Introduction to Chemical Engineering Fluid Mechanics 1st edition Deen Solutions Manual

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Solution: Chapter 2

2.1 Cavitation

(a) Given:

$$\begin{array}{ll} Q = 3.93 \times 10^{-3}\ \mathrm{m^3/s} & h_2 - h_1 = 3\ \mathrm{m} \\ D = 5\ \mathrm{cm} & \rho = 1.0 \times 10^3\ \mathrm{kg/m^3} \\ L = 50\ \mathrm{m} & \nu = 1.0 \times 10^{-6}\ \mathrm{m^2/s} \end{array}$$

Find: $|\Delta\mathcal{P}|$ from hose inlet to pump intake.

Calculate Re and f:

$$U = \frac{4Q}{\pi D^2} = \frac{4(3.93 \times 10^{-3})}{\pi(5 \times 10^{-2})^2} = 2.0\ \mathrm{m/s}$$

$$\mathrm{Re} = \frac{UD}{\nu} = \frac{(2)(0.05)}{1 \times 10^{-6}} = 1.0 \times 10^5$$

Flow is turbulent and Blasius eq. [Eq. (2.2-9)] is applicable.

$$f = 0.0791\ \mathrm{Re}^{-1/4} = 0.0791(1 \times 10^5)^{-1/4} = 4.45 \times 10^{-3}$$

Find $|\Delta\mathcal{P}|$ from Eq. (2.3–6):

$$\begin{aligned} |\Delta\mathcal{P}| &= \frac{2\rho U^2 L}{D} f = \frac{2(1000)(2)^2 50}{0.05}(4.45 \times 10^{-3}) \\ &= 3.56 \times 10^4\ \mathrm{Pa} \end{aligned}$$

(b) Find P_2 (pump intake) for $P_1 = 1.01 \times 10^5$ Pa (nearly empty pool).

$$\begin{aligned} \Delta\mathcal{P} = -|\Delta\mathcal{P}| &= P_2 - P_1 + \rho g(h_2 - h_1) \\ P_2 &= P_1 - |\Delta\mathcal{P}| - \rho g(h_2 - h_1) \\ &= 1.01 \times 10^5 - 3.56 \times 10^4 - (1000)(9.81)(3) \\ &= 3.60 \times 10^4\ \mathrm{Pa} \end{aligned}$$

Although below atmospheric, $P_2 > P_V = 2.34 \times 10^3$ Pa, so cavitation will be absent.

(c) Maximum Q without cavitation.

P_1 is lowest when the pool is nearly empty, so P_2 is also lowest then. $Q = Q_{\max}$ when $P_2 = P_V$ and $P_1 = 1.01 \times 10^5$ Pa.

Find $|\Delta\mathscr{P}|$ at $Q = Q_{\max}$:

$$
\begin{aligned}
|\Delta\mathscr{P}| &= P_1 - P_2 - \rho g(h_2 - h_1)\\
&= 1.01 \times 10^5 - 2.34 \times 10^3 - (1000)(9.81)(3)\\
&= 6.92 \times 10^4 \text{ Pa}
\end{aligned}
$$

Re is now unknown, but $\mathrm{Re}\sqrt{f}$ can be found and the Prandtl–Kármán eq. used [Eq. (2.2-7)].

$$
f = \frac{D}{2\rho U^2}\frac{|\Delta\mathscr{P}|}{L}, \ \mathrm{Re} = \frac{UD}{\nu}
$$

$$
\begin{aligned}
\mathrm{Re}\sqrt{f} &= \frac{D}{\nu}\left(\frac{D|\Delta\mathscr{P}|}{2\rho L}\right)^{1/2}\\
&= \frac{0.05}{1 \times 10^{-6}}\left[\frac{0.05(6.92 \times 10^4)}{2(1000)(50)}\right]^{1/2} = 9.30 \times 10^3
\end{aligned}
$$

$$
f = [4.0 \log(\mathrm{Re}\sqrt{f}) - 0.4]^{-2} = 4.18 \times 10^{-3}
$$

$$
\begin{aligned}
Q_{\max} &= \frac{\pi D^2}{4}U_{\max} = \frac{\pi D^2}{4}\left(\frac{D}{2\rho f}\frac{|\Delta\mathscr{P}|}{L}\right)^{1/2}\\
&= \frac{\pi(0.05)^2}{4}\left[\frac{0.05(6.92 \times 10^4)}{2(1000)(4.18 \times 10^{-3})(50)}\right]^{1/2}\\
&= 5.65 \times 10^{-3} \text{ m}^3/\text{s}
\end{aligned}
$$

2.2 Bottling Honey

Given: Gravity feed with $|\Delta h| = 2$ m.

Tube length $= L = 4$m.

Honey properties as in Table 1.2.

Mass flow rate ≥ 17 oz/min.

Find: Minimum tube diameter D.

Volume flow rate:

$$
17 \text{ oz}\left(\frac{1 \text{ lb}}{16 \text{ oz}}\right)\left(\frac{0.454 \text{ kg}}{\text{lb}}\right) = 0.482 \text{ kg}
$$

$$
w = \text{mass flow rate} = \left(0.482\frac{\text{kg}}{\text{min}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 8.04 \times 10^{-3} \text{ kg/s}
$$

$$
Q = \text{vol. flow rate} = \frac{w}{\rho} = \frac{8.04 \times 10^{-3}}{1.42 \times 10^3} = 5.66 \times 10^{-6} \text{ m}^3/\text{s}
$$

The high viscosity suggests laminar flow. Poiseuille's equation:

$$
Q = \frac{\pi D^4}{128\ \mu}\frac{|\Delta\mathscr{P}|}{L}
$$

$$
D^4 = \frac{128\mu L Q}{\pi|\Delta\mathscr{P}|}, \qquad D = \left(\frac{128\mu L Q}{\pi|\Delta\mathscr{P}|}\right)^{1/4}
$$

$$
|\Delta\mathscr{P}| = |\Delta P| + \rho g|\Delta h| = \rho g|\Delta h| \quad \text{(atm } P \text{ at both ends)}
$$

$$D = \left(\frac{128\nu LQ}{\pi g|\Delta h|}\right)^{1/4} = \left[\frac{128(1.34\times10^{-2})(4)(5.66\times10^{-6})}{\pi(9.81)(2)}\right]^{1/4}$$
$$= 2.82\times10^{-2}\ \text{m} = 2.82\ \text{cm}.$$

This is a very reasonable diameter, indicating that the bottling process is feasible.

Check Re to confirm that the flow is really laminar:

$$U = \frac{4Q}{\pi D^2} = \frac{4}{\pi}\frac{(5.66\times10^{-6})}{(2.82\times10^{-2})^2} = 9.08\times10^{-3}\ \text{m/s}$$
$$\text{Re} = \frac{UD}{\nu} = \frac{(9.08\times10^{-3})(2.82\times10^{-2})}{1.34\times10^{-2}} = 1.91\times10^{-2}$$

This is much smaller than 2100, which confirms the applicability of Poiseuille's equation.

2.3 Filling a Boiler

Given: $V = 7.57\ \text{m}^3$ (volume to be transferred)

Copper tubing with $D = 2.60$ cm and L = 15 m.

$\Delta h = 3$ m

$|\Delta P| = 4.0\times10^5$ Pa

(a) Filling time at 20°C ($\rho = 998\ \text{kg/m}^3$, $\nu = 1.00\times10^{-6}\ \text{m}^2/\text{s}$)

Assume that the flow is turbulent and might be affected by roughness. Although Re is unknown, $\text{Re}\sqrt{f}$ can be found from what is given, making the Colebrook – White eq. [Eq. (2.5.2)] a good choice.

$$f = \frac{D}{2\rho U^2}\frac{|\Delta\mathcal{P}|}{\text{L}},\ \text{Re} = \frac{UD}{\nu}$$
$$\text{Re}\sqrt{f} = \frac{D}{\nu}\left(\frac{D|\Delta\mathcal{P}|}{2\rho L}\right)^{1/2}$$
$$\Delta\mathcal{P} = \Delta P + \rho g\Delta h = -|\Delta P| + \rho g\Delta h.$$
$$|\Delta\mathcal{P}| = -\Delta\mathcal{P} = |\Delta P| - \rho g\Delta h$$
$$= 4.0\times10^5 - (998)(9.81)(3) = 3.71\times10^5\ \text{Pa}$$
$$\text{Re}\sqrt{f} = \frac{0.026}{1.00\times10^{-6}}\left[\frac{0.026(3.71\times10^5)}{2(998)(15)}\right]^{1/2} = 1.48\times10^4$$
$$\frac{k}{3.7D} = \frac{1.5\times10^{-6}}{3.7(0.026)} = 1.56\times10^{-5}\ (k \text{ for "drawn metals", Table 2.2})$$
$$f = \left[4\log\left(\frac{1.26}{\text{Re}\sqrt{f}} + \frac{k}{3.7D}\right)\right]^{-2}$$
$$= \left[4\log\left(\frac{1.26}{1.48\times10^4} + 1.56\times10^{-5}\right)\right]^{-2}$$
$$= 3.91\times10^{-3}$$
$$U = \left(\frac{D}{2\rho f}\frac{|\Delta\mathcal{P}|}{L}\right)^{1/2} = \left[\frac{0.026(3.71\times10^5)}{2(998)(3.91\times10^{-3})(15)}\right]^{1/2} = 9.07\ \frac{\text{m}}{\text{s}}$$

$$Q = \frac{\pi D^2}{4} U = \frac{\pi (0.026)^2}{4}(9.07) = 4.82 \times 10^{-3} \frac{\text{m}^3}{\text{s}}$$

$$t = \frac{V}{Q} = \frac{7.57}{4.82 \times 10^{-3}} = 1.57 \times 10^3 \text{ s} \left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 26.2 \text{ min}$$

Check Re to confirm turbulent flow:

$$\text{Re} = \frac{UD}{\nu} = \frac{9.07(0.026)}{1.00 \times 10^{-6}} = 2.36 \times 10^5 > 2100.$$

(b) Filling time at 52°C ($\rho = 987$ kg/m^3, $\nu = 5.29 \times 10^{-4}/987 = 0.536 \times 10^{-6}$ m^2/s).

$$\text{Re}\sqrt{f} \propto \nu^{-1}\rho^{-1/2}$$

$$\text{Re}\sqrt{f} = 1.48 \times 10^4 \left(\frac{1.00}{0.536}\right)\left(\frac{998}{987}\right)^{1/2} = 2.78 \times 10^4$$

$$\frac{k}{3.7D} = 1.56 \times 10^{-5} \text{ as before}$$

$$f = \left[4 \log\left(\frac{1.26}{2.78 \times 10^4} + 1.56 \times 10^{-5}\right)\right]^{-2} = 3.52 \times 10^{-3}$$

$$U \propto (\rho f)^{-1/2}$$

$$U = 9.07 \left[\frac{998(3.91)}{987(3.52)}\right]^{1/2} = 9.61 \frac{\text{m}}{\text{s}}$$

$$t \propto \frac{1}{U}$$

$$t = 26.2 \left(\frac{9.07}{9.61}\right) = 24.7 \text{ min}$$

Only 1.5 min is saved.

2.4 Syringe Pump

(a) Best syringe size from among 1 ml, 5 ml, and 50 ml.

Total volume to be delivered is

$$V = \left(\frac{100 \text{ µL}}{\text{min}}\right)(30 \text{ min}) = \left(\frac{0.1 \text{ mL}}{\text{min}}\right)(30 \text{ min}) = 3 \text{ mL}.$$

The 1 ml syringe is too small.

Pusher velocity needed for other syringes:

$$U_s = \frac{4Q}{\pi D_s^2}$$

$$U_s(5 \text{ ml}) = \frac{4(0.1 \text{ cm}^3/\text{min})}{\pi (1.21 \text{ cm})^2} = 8.70 \times 10^{-2} \frac{\text{cm}}{\text{min}} = 0.87 \frac{\text{mm}}{\text{min}}$$

$$U_s(50 \text{ ml}) = \frac{4(0.1 \text{ cm}^3/\text{min})}{\pi (2.67 \text{ cm})^2} = 1.79 \times 10^{-2} \frac{\text{cm}}{\text{min}} = 0.18 \frac{\text{mm}}{\text{min}}$$

Either would work, but 5 ml is preferable because its U_s is not as close to the limits of the pump.

(b) $|\Delta P|$ in horizontal tubing

$$\mathrm{Re} = \frac{UD}{\nu} = \left(\frac{4Q}{\pi D^2}\right)\frac{D}{\nu} = \frac{4Q}{\pi D\nu}$$

$$Q = \left(100 \times 10^{-6}\,\frac{L}{\mathrm{min}}\right)\left(10^{-3}\,\frac{\mathrm{m}^3}{L}\right)\left(\frac{1\ \mathrm{min}}{60\mathrm{s}}\right) = 1.67 \times 10^{-9}\,\frac{\mathrm{m}^3}{\mathrm{s}}$$

$$\mathrm{Re} = \frac{4(1.67 \times 10^{-9})}{\pi(8.6 \times 10^{-4})(1 \times 10^{-6})} = 2.47\ \text{(laminar flow)}$$

$$|\Delta P| = |\Delta\mathcal{P}| = \frac{128\ \mu LQ}{\pi D^4}\ \text{(Poiseuille)}$$

$$= \frac{128(1 \times 10^{-3})(0.5)(1.67 \times 10^{-9})}{\pi(8.6 \times 10^{-4})^4} = 62.2\ \mathrm{Pa}$$

As shown, $|\Delta P|$ depends only on Q and the tubing dimensions. The syringe size does not matter.

(c) $|\Delta P|$ for outlet above inlet, $h_2 - h_1 = 10$ cm.

$|\Delta\mathcal{P}|$ (from Poiseuille eq.) will be the same as before.

$$\Delta\mathcal{P} = \Delta P + \rho g \Delta h$$

$$-\Delta P = -\Delta\mathcal{P} + \rho g\Delta h = |\Delta\mathcal{P}| + \rho g \Delta h \quad (\Delta\mathcal{P} < 0)$$

$$\rho g \Delta h = (1 \times 10^3)(9.81)(0.1) = 981\ \mathrm{Pa}$$

$$-\Delta P = 62 + 981 = 1043\ \mathrm{Pa} = |\Delta P|$$

Moving the liquid upward requires a larger $|\Delta P|$. Because P_{outlet} is fixed at atmospheric, P_{inlet} must be larger.

2.5 Flue Gases

Given: D, L, and $|\Delta P|$

Find: w (mass flow rate)

Assume that the flow is turbulent, in which case it might be affected by roughness. Although Re is not known initially, $\mathrm{Re}\sqrt{f}$ can be calculated, making the Colebrook – White equation [Eq. (2.5-2)] a good choice.

$$\frac{1}{\sqrt{f}} = -4\log\left(\frac{1.26}{\mathrm{Re}\sqrt{f}} + \frac{k}{3.7D}\right) \qquad \text{Eq. (2.5-2)}$$

$$\mathrm{Re}\sqrt{f} = \frac{UD}{\nu}\left(\frac{D|\Delta P|}{2\rho U^2 L}\right)^{1/2} = \left(\frac{D^3|\Delta P|}{2\nu^2\rho L}\right)^{1/2}$$

$$\nu = \frac{\mu}{\rho} = \frac{2.6 \times 10^{-5}}{0.63} = 4.13 \times 10^{-5}\,\frac{\mathrm{m}^2}{\mathrm{s}}$$

$$\mathrm{Re}\sqrt{f} = \left[\frac{(6.4 \times 10^{-2})^3(63)}{2(4.13 \times 10^{-5})^2(0.63)(6.1)}\right]^{1/2} = 1.12 \times 10^3$$

$$k = 4.6 \times 10^{-5}\ \text{m for steel (Table 2.2)}$$

$$\frac{k}{D} = \frac{4.6 \times 10^{-5}}{6.4 \times 10^{-2}} = 7.19 \times 10^{-4}$$

$$
\begin{aligned}
f &= \left[4\log\left(\frac{1.26}{\mathrm{Re}\sqrt{f}} + \frac{k}{3.7D}\right)\right]^{-2} \\
&= \left[4\log\left(\frac{1.26}{1.12\times 10^3} + \frac{7.19\times 10^{-4}}{3.7}\right)\right]^{-2} \\
&= 7.53\times 10^{-3}
\end{aligned}
$$

From Fig. 2.6, $\mathrm{Re} \cong 10^4$, confirming that the flow is turbulent.

Find U from f:

$$
U = \left(\frac{D}{2\rho f}\frac{|\Delta P|}{L}\right)^{1/2} = \left[\frac{(6.4\times 10^{-2})(63)}{2(0.63)(7.53\times 10^{-3})(6.1)}\right]^{1/2} = 8.35\ \frac{\mathrm{m}}{\mathrm{s}}
$$

Check Re:

$$
\mathrm{Re} = \frac{UD}{\nu} = \frac{(8.35)(6.4\times 10^{-2})}{4.13\times 10^{-5}} = 1.29\times 10^4 \quad \text{(turbulent, as assumed)}
$$

Find w:

$$
w = \rho Q = \frac{\pi D^2}{4}U\rho = \frac{\pi}{4}(6.4\times 10^{-2})^2(8.35)(0.63) = 1.69\times 10^{-2}\ \frac{\mathrm{kg}}{\mathrm{s}}
$$

2.6 Hydraulic Fracturing

Given Q, D, L, Δh, and P_2 (outlet), find P_1 (inlet). Start with Re:

$$
\begin{aligned}
\mathrm{Re} &= \frac{UD}{\nu} = \left(\frac{4Q}{\pi D^2}\right)\frac{D}{\nu} = \frac{4Q}{\pi \nu D} \\
\nu &= \frac{0.467\times 10^{-3}}{0.983\times 10^3} = 4.75\times 10^{-7}\ \frac{\mathrm{m}^2}{\mathrm{s}} \quad \text{for } 60^\circ\text{C water.} \\
\mathrm{Re} &= \frac{4}{\pi}\frac{0.21}{(4.75\times 10^{-7})(0.15)} = 3.75\times 10^6 \quad \text{(turbulent flow)}
\end{aligned}
$$

See if roughness is important:

$$
\begin{aligned}
k &= 4.6\times 10^{-5}\ \text{m for steel (Table 2.2)} \\
\frac{k}{D} &= \frac{4.6\times 10^{-5}}{0.15} = 3.1\times 10^{-4}
\end{aligned}
$$

Inspection of Fig. 2.6 indicates that roughness significantly increases f at this Re and k/D.

For turbulent flow with roughness, Eq. (2.5–3) is convenient for finding f:

$$
\begin{aligned}
f &= \left\{3.6\log\left[\frac{6.9}{\mathrm{Re}} + \left(\frac{k}{3.7D}\right)^{1.11}\right]\right\}^{-2} \\
&= \left\{3.6\log\left[\frac{6.9}{3.75\times 10^6} + \left(\frac{3.1\times 10^{-4}}{3.7}\right)^{1.11}\right]\right\}^{-2} = 3.81\times 10^{-3}
\end{aligned}
$$

Find $|\Delta\mathcal{P}|$ from f:

$$
|\Delta\mathcal{P}| = \frac{2\rho U^2 L f}{D} \qquad U = \frac{4Q}{\pi D^2} = 11.9\ \frac{\mathrm{m}}{\mathrm{s}}
$$

$$= \frac{2(0.983 \times 10^3)(11.9)^2(2.3 \times 10^3)(3.81 \times 10^{-3})}{0.15}$$
$$= 1.62 \times 10^7 \text{ Pa} = 16.2 \text{ MPa}$$

Find P_1 (top of pipe):

$$\begin{aligned}
\Delta\mathcal{P} &= \Delta P + \rho g \Delta h \\
-\Delta P &= P_1 - P_2 = -\Delta\mathcal{P} + \rho g \Delta h = |\Delta\mathcal{P}| - \rho g(\Delta h)\ (\Delta\mathcal{P} < 0, \Delta h < 0) \\
\rho g|\Delta h| &= (0.983 \times 10^3)(9.81)(2.3 \times 10^3) = 2.22 \times 10^7 \text{ Pa} = 22.2 \text{ M Pa} \\
P_1 &= P_2 + |\Delta\mathcal{P}| - \rho g|\Delta h| \\
&= 44 + 16.2 - 22.2 = 38 \text{ M Pa}
\end{aligned}$$

Note that even though the downward flow encounters a large resistance in the pipe, $P_1 < P_2$ due to the large $\rho g|\Delta h|$.

2.7 Drag Reduction

Given:

$$\begin{aligned}
U &= 3.0 \text{ m/s} & \rho &= 850 \text{ kg/m}^3 \\
D &= 1.0 \text{ m} & \nu &= 1.06 \times 10^{-5} \text{ m}^2\text{/s} \\
L &= 100 \text{ km}
\end{aligned}$$

Find: $|\Delta P|(= |\Delta\mathcal{P}|)$ for maximum drag reduction.

In Example 2.3–2 it was found that $\text{Re} = 2.83 \times 10^5$ and roughness was assumed (implicitly) to be negligible. Using k for steel in Table 2.2, $k/D = 4.6 \times 10^{-5}$. Figure 2.6 confirms that roughness effects are indeed small at this Re and k/D.

Approach: Find f iteratively from Eq. (P2.7-1)

Find $|\Delta P|$ from f and the given data.

For Re $= 2.83 \times 10^5$, Fig. P2.7 gives $f \cong 6 \times 10^{-4}$. An improved value from Eq. (P2.7-1) is

$$\begin{aligned}
\text{Re}\sqrt{f} &= (2.83 \times 10^5)(6 \times 10^{-4})^{1/2} = 6.93 \times 10^3 \\
f &= [19 \log(\text{Re}\sqrt{f}) - 32.4]^{-2} = [19 \log(6.93 \times 10^3) - 32.4]^{-2} \\
&= 6.07 \times 10^{-4}
\end{aligned}$$

A second refinement is

$$\begin{aligned}
\text{Re}\sqrt{f} &= (2.83 \times 10^5)(6.07 \times 10^{-4})^{1/2} = 6.97 \times 10^3 \\
f &= [19 \log(6.97 \times 10^3) - 32.4]^{-2} = 6.06 \times 10^{-4}
\end{aligned}$$

which is adequate convergence. Then

$$|\Delta P| = \frac{2\rho U^2 L}{D} f \propto f$$

$|\Delta P| = |\Delta P|_0 = 5.59 \times 10^6$ Pa and $f = f_0 = 3.66 \times 10^{-3}$ without additives (Example 2.3-2).

$$
\begin{aligned}
|\Delta P|_{\text{MDR}} &= |\Delta P|_0 \left(\frac{f_{\text{MDR}}}{f_0}\right) = 5.59 \times 10^6 \left(\frac{6.06 \times 10^{-4}}{3.66 \times 10^{-3}}\right) \\
&= 9.26 \times 10^5 \text{ Pa}
\end{aligned}
$$

This is an 83% reduction. Polymeric additives have been used, for example, in the Trans-Alaska Pipeline.

2.8 Economic Pipe Diameter

(a) Pipe diameter D^* that minimizes total cost

$$C = K_1 D^n \mathrm{L} + K_2 Q|\Delta\mathcal{P}|$$

Relate $Q|\Delta\mathcal{P}|$ to D:

$$
\begin{aligned}
|\Delta\mathcal{P}| &= \frac{2\rho U^2 fL}{D} = \frac{2\rho f L}{D}\left(\frac{4Q}{\pi D^2}\right)^2 = \frac{32}{\pi^2}\frac{\rho f L Q^2}{D^5} \\
f &\cong a\mathrm{Re}^{-b} = a\left(\frac{4}{\pi}\frac{Q}{\nu D}\right)^{-b} = a\left(\frac{\pi}{4}\right)^b\left(\frac{\nu D}{Q}\right)^b \\
Q|\Delta\mathcal{P}| &= \frac{32a}{\pi^2}\left(\frac{\pi}{4}\right)^b \frac{\rho L Q^3}{D^5}\frac{\nu^b D^b}{Q^b}
\end{aligned}
$$

Let $c = \dfrac{32a}{\pi^2}\left(\dfrac{\pi}{4}\right)^b$

$$Q|\Delta\mathcal{P}| = c\rho L \nu^b Q^{3-b} D^{b-5}$$

Total cost is now

$$C(D) = K_1 D^n L + K_2 c\rho L\nu^b Q^{3-b} D^{b-5}$$

To minimize C, set $dC/dD = 0$:

$$
\begin{aligned}
\frac{dC}{dD} &= 0 = nK_1 D^{n-1} L + K_2 c\rho L \nu^b Q^{3-b}(b-5)D^{b-6} \\
nK_1 D^{n-1} &= (5-b)(K_2 c\rho\nu^b Q^{3-b} D^{b-6} \\
D^{n-1-b+6} &= D^{5+n-b} = \frac{(5-b)K_2 c}{nK_1}\rho\nu^b Q^{3-b} \\
&= \frac{(5-b)K_2 c}{nK_1}\rho^{1-b}\mu^b Q^{3-b} \\
D^* &= \left[\frac{(5-b)K_2 c}{nK_1}\right]^{1/(5+n-b)} \rho^{(1-b)/(5+n-b)}\mu^{b/(5+n-b)}Q^{(3-b)/(5+n-b)}
\end{aligned}
$$

(b) Dependence of D^* on ρ and μ

Use $n = 1.5$ and $b = 0.16$.

ρ exponent is $\dfrac{1-b}{5+n-b} = \dfrac{0.84}{6.34} = 0.132$

μ exponent is $\dfrac{b}{5+n-b} = \dfrac{0.16}{6.34} = 0.025$

Both are small, implying insensitivity of D^* to property values.

(c) Dependence of U^* on Q

$$D^* \propto Q^{(3-b)/(5+n-b)} = Q^{2.84/6.34} = Q^{0.45}$$

$$U^* \propto \frac{Q}{(D^*)^2} \propto \frac{Q}{Q^{2(3-b)/(5+n-b)}}$$

$$1 - \frac{2(3-b)}{(5+n-b)} = \frac{5+n-b-6+2b}{5+n-b} = \frac{-1+n+b}{5+n-b}$$

$$= \frac{-1+1.5+0.16}{5+1.5-0.16} = 0.10$$

$$U^* \propto Q^{0.10}$$

Whereas D^* is somewhat sensitive to Q, U^* is relatively insensitive.

2.9 Microfluidic Device

(a) Flow-rate ratios.

From flow in = flow out, $Q_1 = Q_5$.

From $L_2 = L_4$, $Q_2 = Q_4$

From $L_3 = L_2/2$, $Q_3 = 2Q_2$.

$$Q_1 = Q_2 + Q_3 + Q_4 = Q_2 + 2Q_2 + Q_2 = 4Q_2$$

$$\frac{Q_2}{Q_1} = \frac{1}{4}, \frac{Q_3}{Q_1} = \frac{1}{2}, \frac{Q_4}{Q_1} = \frac{1}{4}, \frac{Q_5}{Q_1} = 1$$

(b) Calculate Q_1 = overall flow rate for $P_A - P_D = 3000$ Pa, $L_1 = 1$ mm, and square channels with side length 100 μm.

Let r_{ij} = resistance between nodes i and j

r_k = resistance of channel k

There are 3 parallel channels between B and C. With $r_i \propto L_i$,

$$\frac{1}{r_{BC}} = \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{1}{r_2} + \frac{2}{r_2} + \frac{1}{r_2} = \frac{4}{r_2}$$

With $L_2 = 4L_1$, $r_2 = 4r_1$ and

$$\frac{1}{r_{BC}} = \frac{1}{r_1} \quad \text{or} \quad r_{BC} = r_1.$$

With $L_5 = L_1$, $r_5 = r_1$. Adding the series resistances,

$$r_{AD} = r_1 + r_{BC} + r_5 = 3r_1$$

Friction factor for laminar flow:

$$f = \frac{D_H}{2\rho U^2}\frac{|\Delta P|}{L} = \frac{c}{\mathrm{Re}_H} \qquad c = \text{const.}$$

For square channels, $c = 14.23$, $D_H = a =$ side length (Table 2.1 with $a = b$), and $\text{Re}_4 = UD_H/\nu = Ua/\nu$. Thus

$$\frac{a}{2\rho U^2}\frac{|\Delta P|}{L} = \frac{c\nu}{Ua} \quad \text{or} \quad U = \frac{a^2}{2c\mu}\frac{|\Delta P|}{L}$$

Apply to segment 1:

$$|\Delta P|_1 = \frac{|\Delta P|_{AD}}{3} = 1000 \text{ Pa} \quad \text{because } r_1 = r_{AD}/3.$$
$$L_1 = 1 \times 10^{-3} \text{ m}$$
$$a = 1 \times 10^{-4} \text{ m}$$
$$U_1 = \frac{(1 \times 10^{-4})^2}{2(14.23)(1 \times 10^{-3})}\frac{1000}{1 \times 10^{-3}} = 0.351 \frac{\text{m}}{\text{s}}$$
$$Q_1 = U_1 a^2 = (0.351)(1 \times 10^{-4}\text{m})^2 = 3.51 \times 10^{-9} \frac{\text{m}^3}{\text{s}}$$

2.10 Murray's Law

(a) Radius that minimizes total energy expended (E_t) for one vessel.

$$E_t = Q|\Delta\mathcal{P}| + b\pi R^2 L$$
$$|\Delta\mathcal{P}| = \frac{8\mu LQ}{\pi R^4} \quad \text{(Poiseuille)}$$
$$E_t = \frac{8\mu LQ^2}{\pi R^4} + b\pi R^2 L \equiv AR^{-4} + BR^2$$

Minimize E_t with respect to R, for constant $A = 8\mu LQ^2/\pi$ and $B = 6\pi L$:

$$\frac{dE_t}{dR} = -4AR^{-5} + 2BR = 0$$
$$R^6 = \frac{2A}{B} = \frac{16\mu LQ^2}{\pi}\left(\frac{1}{b\pi L}\right) = \frac{16\mu Q^2}{b\pi^2}$$
$$R = \left(\frac{16\mu Q^2}{b\pi^2}\right)^{1/6} \quad \text{(independent of } L\text{)}$$

(b) Obtain Murray's law by applying result from part (a) to each vessel at a bifurcation.

$Q_1 = Q_2 + Q_3$ (vessels labeled as in Fig. P2.10)

$R_i \propto Q_i^{1/3}$ or $Q_i \propto R_i^3$ (optimum found above)

$R_1^3 = R_2^3 + R_3^3$

(c) Extend Murray's law to generations j and k

Let $Q_{ij} =$ flow rate in vessel i at generation j

Same total flow rate at each generation:

$$\sum_{i=1}^{N_j} Q_{ij} = \sum_{i=1}^{N_k} Q_{ik}$$

Energy minimization:

$$Q_{ij} \propto R_{ij}^3$$

Combine:

$$\sum_{i=1}^{N_j} R_{ij}^3 = \sum_{i=1}^{N_k} R_{ik}^3$$

2.11 Open-Channel Flow

(a) Dimensional analysis

Governing quantities with dimensions:

$$\begin{array}{ll}
[U] = \mathrm{LT}^{-1} & [g] = \mathrm{LT}^{-2} \\
[H] = \mathrm{L} & [\rho] = \mathrm{ML}^{-3} \\
[W] = \mathrm{L} & [\mu] = \mathrm{ML}^{-1}\mathrm{T}^{-1} \\
[k] = \mathrm{L} & [S] = \text{dimensionless (slope)}
\end{array}$$

7 quantities − 3 dimensions = 4 dimensionless groups

For N_1 choose U (the unknown) for the numerator and g and H for the denominator:

$$N_1 = \frac{U}{g^a H^b}$$
$$[U] = \mathrm{LT}^{-1} = [g^a H^b] = (\mathrm{LT}^{-2})^a \mathrm{L}^b = \mathrm{L}^{a+b}\mathrm{T}^{-2a}$$

L: $1 = a + b$
T: $-1 = -2a$

$$a = 1/2 = b$$
$$N_1 = \frac{U}{(gH)^{1/2}} (= \mathrm{Fr}^{1/2})$$

Note: The reason for preferring H over W or k is that U may be expected to be independent of W for $W/H \to \infty$ (as in a parallel-plate channel) and independent of k for $k/H \to 0$ (smooth surfaces), whereas H will always affect U. That g is a key quantity in a gravity-driven flow is obvious.

A known group with ρ and μ is Re:

$$N_2 = \frac{UH\rho}{\mu} = \mathrm{Re}$$

The remaining two groups must include W and k. Let

$$N_3 = \frac{W}{H} = \phi \quad \text{aspect ratio}$$
$$N_4 = \frac{k}{H} \quad \text{roughness parameter}$$

Conclusion:

$$\frac{U}{(gH)^{1/2}} = F(S, \mathrm{Re}, \phi, k/H)$$

U will certainly increase with increasing slope and decrease with increasing roughness, but the form of F is largely unknown.

(b) Force balance for length L of channel

Let τ_w be the shear stress averaged over the solid surfaces. Then

shear force $= \tau_w$ (side area + bottom area) $= \tau_w(2H + w)L$

gravitational force = (fluid mass) · (grav, accel. in flow direction)

$$= (\rho LHW)g \sin\theta = \rho LHWgS$$

$$\rho LHWgS = \tau_w(2H + W)L$$

$$\tau_w = \rho gS\left(\frac{HW}{2H + W}\right) = \frac{\rho gHS\phi}{2 + \phi}$$

(c) Force balance using hydraulic diameter

Let $D_H = \dfrac{4A}{C}$ with $A =$ cross-sectional area

$C =$ wetted perimeter (solid only).

For the rectangular channel,

$$D_H = \frac{4HW}{2H + W} = \frac{4H\phi}{2 + \phi}$$

The general expression for τ_w is

$$\rho ALgS = \tau_w CL$$

$$\tau_w = \frac{\rho gSA}{C} = \frac{\rho gSD_H}{4}$$

(d) Friction factor independent of flow rate (f independent of Re)

$$f = \frac{2\tau_w}{\rho U^2} = \frac{2}{\rho U^2}\left(\frac{\rho gSD_H}{4}\right) = \frac{gSD_H}{2U^2}$$

Arrange as in part (a), but with D_H instead of H:

$$\frac{U}{(gD_H)^{1/2}} = \left(\frac{S}{2f}\right)^{1/2}$$

It is seen now that $U \propto S^{1/2}$. Although independent of Re_H and presumably ϕ, in a rough channel f will depend on k/D_H [Eq. (P2.11–2)].