Principles of Seismology 2nd edition UdÃ-as Solutions Manual

SKU: 9781107138698-SOLUTIONS

Chapter 4. Solved Problems

P4.2. Given the stress tensor

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

find the principal stresses, the principal axes, the invariants I_1, I_2, and I_3, the deviatoric stress tensor, and the invariants J_2 and J_3.

Solution

To calculate the principal stresses ($\sigma_1, \sigma_2, \sigma_3$) and principal axes ($\nu^1_i, \nu^2_i, \nu^3_i$), we calculate the eigenvalues and eigenvectors of the matrix. They are found through the equation

$$(\tau_{ij} - \sigma\delta_{ij})\nu_i = 0 \qquad (4.2.1)$$

Putting the determinant equal to zero we obtain

$$\begin{vmatrix} 2-\sigma & -1 & 1 \\ -1 & -\sigma & 1 \\ 1 & 1 & 2-\sigma \end{vmatrix} = 0$$

$$\sigma^3 - 4\sigma^2 + \sigma + 6 = 0$$

The three roots of the equation are the principal stresses

$$\sigma_1 = -1$$
$$\sigma_2 = 2$$
$$\sigma_3 = 3$$

From these values we obtain, $\sigma_o = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{4}{3}$

The principal axes of stress are the eigenvectors ν_i associated with the three eigenvalues. Using equation (4.2.1) and substituting the obtained values of $\sigma_1, \sigma_2, \sigma_3$, we obtain the for each one the values of ν_1, ν_2, ν_3, adding the condition that $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$

For $\sigma_1 = -1$

$$(v_1^1, v_2^1, v_3^1) = \left(-{}^{1}\!/_{\sqrt{6}}, -{}^{2}\!/_{\sqrt{6}}, {}^{1}\!/_{\sqrt{6}}\right)$$

For $\sigma_2 = 2$

$$(v_1^2, v_2^2, v_3^2) = \left({}^{1}\!/_{\sqrt{3}}, {}^{1}\!/_{\sqrt{3}}, {}^{1}\!/_{\sqrt{3}}\right)$$

For $\sigma_3 = 3$

$$(v_1^3, v_2^3, v_3^3) = \left({}^{1}\!/_{\sqrt{2}}, 0, {}^{1}\!/_{\sqrt{2}}\right)$$

The invariants of the matrix are the coefficients of the characteristic equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

which in terms of the values of $\sigma_1, \sigma_2, \sigma_3$ are.

$$\boldsymbol{I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 4}$$
$$\boldsymbol{I_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 = 1}$$
$$\boldsymbol{I_3 = \sigma_1\sigma_2\sigma_3 = -6}$$

The deviatoric stress tensor is defined as,

$$\tau'_{ij} = \tau_{ij} - \sigma_0\delta_{ij}$$

To calculate its eigenvalues we proceed as we did before,

$$\begin{vmatrix} \frac{2}{3} - s & -1 & 1 \\ -1 & -\frac{4}{3} - s & 1 \\ 1 & 1 & \frac{2}{3} - s \end{vmatrix} = 0 \Rightarrow s^3 - \frac{13}{3}s + \frac{38}{27} = 0$$

Solving the cubic equation we obtain the eigenvalues : $s_1 = -2.22$, $s_2 = 0.33$, $s_3 = 1.89$.

Comparing with the characteristic equation

$$\boldsymbol{s^3 - J_1 s^2 + J_2 s - J_3 = 0}$$

The invariants are

$$J_1 = 0$$

$$J_2 = -\frac{13}{3}$$

$$J_3 = -\frac{38}{27}$$

P4.5. Given a stress tensor

$$\begin{pmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3 \\ 0 & 2x_3 & 0 \end{pmatrix}$$

determine the stress vector T at point (2, 1, $\sqrt{3}$) through a plane tangential to the cylindrical surface $x_2^2 + x_3^2 = 4$.

Solution

First we calculate the value of the stress tensor at the given point

$$\tau_{ij}(2,1,\sqrt{3}) = \begin{pmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{pmatrix}$$

A unit vector normal to the surface $f = x_2^2 + x_3^2 - 4 = 0$ at the given point is

$$\upsilon_i = \frac{gradf}{|gradf|} = \left(\frac{\frac{\partial f}{\partial x_1}}{\left|\frac{\partial f}{\partial x_i}\right|}, \frac{\frac{\partial f}{\partial x_2}}{\left|\frac{\partial f}{\partial x_i}\right|}, \frac{\frac{\partial f}{\partial x_3}}{\left|\frac{\partial f}{\partial x_i}\right|} \right) = \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2} \right)$$

Then, the stress vector acting at the point through that surface is given by,

$$T_i^{\nu} = \tau_{ij}\upsilon_j = \begin{pmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \left(\frac{5}{2}, \quad 3, \quad \sqrt{3} \right)$$

P4.6. Given the stress tensor

$$\begin{pmatrix} 1/4 & -5/4 & (1/2)^{3/2} \\ -5/4 & 1/4 & -(1/2)^{3/2} \\ (1/2)^{3/2} & -(1/2)^{3/2} & 3/2 \end{pmatrix}$$

(a) Determine the principal stresses (eigenvalues);

(b) Determine the angles with the coordinate axes of the eigenvector corresponding to the greatest principal stress.

Solution

As in problem 4.2 to find the principal stresses we calculate the eigenvalues of the stress matrix.

The resulting cubic equation is :

$\sigma^3 - 2\sigma^2 - \sigma + 2 = 0$

Its roots are

$$\sigma_1 = 2$$
$$\sigma_2 = 1$$
$$\sigma_3 = -1$$

To find the eigenvectors corresponding to the greater eigenvalue $\sigma_1 = 2$ we solve the equation

$$(\tau_{ij} - 2\delta_{ij})\nu_i = 0$$

and the condition $\upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2 = 1$

The result is

$$\upsilon_1 = -\upsilon_2 = \frac{1}{2}$$
$$\upsilon_3 = \frac{1}{\sqrt{2}}$$

From these values we obtain ϑ, the angle with the vertical axis (x_3) and φ the angle which forms its projection on the horizontal plane with x_1

$$\upsilon_1 = \sin\vartheta\cos\varphi = \frac{1}{2}$$

$$\upsilon_2 = \sin\vartheta\sin\varphi = -\frac{1}{2} \Rightarrow \varphi = 315^\circ, \qquad \vartheta = 45^\circ$$

$$\upsilon_3 = \cos\vartheta = \frac{1}{\sqrt{2}}$$

P4.9. Derive for the stress–strain relation for an isotropic medium that

$$e_{11} = \frac{(\lambda+\mu)\tau_{11}}{\mu(3\lambda+2\mu)} - \frac{\lambda(\tau_{22}+\tau_{33})}{2\mu(3\lambda+2\mu)}$$

Solution.

We use the equation

$$\tau_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} \qquad (4.8.1)$$

$$\tau_{11} = \lambda\theta + 2\mu e_{11} \qquad (4.8.2)$$

$$\tau_{22} = \lambda\theta + 2\mu e_{22} \qquad (4.8.3)$$

$$\tau_{33} = \lambda\theta + 2\mu e_{33} \qquad (4.8.4)$$

Where $\theta = e_{11} + e_{22} + e_{33}$

From equations (4.8.3) and (4.8.4) we can write

$$e_{22} - e_{33} = \frac{\tau_{22} - \tau_{33}}{2\mu} \qquad (4.8.5)$$

We can write eq. (4.8.2) as

$$\tau_{11} = \lambda(e_{22} + e_{33}) + (\lambda + 2\mu)e_{11} \qquad (4.8.6)$$

If we divide (4.8.6) by e_{11} $\Rightarrow$ $(e_{22} + e_{33}) = \frac{-e_{11}(\lambda+2\mu)+\tau_{11}}{\lambda}$ (4.8.7)

From (4.8.3) and (4.8.4) and using (4.8.7) we obtain

$$\tau_{22} + \tau_{33} = 2\lambda\theta + 2\mu\left(\frac{\tau_{11} - e_{11}(\lambda + 2\mu)}{\lambda}\right)$$

$$\lambda\theta = \frac{1}{2}(\tau_{22} + \tau_{33}) - \mu\left(\frac{\tau_{11} - e_{11}(\lambda + 2\mu)}{\lambda}\right)$$

(4.8.8)

Substituting (4.8.8) in (4.8.2) we obtain

$$e_{11} = \frac{(\lambda+\mu)\tau_{11}}{\mu(3\lambda+2\mu)} - \frac{\lambda(\tau_{22}+\tau_{33})}{2\mu(3\lambda+2\mu)}$$