are found from

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3\gamma P_L}{\rho_L R_0^2}}$$

$$f_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi}\left\{\frac{24\sigma}{R_0^3\left[3\rho_G + 2\rho_L\right]}\right\}^{1/2}$$

The results are as follows

R_0 (m)	f_0 (Hz)	f_2 (Hz)
1.0×10^{-5}	3.28×10^{5}	1.47×10^{5}
1.0×10^{-4}	3.28×10^{4}	4,641
5×10^{-4}	6,567	415

```
{P 2.17}
R_01=10.e-6
R_02=100.e-6
R_03=500.e-6
gamma=1.4
P_L=101330.
T_L=298.
rho_L=1000.
sigma=0.071
omega_01=sqrt(3.*gamma*P_L/(rho_L*R_01^2))
f_01=omega_01/(2.*pi)
omega_02=sqrt(3.*gamma*P_L/(rho_L*R_02^2))
f_02=omega_02/(2.*pi)
omega_03=sqrt(3.*gamma*P_L/(rho_L*R_03^2))
f_03=omega_03/(2.*pi)
rho_a=density(air, P=P_L, T=T_L)
omega_21=sqrt((24.*sigma)/(R_01^3*(3.*rho_a+2.*rho_L)))
omega_22=sqrt((24.*sigma)/(R_02^3*(3.*rho_a+2.*rho_L)))
omega_23=sqrt((24.*sigma)/(R_03^3*(3.*rho_a+2.*rho_L)))
f_21=omega_21/(2.*pi)
f_22=omega_22/(2.*pi)
f_23=omega_23/(2.*pi)
```

$R_{01} = 0.00001$

$R_{02} = 0.0001$

$R_{03} = 0.0005$

$\gamma = 1.4$

$P_L = 101330$

$T_L = 298$

$\rho_L = 1000$

$\sigma = 0.071$

$$\omega_{01} = \sqrt{3 \cdot \gamma \cdot \frac{P_L}{\rho_L \cdot R_{01}^{\ 2}}}$$

$$f_{01} = \frac{\omega_{01}}{2 \cdot \pi}$$

$$\omega_{02} = \sqrt{3 \cdot \gamma \cdot \frac{P_L}{\rho_L \cdot R_{02}^{\ 2}}}$$

$$f_{02} = \frac{\omega_{02}}{2 \cdot \pi}$$

$$\omega_{03} = \sqrt{3 \cdot \gamma \cdot \frac{P_L}{\rho_L \cdot R_{03}^{\ 2}}}$$

$$f_{03} = \frac{\omega_{03}}{2 \cdot \pi}$$

$$\rho_a = \rho\left[\text{'Air'}, P = P_L, T = T_L\right]$$

$$\omega_{21} = \sqrt{\frac{24 \cdot \sigma}{R_{01}^{3} \cdot \left[3 \cdot \rho_a + 2 \cdot \rho_L\right]}}$$

$$\omega_{22} = \sqrt{\frac{24 \cdot \sigma}{R_{02}^{3} \cdot \left[3 \cdot \rho_a + 2 \cdot \rho_L\right]}}$$

$$\omega_{23} = \sqrt{\frac{24 \cdot \sigma}{R_{03}^{3} \cdot \left[3 \cdot \rho_a + 2 \cdot \rho_L\right]}}$$

$$f_{21} = \frac{\omega_{21}}{2 \cdot \pi}$$

$$f_{22} = \frac{\omega_{22}}{2 \cdot \pi}$$

$$f_{23} = \frac{\omega_{23}}{2 \cdot \pi}$$

SOLUTION

Unit Settings: [J]/[K]/[Pa]/[kg]/[degrees]

$f_{01} = 328332$	$f_{02} = 32833$	$f_{03} = 6567$
$f_{21} = 146776$	$f_{22} = 4641$	$f_{23} = 415.1$
$\gamma = 1.4$	$\omega_{01} = 2.063E+06$	$\omega_{02} = 206297$
$\omega_{03} = 41259$	$\omega_{21} = 922219$	$\omega_{22} = 29163$
$\omega_{23} = 2608$	$P_L = 101330$	$\rho_a = 1.185$
$\rho_L = 1000$	$R_{01} = 0.00001$	$R_{02} = 0.0001$
$R_{03} = 0.0005$	$\sigma = 0.071$	$T_L = 298$

1 potential unit problem was detected.

```
{P 2.17}
R_01=10.e-6
R_02=100.e-6
R_03=500.e-6
gamma=1.4
P_L=101330.
T_L=298.
rho_L=1000.
sigma=0.071
omega_01=sqrt(3.*gamma*P_L/(rho_L*R_01^2))
f_01=omega_01/(2.*pi)
omega_02=sqrt(3.*gamma*P_L/(rho_L*R_02^2))
f_02=omega_02/(2.*pi)
omega_03=sqrt(3.*gamma*P_L/(rho_L*R_03^2))
f_03=omega_03/(2.*pi)
rho_a=density(air, P=P_L, T=T_L)
omega_21=sqrt((24.*sigma)/(R_01^3*(3.*rho_a+2.*rho_L)))
omega_22=sqrt((24.*sigma)/(R_02^3*(3.*rho_a+2.*rho_L)))
omega_23=sqrt((24.*sigma)/(R_03^3*(3.*rho_a+2.*rho_L)))
f_21=omega_21/(2.*pi)
f_22=omega_22/(2.*pi)
f_23=omega_23/(2.*pi)
```

$R_{01} = 0.00001$

$R_{02} = 0.0001$

$R_{03} = 0.0005$

$\gamma = 1.4$

$P_L = 101330$

$T_L = 298$

$\rho_L = 1000$

$\sigma = 0.071$

$$\omega_{01} = \sqrt{3 \cdot \gamma \cdot \frac{P_L}{\rho_L \cdot R_{01}^2}}$$

$$f_{01} = \frac{\omega_{01}}{2 \cdot \pi}$$

$$\omega_{02} = \sqrt{3 \cdot \gamma \cdot \frac{P_L}{\rho_L \cdot R_{02}^2}}$$

$$f_{02} = \frac{\omega_{02}}{2 \cdot \pi}$$

$$\omega_{03} = \sqrt{3 \cdot \gamma \cdot \frac{P_L}{\rho_L \cdot R_{03}^2}}$$

$$f_{03} = \frac{\omega_{03}}{2 \cdot \pi}$$

$$\rho_a = \rho\left[\text{'Air'}, P = P_L, T = T_L\right]$$

$$\omega_{21} = \sqrt{\frac{24 \cdot \sigma}{R_{01}^{3} \cdot \left[3 \cdot \rho_a + 2 \cdot \rho_L\right]}}$$

$$\omega_{22} = \sqrt{\frac{24 \cdot \sigma}{R_{02}^{3} \cdot \left[3 \cdot \rho_a + 2 \cdot \rho_L\right]}}$$

$$\omega_{23} = \sqrt{\frac{24 \cdot \sigma}{R_{03}^{3} \cdot \left[3 \cdot \rho_a + 2 \cdot \rho_L\right]}}$$

$$f_{21} = \frac{\omega_{21}}{2 \cdot \pi}$$

$$f_{22} = \frac{\omega_{22}}{2 \cdot \pi}$$

$$f_{23} = \frac{\omega_{23}}{2 \cdot \pi}$$

SOLUTION

Unit Settings: [J]/[K]/[Pa]/[kg]/[degrees]

f_{01} = 328332	f_{02} = 32833	f_{03} = 6567
f_{21} = 146776	f_{22} = 4641	f_{23} = 415.1
γ = 1.4	ω_{01} = 2.063E+06	ω_{02} = 206297
ω_{03} = 41259	ω_{21} = 922219	ω_{22} = 29163
ω_{23} = 2608	P_L = 101330	ρ_a = 1.185
ρ_L = 1000	R_{01} = 0.00001	R_{02} = 0.0001
R_{03} = 0.0005	σ = 0.071	T_L = 298

1 potential unit problem was detected.

Problem 2.7. For a microbubble surrounded and at thermal and mechanical equilibrium with liquid, prove that the liquid must be superheated according to:

$$P_{sat} - P_L \approx \frac{2\sigma}{R} \frac{\rho_L + \rho_v}{\rho_L}$$

where P_L is the pressure in the liquid phase and P_{sat} is the saturation pressure corresponding to the temperature of the temperature of the bubble and its surroundings.

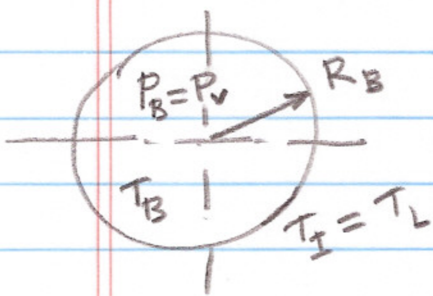
Also, for a micro droplet surrounded and at thermal and mechanical equilibrium with its vapor, prove that the vapor must be supercooled according to

$$P_{sat} - P_v \approx -\frac{2\sigma}{R} \frac{\rho_v}{\rho_L}$$

where P_v is the pressure in the vapor phase and P_{sat} is the saturation pressure corresponding to the temperature of the temperature of the bubble and its surroundings.

Problem 2.7

P_L, T_L



$T_B = T_L$ (from thermal equilibrium)

From Eq. (2.50)

$$P_B = P_{sat}(T_L) \exp\left(-\frac{2\sigma}{\rho_L \frac{R_u}{M} T_L R}\right) \qquad (1)$$

Since $\dfrac{2\sigma}{\rho_L \frac{R_u}{M} T_L R} << 1$, then

$$P_B \approx P_{sat}\Big|_{T_L}\left[1 - \frac{2\sigma}{\rho_L \frac{R_u}{M} T_L R}\right]$$

$$P_{sat}\Big|_{T_L} - P_B \approx P_{sat}\Big|_{T_L} \frac{2\sigma}{\rho_L \frac{R_u}{M} T_L R} \approx \frac{2\sigma \rho_g}{\rho_L R} \qquad (2)$$

Mechanical equilibrium requires that

$$P_B - P_\infty = \frac{2\sigma}{R} \qquad (3)$$

$(2)\ \&\ (3) \Rightarrow \quad P_{sat}\Big|_{T_L} - P_\infty = \dfrac{2\sigma(\rho_L - \rho_g)}{\rho_L R} \approx \dfrac{2\sigma}{R} \qquad (4)$

Now, from Clapeyron's relation:

$$\frac{dP}{dT} = \frac{h_{fg}}{T_{sat} v_{fg}}$$

$$\Rightarrow \quad \Delta T \approx \Delta P \; \frac{T_{sat} v_g}{h_{fg}} \qquad (5)$$

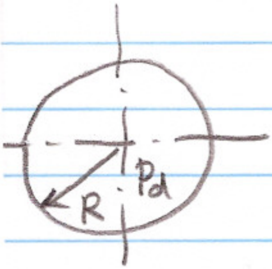
(4) & (5) $\Rightarrow$

$$T_L - T_{sat}\Big|_{P_\infty} = \frac{2\sigma T_{sat} v_g}{R\, h_{fg}} \qquad (6)$$

This expression represents the liquid superheat needed for the bubble with radius R to be at equilibrium with its surrounding liquid.

The attached Table 1 shows some numbers.

Now, for a droplet



$$P_\infty = P_v$$

$$T_\infty = T_d = T_v$$

$$\ln \frac{P_v}{P_{sat}|_{T_v}} = + \frac{2\sigma}{\rho_L \frac{R_u}{M} T_v R} \qquad (7)$$

$$\Rightarrow P_v \approx P_{sat}|_{T_v} \left[1 + \frac{2\sigma}{\rho_L \frac{R_u}{M} T_v R}\right] \approx P_{sat}|_{T_v} + \frac{2\sigma \rho_v}{\rho_L R} \qquad (8)$$

$$P_v - P_{sat}|_{T_v} = \frac{2\sigma \rho_v}{\rho_L R}$$

$$\Rightarrow T_{sat}|_{P_v} - T_v \approx \frac{2\sigma T_v}{\rho_L R h_{fg}} \qquad (9)$$

Thus, for a microdroplet with radius R to be stable in pure vapor, the vapor must be supercooled by the above amount.

The attached Table 2 shows some numerical results.