Fundamentals of Differential Equations 9th edition Nagle Solutions Manual

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CHAPTER 2: First Order Differential Equations

EXERCISES 2.2: Separable Equations

2. This equation is separable because we can separate variables by multiplying both sides by dx and dividing by $4y^2 - 3y + 1$.

4. This equation is separable because

$$\frac{dy}{dx} = \frac{ye^{x+y}}{x^2+2} = \left(\frac{e^x}{x^2+2}\right) ye^y = g(x)p(y).$$

6. Writing the equation in the form

$$\frac{ds}{dt} = \frac{s+1}{st} - s^2,$$

we see that the right-hand side cannot be represented in the form $g(t)p(s)$. Therefore, the equation is not separable.

8. To separate variables, we divide the equation by x and multiply by dt. Integrating yields

$$\begin{aligned} &\frac{dx}{x} = 3t^2 dt \qquad \Rightarrow \qquad \ln|x| = t^3 + C_1 \qquad \Rightarrow \qquad |x| = e^{t^3+C_1} = e^{C_1}e^{t^3} \\ &\Rightarrow \qquad |x| = C_2 e^{t^3} \qquad \Rightarrow \qquad x = \pm C_2 e^{t^3} = Ce^{t^3}, \end{aligned}$$

where C_1 is an arbitrary constant and, therefore, $C_2 := e^{C_1}$ is an arbitrary *positive* constant, $C = \pm C_2$ is any *nonzero* constant. Separating variables, we lost a solution $x \equiv 0$, which can be included in the above formula by taking $C = 0$. Thus, $x = Ce^{t^3}$, C – arbitrary constant, is a general solution.

10. Separating variables yields

$$y^2\, dy = \frac{xdx}{\sqrt{1+x}} \qquad \Rightarrow \qquad \int y^2\, dy = \int \frac{xdx}{\sqrt{1+x}}.$$

The second integral can be evaluated using a substitution $u = 1 + x$.

$$\int \frac{xdx}{\sqrt{1+x}} = \int \frac{(u-1)du}{\sqrt{u}} = \int \left(u^{1/2} - u^{-1/2}\right) du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C.$$

Thus,

$$\frac{y^3}{3} = \frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C \qquad \Rightarrow \qquad y = \sqrt[3]{2(1+x)^{3/2} - 6(1+x)^{1/2} + C}\,.$$

12. Separating variables, we obtain

$$\frac{dy}{\sec^2 y} = \frac{dx}{1+x^2}\,.$$

Using the trigonometric identities $\sec y = 1/\cos y$ and $\cos^2 y = (1+\cos 2y)/2$ and integrating, we get

$$\begin{aligned}
&\frac{dy}{\sec^2 y} = \frac{dx}{1+x^2} \qquad \Rightarrow \qquad \frac{(1+\cos 2y)dy}{2} = \frac{dx}{1+x^2}\\
\Rightarrow \qquad & \int \frac{(1+\cos 2y)dy}{2} = \int \frac{dx}{1+x^2}\\
\Rightarrow \qquad & \frac{1}{2}\left(y + \frac{1}{2}\sin 2y\right) = \arctan x + C_1\\
\Rightarrow \qquad & 2y + \sin 2y = 4\arctan x + 4C_1 \qquad \Rightarrow \qquad 2y + \sin 2y = 4\arctan x + C.
\end{aligned}$$

The last equation defines implicit solutions to the given differential equation.

14. Writing the given equation in the form $dx/dt = x^3 + x$, we separate the variables to get

$$\frac{dx}{x^3+x} = \frac{dx}{x\left(x^2+1\right)} = \int \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) x\,dx = dt\,.$$

Thus,

$$\frac{1}{2}\left[\ln\left(x^2\right) - \ln\left(x^2+1\right)\right] = t + C_1 \qquad \Rightarrow \qquad \frac{x^2}{x^2+1} = e^{2C_1}e^{2t} = Ce^{2t}\,,$$

where $C > 0$ is an arbitrary constant. (Note that, separating variables, we lost a solution $x \equiv 0$, which can be included in the above formula by letting $C = 0$.) Solving for x, we get

$$x = \pm\sqrt{\frac{Ce^{2t}}{1-Ce^{2t}}}\,, \quad C \geq 0\,.$$

16. To separate variables, we move the term, containing dx, to the right-hand side of the equation and divide both sides by y. This yields

$$y^{-1}dy = -ye^{\cos x}\sin x\,dx \qquad \Rightarrow \qquad y^{-2}dy = -e^{\cos x}\sin x\,dx.$$

Integrating the last equation, we obtain

$$\int y^{-2}dy = \int \left(-e^{\cos x}\sin x\right)dx \qquad \Rightarrow \qquad -y^{-1} + C = \int e^u du \quad (u = \cos x)$$

$$\Rightarrow \qquad -\frac{1}{y}+C=e^{u}=e^{\cos x} \qquad \Rightarrow \qquad y=\frac{1}{C-e^{\cos x}}\,,$$

where C is an arbitrary constant.

18. First we find a general solution to the equation. Separating variables and integrating, we get

$$\begin{aligned}
&\frac{dy}{dx}=x^3(1-y) \qquad \Rightarrow \qquad \frac{dy}{1-y}=x^3dx\\
\Rightarrow \qquad &\int\frac{dy}{1-y}=\int x^3dx \qquad \Rightarrow \qquad -\ln|1-y|+C_1=\frac{x^4}{4}\\
\Rightarrow \qquad &|1-y|=\exp\left(C_1-\frac{x^4}{4}\right)=Ce^{-x^4/4}.
\end{aligned}$$

We use the initial condition, $y(0)=3$, to find C. Thus, substitution $y=3$ and $x=0$ into the last equation yields

$$|1-3|=Ce^{-0^4/4} \qquad \Rightarrow \qquad 2=C.$$

Therefore, $|1-y|=2e^{-x^4/4}$. Finally, since $1-y(0)=1-3<0$, on an interval containing $x=0$ one has $1-y(x)<0$ so that $|1-y(x)|=y(x)-1$. Thus, the answer

$$y-1=2e^{-x^4/4} \qquad \text{or} \qquad y=2e^{-x^4/4}+1.$$

20. Separating variables yields

$$(y+1)dy=\frac{4x^2-x-2}{x^2(x+1)}\,dx\,. \tag{2.1}$$

For integrating the right-hand side, we use partial fractions.

$$\frac{4x^2-x-2}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

Solving for A, B, and C, we get $A=1$, $B=-2$, and $C=3$. Thus, integrating (2.1), we obtain

$$\frac{1}{2}\,(y+1)^2=\ln x+\frac{2}{x}+3\ln(x+1)+C.$$

With $y(1)=1$, $C=-3\ln 2$ so that the answer is

$$\frac{1}{2}\,(y+1)^2=\ln x+\frac{2}{x}+3\ln(x+1)-3\ln 2$$

or, solving for y,

$$y=-1+\sqrt{\ln\left(x^2\right)+\frac{4}{x}+\ln\left[(x+1)^6\right]-6\ln 2}\,.$$

22. Writing $2ydy = -x^2dx$ and integrating, we find

$$y^2 = -\frac{x^3}{3} + C.$$

With $y(0) = 2$,

$$(2)^2 = -\frac{(0)^3}{3} + C \qquad \Rightarrow \qquad C = 4,$$

and so

$$y^2 = -\frac{x^3}{3} + 4 \qquad \Rightarrow \qquad y = \sqrt{-\frac{x^3}{3} + 4}.$$

We note that, taking the square root, we chose the positive sign because $y(0) > 0$.

24. For a general solution, we separate variables and integrate.

$$\int e^{2y}dy = \int 8x^3dx \qquad \Rightarrow \qquad \frac{e^{2y}}{2} = 2x^4 + C_1 \qquad \Rightarrow \qquad e^{2y} = 4x^4 + C.$$

We substitute now the initial condition, $y(1) = 0$, and obtain

$$1 = 4 + C \qquad \Rightarrow \qquad C = -3.$$

Hence, the answer is given by

$$e^{2y} = 4x^4 - 3 \qquad \Rightarrow \qquad y = \frac{1}{2}\ln\left(4x^4 - 3\right).$$

26. We separate variables and obtain

$$\int \frac{dy}{\sqrt{y}} = -\int \frac{dx}{1+x} \qquad \Rightarrow \qquad 2\sqrt{y} = -\ln|1+x| + C = -\ln(1+x) + C,$$

because at initial point, $x = 0$, $1 + x > 0$. Using the fact that $y(0) = 1$, we find C.

$$2 = 0 + C \qquad \Rightarrow \qquad C = 2,$$

and so $y = \left[2 - \ln(1+x)\right]^2/4$ is the answer.

28. We have

$$\frac{dy}{dt} = 2y(1-t) \qquad \Rightarrow \qquad \frac{dy}{y} = 2(1-t)dt \qquad \Rightarrow \qquad \ln|y| = -(t-1)^2 + C$$
$$\Rightarrow \qquad y = \pm e^C e^{-(t-1)^2} = C_1 e^{-(t-1)^2},$$

where $C_1 \neq 0$ is any constant. Separating variables, we lost the solution $y \equiv 0$, which can be included into the above formula by letting $C_1 = 0$. So, a general solution to the given equation is

$$y = C_1 e^{-(t-1)^2}, \quad C_1 \text{ is an arbitrary constant.}$$

Substituting $t = 0$ and $y = 3$, we find

$$3 = C_1 e^{-1} \quad \Rightarrow \quad C_1 = 3e \quad \Rightarrow \quad y = 3e^{1-(t-1)^2} = 3e^{2t-t^2}.$$

The graph of this function is given in Fig. **2–A** at the end of the chapter.

Since $y(t) > 0$ for any t, from the given equation we have $y'(t) > 0$ for $t < 1$ and $y'(t) < 0$ for $t > 1$. Thus $t = 1$ is the point of absolute maximum with $y_{\max} = y(1) = 3e$.

30. **(a)** Dividing by $(y+1)^{2/3}$, multiplying by dx, and integrating, we obtain

$$\int \frac{dy}{(y+1)^{2/3}} = \int (x-3)dx \quad \Rightarrow \quad 3(y+1)^{1/3} = \frac{x^2}{2} - 3x + C$$

$$\Rightarrow \quad y = -1 + \left(\frac{x^2}{6} - x + C_1\right)^3.$$

(b) Substituting $y \equiv -1$ into the original equation yields

$$\frac{d(-1)}{dx} = (x-3)(-1+1)^{2/3} = 0,$$

and so the equation is satisfied.

(c) For $y \equiv -1$ for the solution in part (a), we must have

$$\left(\frac{x^2}{6} - x + C_1\right)^3 \equiv 0 \quad \Leftrightarrow \quad \frac{x^2}{6} - x + C_1 \equiv 0,$$

which is impossible since a quadratic polynomial has at most two zeros.

32. **(a)** The direction field of the given differential equation is shown in Fig. **2–B** at the end of the chapter. Using this picture we predict that $\lim_{x\to\infty} \phi(x) = 1$.

(b) In notation of Section 1.4, we have $x_0 = 0$, $y_0 = 1.5$, $f(x,y) = y^2 - 3y + 2$, and $h = 0.1$. With this step size, we need $(1-0)/0.1 = 10$ steps to approximate $\phi(1)$. The results of computation are given in Table **2–A** at the end of the chapter. From this table we conclude that $\phi(1) \approx 1.26660$.

(c) Separating variables and integrating, we obtain

$$\frac{dy}{y^2-3y+2}=dx \quad \Rightarrow \quad \int \frac{dy}{y^2-3y+2}=\int dx \quad \Rightarrow \quad \ln\left|\frac{y-2}{y-1}\right|=x+C\,,$$

where we have used a partial fractions decomposition

$$\frac{1}{y^2-3y+2}=\frac{1}{y-2}-\frac{1}{y-1}$$

to evaluate the integral. The initial condition, $y(0)=1.5$, implies that $C=0$, and so

$$\ln\left|\frac{y-2}{y-1}\right|=x \qquad \Rightarrow \qquad \left|\frac{y-2}{y-1}\right|=e^x \qquad \Rightarrow \qquad \frac{y-2}{y-1}=-e^x\,.$$

(We have chosen the negative sign because of the initial condition.) Solving for y yields

$$y=\phi(x)=\frac{e^x+2}{e^x+1}$$

The graph of this solution is shown in Fig. **2–B** at the end of the chapter.

(d) We find

$$\phi(1)=\frac{e+2}{e+1}\approx 1.26894\,.$$

Thus, the approximate value $\phi(1)\approx 1.26660$ found in part (b) differs from the actual value by less than 0.003.

(e) We find the limit of $\phi(x)$ at infinity writing

$$\lim_{x\to\infty}\frac{e^x+2}{e^x+1}=\lim_{x\to\infty}\left(1+\frac{1}{e^x+1}\right)=1\,,$$

which confirms our guess in part (a).

34. (a) Separating variables and integrating, we get

$$\frac{dT}{T-M}=-kdt \quad \Rightarrow \quad \int \frac{dT}{T-M}=-\int kdt \quad \Rightarrow \quad \ln|T-M|=-kt+C_1$$
$$\Rightarrow \quad |T-M|=e^{C_1}e^{-kt} \quad \Rightarrow \quad T-M=\pm e^{C_1}e^{-kt}=Ce^{-kt},$$

where C is any nonzero constant. We can include the lost solution $T\equiv M$ into this formula by letting $C=0$. Thus, a general solution to the equation is

$$T=M+Ce^{-kt}.$$

(b) Given that $M = 70°$, $T(0) = 100°$, $T(6) = 80°$, we form a system to determine C and k.

$$\begin{cases} 100 &= 70 + C \\ 80 &= 70 + Ce^{-6k} \end{cases} \quad \Rightarrow \quad \begin{cases} C &= 30 \\ k &= -(1/6)\ln[(80-70)/30] = (1/6)\ln 3. \end{cases}$$

Therefore,

$$T = 70 + 30e^{-(t\ln 3)/6} = 70 + (30)3^{-t/6},$$

and after 20 min the reading is

$$T(20) = 70 + (30)3^{-20/6} \approx 70.77°.$$

36. A general solution to the cooling equation found in Problem 34, that is, $T = M + Ce^{-kt}$. Since $T(0) = 100°$, $T(5) = 80°$, and $T(10) = 65°$, we determine M, C, and k from the system

$$\begin{cases} M + C &= 100 \\ M + Ce^{-5k} &= 80 \\ M + Ce^{-10k} &= 65 \end{cases} \quad \Rightarrow \quad \begin{cases} C(1 - e^{-5k}) &= 20 \\ Ce^{-5k}(1 - e^{-5k}) &= 15 \end{cases} \quad \Rightarrow \quad e^{-5k} = 3/4.$$

To find M, we can now use the first two equations in the above system.

$$\begin{cases} M + C &= 100 \\ M + (3/4)C &= 80 \end{cases} \quad \Rightarrow \quad M = 20.$$

38. With $m = 100$, $g = 9.8$, and $k = 5$, the equation becomes

$$100\,\frac{dv}{dt} = 100(9.8) - 5v \quad \Rightarrow \quad 20\,\frac{dv}{dt} = 196 - v.$$

Separating variables and integrating yields

$$\int \frac{dv}{v - 196} = -\frac{1}{20}\int dt \quad \Rightarrow \quad \ln|v - 196| = -\frac{t}{20} + C_1 \quad \Rightarrow \quad v = 196 + Ce^{-t/20},$$

where C is an arbitrary nonzero constant. With $C = 0$, this formula also gives the (lost) constant solution $v(t) \equiv 196$. From the initial condition, $v(0) = 10$, we find C.

$$196 + C = 10 \quad \Rightarrow \quad C = -186 \quad \Rightarrow \quad v(t) = 196 - 186e^{-t/20}.$$

The terminal velocity of the object can be found by letting $t \to \infty$.

$$v_\infty = \lim_{t\to\infty}\left(196 - 186e^{-t/20}\right) = 196\ \text{(m/sec)}.$$

40. **(a)** Substituting $\rho = Mp/RT$ into $dp/dz = -\rho g$ yields

$$\frac{dp}{dz} = -\left(\frac{Mp}{RT}\right) g = -\frac{Mg}{RT}\, p.$$

Separating variables and integrating, we find

$$\frac{dp}{p} = -\frac{Mg}{RT}\, dz \tag{2.2}$$

$$\Rightarrow \qquad \ln p = -\frac{Mg}{RT}\, z + C_1 \qquad \Rightarrow \qquad p(z) = Ce^{-(Mg/RT)z}. \tag{2.3}$$

For $z = z_0$,

$$p\left(z_0\right) = Ce^{-(Mg/RT)z_0} \qquad \Rightarrow \qquad C = p\left(z_0\right) e^{(Mg/RT)z_0}\,.$$

Thus,

$$p(z) = p\left(z_0\right) e^{(Mg/RT)(z_0 - z)} = p\left(z_0\right) e^{-Mg(z-z_0)/(RT)}\,.$$

(b) If $T = T(z)$ varies, then integrating (2.2) from z_0 to z we obtain

$$\ln \frac{p(z)}{p\left(z_0\right)} = -\frac{Mg}{R} \int\limits_{z_0}^{z} \frac{d\zeta}{T(\zeta)}$$

$$\Rightarrow \qquad p(z) = p\left(z_0\right) \exp\left[-\frac{Mg}{R} \int\limits_{z_0}^{z} \frac{d\zeta}{T(\zeta)}\right].$$

(c) With the given formula for $T(z)$, the integral in part (b) gives

$$\int\limits_{z_0}^{z} \frac{d\zeta}{288 - 0.0065\left(\zeta - z_0\right)} = -\frac{1}{0.0065} \ln \frac{288 - 0.0065\left(z - z_0\right)}{288}\,.$$

Therefore,

$$p(z) = p\left(z_0\right) \left[\frac{288 - 0.0065\left(z - z_0\right)}{288}\right]^{Mg/(0.0065R)}\,.$$

Substituting the given data into this equation and computing, we get the height

$$z - z_0 \approx 168 \text{ (m)}\,.$$

EXERCISES 2.3: Linear Equations

2. Neither.

4. Linear.

6. Linear.

8. Writing the equation in standard form,

$$\frac{dy}{dx} - \frac{y}{x} = 2x + 1,$$

we see that

$$P(x) = -\frac{1}{x} \qquad \Rightarrow \qquad \mu(x) = \exp\left[\int \left(-\frac{1}{x}\right) dx\right] = \exp\left(-\ln x\right) = \frac{1}{x}\,.$$

Multiplying the given equation by $\mu(x)$, we get

$$\frac{d}{dx}\left(\frac{y}{x}\right) = 2 + \frac{1}{x} \qquad \Rightarrow \qquad y = x\int \left(2 + \frac{1}{x}\right) dx = x\left(2x + \ln|x| + C\right).$$

10. From the standard form of the given equation,

$$\frac{dy}{dx} + \frac{2}{x}y = x^{-4},$$

we find that

$$\mu(x) = \exp\left[\int (2/x)dx\right] = \exp\left(2\ln|x|\right) = x^2$$

$$\Rightarrow \quad \frac{d}{dx}\left(x^2 y\right) = x^{-2} \qquad \Rightarrow \qquad y = x^{-2}\int x^{-2}dx = x^{-2}\left(-x^{-1} + C\right) = \frac{Cx - 1}{x^3}\,.$$

12. Here, $P(x) = 4$, $Q(x) = x^2 e^{-4x}$. So, $\mu(x) = e^{4x}$ and

$$\frac{d}{dx}\left(e^{4x}y\right) = x^2 \qquad \Rightarrow \qquad y = e^{-4x}\int x^2 dx = e^{-4x}\left(\frac{x^3}{3} + C\right).$$

14. In standard form, we have

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^2} - 3x\,.$$

Therefore, $P(x) = 3/x$,

$$\mu(x) = \exp\left(\int \frac{3\,dx}{x}\right) = x^3$$

and

$$y = \frac{1}{x^3}\int \left(x\sin x - 3x^4\right) dx = \frac{1}{x^3}\left(\sin x - x\cos x - \frac{3}{5}x^5 + C\right).$$

(Integration by parts was used to integrate the first term.)

16. Here, $|x| < 1$ and

$$P(x) = -\frac{x^2}{1-x^2} = 1 - \frac{1}{1-x^2}\,, \quad Q(x) = \frac{1+x}{\sqrt{1-x^2}}\,.$$

Therefore,

$$\mu(x) = \exp\left[\int \left(1 - \frac{1}{1-x^2}\right) dx\right] = e^x \sqrt{\frac{1-x}{1+x}}$$

so that $\mu(x)Q(x) = e^x$ and

$$y = e^{-x}\sqrt{\frac{1+x}{1-x}}\,(e^x + C) = \sqrt{\frac{1+x}{1-x}}\,(1 + Ce^{-x})\,.$$

18. Since $\mu(x) = \exp\left(\int 4dx\right) = e^{4x}$, we have

$$\begin{aligned}
&\frac{d}{dx}\left(e^{4x}y\right) = e^{4x}e^{-x} = e^{3x} \\
\Rightarrow \qquad & y = e^{-4x}\int e^{3x}dx = \frac{e^{-x}}{3} + Ce^{-4x}.
\end{aligned}$$

Substituting the initial condition, $y = 4/3$ at $x = 0$, yields

$$\frac{4}{3} = \frac{1}{3} + C \qquad \Rightarrow \qquad C = 1,$$

and so $y = e^{-x}/3 + e^{-4x}$ is the solution to the given initial value problem.

20. We have

$$\begin{aligned}
&\mu(x) = \exp\left(\int \frac{3\,dx}{x}\right) = \exp\left(3\ln x\right) = x^3 \\
\Rightarrow \qquad & x^3 y = \int x^3\,(3x-2)\,dx = \frac{3x^5}{5} - \frac{x^4}{2} + C \\
\Rightarrow \qquad & y = \frac{3x^2}{5} - \frac{x}{2} + Cx^{-3}.
\end{aligned}$$

With $y(1) = 1$,

$$1 = y(1) = \frac{3}{5} - \frac{1}{2} + C \qquad \Rightarrow \qquad C = \frac{9}{10} \qquad \Rightarrow \qquad y = \frac{3x^2}{5} - \frac{x}{2} + \frac{9}{10x^3}\,.$$

22. From the standard form of this equation,

$$\frac{dy}{dx} + y\cot x = x,$$

we find

$$\mu(x) = \exp\left(\int \cot x\,dx\right) = \exp\left(\ln \sin x\right) = \sin x.$$

(Alternatively, one can notice that the left-hand side of the original equation is the derivative of the product $y \sin x$.) So, using integration by parts, we obtain

$$\begin{aligned} y \sin x &= \int x \sin x \, dx = -x \cos x + \sin x + C \\ \Rightarrow \qquad y &= -x \cot x + 1 + C \csc x. \end{aligned}$$

We find C using the initial condition $y(\pi/2) = 2$:

$$2 = -\frac{\pi}{2} \cot \frac{\pi}{2} + 1 + C \csc\left(\frac{\pi}{2}\right) = 1 + C \qquad \Rightarrow \qquad C = 1,$$

and the solution is given by

$$y = -x \cot x + 1 + \csc x.$$

24. **(a)** The equation (12) of the text becomes

$$\frac{dy}{dt} + 20y = 50e^{-10t} \qquad \Rightarrow \qquad \mu(t) = e^{20t}$$
$$\Rightarrow \qquad y = e^{-20t} \int e^{20t} \left(50e^{-10t}\right) dt = e^{-20t} \int 50e^{10t} dt = 5e^{-10t} + Ce^{-20t}.$$

Since $y(0) = 40$, we have

$$40 = 5 + C \qquad \Rightarrow \qquad C = 35 \qquad \Rightarrow \qquad y = 5e^{-10t} + 35e^{-20t}.$$

The term $5e^{-10t}$ will eventually dominate.

(b) This time, the equation (12) has the form

$$\frac{dy}{dt} + 10y = 50e^{-10t} \qquad \Rightarrow \qquad \mu(t) = e^{10t}$$
$$\Rightarrow \qquad y = e^{-10t} \int e^{10t} \left(50e^{-10t}\right) dt = e^{-10t}(50t + C).$$

Substituting the initial condition yields

$$40 = y(0) = C \qquad \Rightarrow \qquad y = e^{-10t}(50t + 40).$$

26. Here

$$P(x) = \frac{\sin x \cos x}{1 + \sin^2 x}$$
$$\Rightarrow \qquad \mu(x) = \exp\left(\int \frac{\sin x \cos x \, dx}{1 + \sin^2 x}\right) = \exp\left[\frac{1}{2} \ln\left(1 + \sin^2 x\right)\right] = \sqrt{1 + \sin^2 x}.$$

Thus,

$$y\sqrt{1+\sin^2 x} = \int_0^x \sqrt{1+\sin^2 t}\,dt \quad \Rightarrow \quad y = (1+\sin^2 x)^{-1/2}\int_0^x (1+\sin^2 t)^{1/2}dt$$

and

$$y(1) = (1+\sin^2 1)^{-1/2}\int_0^1 (1+\sin^2 t)^{1/2}dt.$$

We now use the Simpson's Rule to find that $y(1) \approx 0.860$.

28. (a) Substituting $y = e^{-x}$ into the equation (16) yields

$$\frac{d(e^{-x})}{dx} + e^{-x} = -e^{-x} + e^{-x} = 0.$$

So, $y = e^{-x}$ is a solution to (16).

The function $y = x^{-1}$ is a solution to (17) because

$$\frac{d(x^{-1})}{dx} + (x^{-1})^2 = (-1)x^{-2} + x^{-2} = 0.$$

(b) For any constant C,

$$\frac{d(Ce^{-x})}{dx} + Ce^{-x} = -Ce^{-x} + Ce^{-x} = 0.$$

Thus $y = Ce^{-x}$ is a solution to (16).

Substituting $y = Cx^{-1}$ into (17), we obtain

$$\frac{d(Cx^{-1})}{dx} + (Cx^{-1})^2 = (-C)x^{-2} + C^2x^{-2} = C(C-1)x^{-2},$$

and so we must have $C(C-1) = 0$ in order that $y = Cx^{-1}$ is a solution to (17). Thus, either $C = 0$ or $C = 1$.

(c) For the function $y = C\hat{y}$, one has

$$\frac{d(C\hat{y})}{dx} + P(x)\,(C\hat{y}) = C\frac{d\hat{y}}{dx} + C\,(P(x)\hat{y}) = C\left(\frac{d\hat{y}}{dx} + P(x)\hat{y}\right) = 0$$

if $\hat{y}$ is a solution to $y' + P(x)y = 0$.

30. (a) Multiplying both sides of (18) by y^2, we get

$$y^2\frac{dy}{dx} + 2y^3 = x.$$

If $v = y^3$, then $v' = 3y^2y'$. Thus, $y^2y' = v'/3$, and we have

$$\frac{1}{3}\frac{dv}{dx} + 2v = x,$$

which is equivalent to (19).

(b) The equation (19) is linear with $P(x) = 6$ and $Q(x) = 3x$. So,

$$\mu(x) = \exp\left(\int 6dx\right) = e^{6x}$$

$$\Rightarrow \qquad v(x) = e^{-6x}\int (3xe^{6x})dx = \frac{e^{-6x}}{2}\left(xe^{6x} - \int e^{6x}dx\right)$$

$$= \frac{e^{-6x}}{2}\left(xe^{6x} - \frac{e^{6x}}{6} + C_1\right) = \frac{x}{2} - \frac{1}{12} + Ce^{-6x},$$

where $C = C_1/2$ is an arbitrary constant. The back substitution yields

$$y = \sqrt[3]{\frac{x}{2} - \frac{1}{12} + Ce^{-6x}}.$$

32. In the given equation, $P(x) = 2$, which implies that $\mu(x) = e^{2x}$. Following guidelines, first we solve the equation on $[0, 3]$. On this interval, $Q(x) \equiv 2$. Therefore,

$$y_1(x) = e^{-2x}\int (2)e^{2x}dx = 1 + C_1e^{-2x}.$$

Since $y_1(0) = 0$, we get

$$1 + C_1e^0 = 0 \qquad \Rightarrow \qquad C_1 = -1 \qquad \Rightarrow \qquad y_1(x) = 1 - e^{-2x}.$$

For $x > 3$, $Q(x) = -2$ and so

$$y_2(x) = e^{-2x}\int (-2)e^{2x}dx = -1 + C_2e^{-2x}.$$

We now choose C_2 so that

$$y_2(3) = y_1(3) = 1 - e^{-6} \qquad \Rightarrow \qquad -1 + C_2e^{-6} = 1 - e^{-6} \qquad \Rightarrow \qquad C_2 = 2e^6 - 1.$$

Therefore, $y_2(x) = -1 + (2e^6 - 1)e^{-2x}$, and the continuous solution to the given initial value problem on $[0, \infty)$ is

$$y(x) = \begin{cases} 1 - e^{-2x}, & 0 \le x \le 3, \\ -1 + (2e^6 - 1)e^{-2x}, & x > 3. \end{cases}$$

The graph of this function is shown in Fig. **2–D** at the end of the chapter.

34. **(a)** Since $P(x)$ is continuous on (a, b), its antiderivatives given by $\int P(x)dx$ are continuously differentiable, and therefore continuous, functions on (a, b). Since the function e^x is continuous on $(-\infty, \infty)$, composite functions $\mu(x) = e^{\int P(x)dx}$ are continuous on (a, b). The range of the exponential function is $(0, \infty)$. This implies that $\mu(x)$ is positive with any choice of the integration constant. Using the chain rule, we conclude that

$$\frac{d\mu(x)}{dx} = e^{\int P(x)dx} \frac{d}{dx}\left(\int P(x)dx\right) = \mu(x)P(x)$$

for any x on (a, b).

(b) Differentiating (8), we apply the product rule and obtain

$$\frac{dy}{dx} = -\mu^{-2}\mu'\left(\int \mu Q\,dx + C\right) + \mu^{-1}\mu Q = -\mu^{-1}P\left(\int \mu Q\,dx + C\right) + Q\,,$$

and so

$$\frac{dy}{dx} + Py = \left[-\mu^{-1}P\left(\int \mu Q\,dx + C\right) + Q\right] + P\left[\mu^{-1}\left(\int \mu Q\,dx + C\right)\right] = Q\,.$$

(c) Suggested choice of the antiderivative and the constant C yields

$$y(x_0) = \mu(x)^{-1}\left(\int_{x_0}^{x} \mu Q dx + y_0\mu(x_0)\right)\Bigg|_{x=x_0} = \mu(x_0)^{-1}y_0\mu(x_0) = y_0\,.$$

(d) We assume that $y(x)$ is a solution to the initial value problem (15). Since $\mu(x)$ is a continuous positive function on (a, b), the equation (5) is equivalent to (4). Since, from the part (a), the left-hand side of (5) is the derivative of the product $\mu(x)y(x)$, this function must be an antiderivative of the right-hand side, which is $\mu(x)Q(x)$. Thus, we come up with (8), where the integral means one of the antiderivatives, for example, the one suggested in the part (c) (which has zero value at x_0). Substituting $x = x_0$ into (8), we conclude that

$$y_0 = y(x_0) = \mu(x_0)^{-1}\left(\int \mu Q dx + C\right)\Bigg|_{x=x_0} = C\mu(x_0)^{-1},$$

and so $C = y_0\mu(x_0)$ is uniquely defined.

36. **(a)** If $\mu(x) = \exp\left(\int P dx\right)$ and $y_h(x) = \mu(x)^{-1}$, then

$$\frac{dy_h}{dx} = (-1)\mu(x)^{-2}\frac{d\mu(x)}{dx} = -\mu(x)^{-2}\mu(x)P(x) = -\mu(x)^{-1}P(x)$$

and so

$$\frac{dy_h}{dx} + P(x)y_h = -\mu(x)^{-1}P(x) + P(x)\mu(x)^{-1} = 0,$$

i.e., y_h is a solution to the equation $y' + Py = 0$. Now, the formula (8) yields

$$y = \mu(x)^{-1}\left(\int \mu(x)Q(x)dx + C\right) = y_h(x)v(x) + Cy_h(x) = y_p(x) + Cy_h(x),$$

where $v(x) = \int \mu(x)Q(x)dx$.

(b) Separating variables in (22) and integrating, we obtain

$$\frac{dy}{y} = -\frac{3\,dx}{x} \qquad \Rightarrow \qquad \int \frac{dy}{y} = -\int \frac{3\,dx}{x} \qquad \Rightarrow \qquad \ln|y| = -3\ln x + C.$$

Since we need just one solution y_h, we take $C = 0$

$$\ln|y| = -3\ln x \qquad \Rightarrow \qquad y = \pm x^{-3},$$

and we choose, say, $y_h = x^{-3}$.

(c) Substituting $y_p = v(x)y_h(x) = v(x)x^{-3}$ into (21), we get

$$\frac{dv}{dx}y_h + v\frac{dy_h}{dx} + \frac{3}{x}vy_h = \frac{dv}{dx}y_h + v\left(\frac{dy_h}{dx} + \frac{3}{x}y_h\right) = \frac{dv}{dx}y_h = x^2.$$

Therefore, $dv/dx = x^2/y_h = x^5$.

(d) Integrating yields

$$v(x) = \int x^5 dx = \frac{x^6}{6}\,.$$

(We have chosen zero integration constant.)

(e) The function

$$y = Cy_h + vy_h = Cx^{-3} + \frac{x^3}{6}$$

is a general solution to (21) because

$$\begin{aligned}\frac{dy}{dx} + \frac{3}{x}y &= \frac{d}{dx}\left(Cx^{-3} + \frac{x^3}{6}\right) + \frac{3}{x}\left(Cx^{-3} + \frac{x^3}{6}\right)\\ &= \left(-3Cx^{-4} + \frac{x^2}{2}\right) + \left(3Cx^{-4} + \frac{x^2}{2}\right) = x^2.\end{aligned}$$

38. Dividing both sides of (6) by μ and multiplying by dx yields

$$\frac{d\mu}{\mu} = Pdx \qquad \Rightarrow \qquad \int \frac{d\mu}{\mu} = \int Pdx$$

$$\Rightarrow \qquad \ln|\mu| = \int Pdx \qquad \Rightarrow \qquad \mu = \pm\exp\left(\int Pdx\right).$$

Choosing the positive sign, we obtain (7).

40. **(a)** Figure 2-E at the end of this chapter depicts the direction field for $\alpha = \frac{1}{4}, \beta = \frac{1}{2}$. If $u = 1, u' = -\beta < 0; if\ u = 0, u' = \alpha > 0$. Thus $u(t)$ is confined to [0,1] if it ever enters this interval.

(b) The general solution $u(t) = \frac{\alpha}{\alpha+\beta} + Ce^{-(\alpha+\beta)t}$ approaches $\frac{\alpha}{\alpha+\beta}$ as $t \to \infty$.

EXERCISES 2.4: Exact Equations

2. This equation is not separable because the coefficient $x^{10/3} - 2y$ cannot be written as a product $p(x)q(y)$. Writing the equation in the form

$$\frac{dy}{dx} - 2x^{-1}y = -x^{7/3},$$

we see that the equation is linear. Since $M(x,y) = x^{10/3} - 2y$, $N(x,y) = x$,

$$\frac{\partial M}{\partial y} = -2 \neq \frac{\partial N}{\partial x} = 1,$$

and so the equation is not exact.

4. Here $M(x,y) = ye^{xy} + 2x$, $N(x,y) = xe^{xy} - 2y$. Thus

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(ye^{xy} + 2x\right) = e^{xy} + y\frac{\partial}{\partial y}\left(e^{xy}\right) = e^{xy} + ye^{xy}x = e^{xy}(1 + yx),\\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(xe^{xy} - 2y\right) = e^{xy} + x\frac{\partial}{\partial x}\left(e^{xy}\right) = e^{xy} + xe^{xy}y = e^{xy}(1 + xy),\end{aligned}$$

Since $\partial M/\partial y = \partial N/\partial x$, the equation is exact.

Writing the equation in the form

$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y},$$

we conclude that it is not separable, because the right-hand side cannot be represented as a product of two functions of single variables x and y. Also, the right-hand side is not linear with respect to y, which implies that the equation is not linear with y as the dependent variable. Similarly, choosing x as the dependent variable (taking the reciprocals of both sides), we conclude that the equation is not linear in x either.

6. The differential equation is not separable because $(2xy + \cos y)$ cannot be factored. This equation can be put in standard form by defining x as the dependent variable and y as the independent variable. This gives

$$\frac{dx}{dy} + \frac{2}{y}x = \frac{-\cos y}{y^2},$$

and so we see that it is linear.

If we set $M(x,y) = y^2$ and $N(x,y) = 2xy + \cos y$, we see that the differential equation is also exact, because $M_y(x,y) = 2y = N_x(x,y)$.

8. In this problem, the variables are r and θ, $M(r,\theta) = \theta$, and $N(r,\theta) = 3r - \theta - 1$. Since

$$\frac{\partial M}{\partial \theta} = 1 \neq 3 = \frac{\partial N}{\partial r},$$

the equation is not exact. With r as the dependent variable, the equation takes the form

$$\frac{dr}{d\theta} = -\frac{3r - \theta - 1}{\theta} = -\frac{3}{\theta}\, r + \frac{\theta + 1}{\theta},$$

and so it is linear. Since the right-hand side in the above equation cannot be factored as $p(\theta)q(r)$, the equation is not separable.

10. In this problem, $M(x,y) = 2x + y$, $N(x,y) = x - 2y$. Thus, $M_y = N_x = 1$, and the equation is exact. We find

$$\begin{aligned}
&F(x,y) = \int (2x+y)dx = x^2 + xy + g(y),\\
&\frac{\partial F}{\partial y} = x + g'(y) = N(x,y) = x - 2y\\
&\Rightarrow \qquad g'(y) = -2y \qquad \Rightarrow \qquad g(y) = \int (-2y)dy = -y^2\\
&\Rightarrow \qquad F(x,y) = x^2 + xy - y^2,
\end{aligned}$$

and so $x^2 + xy - y^2 = C$ is a general solution.

12. Taking partial derivatives of $M(x,y) = \cos x \cos y + 2x$ and $N(x,y) = -(\sin x \sin y + 2y)$, we obtain

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\,(\cos x \cos y + 2x) = -\cos x \sin y\,,\\
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\,[-(\sin x \sin y + 2y)] = -\cos x \sin y\,,
\end{aligned}$$

and so the equation is exact.

Integrating $M(x,y)$ with respect to x yields

$$F(x,y) = \int M(x,y)dx \;=\; \int (\cos x \cos y + 2x)\, dx$$

$$= \cos y \int \cos x\,dx + \int 2x\,dx = \sin x \cos y + x^2 + g(y).$$

To find $g(y)$, we compute the partial derivative of $F(x,y)$ with respect to y and compare the result with $N(x,y)$.

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\sin x \cos y + x^2 + g(y)\right] = -\sin x \sin y + g'(y) = -\left(\sin x \sin y + 2y\right)$$

$$\Rightarrow \qquad g'(y) = -2y \qquad \Rightarrow \qquad g(y) = \int (-2y)dy = -y^2.$$

(We have taken zero integration constant.) Therefore,

$$F(x,y) = \sin x \cos y + x^2 - y^2 = C$$

is a general solution to the given equation.

14. In this equation, the variables are y and t, $M(y,t) = t/y$, $N(y,t) = 1 + \ln y$. Since

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\left(\frac{t}{y}\right) = \frac{1}{y} \qquad \text{and} \qquad \frac{\partial N}{\partial y} = \frac{\partial}{\partial y}\left(1 + \ln y\right) = \frac{1}{y}\,,$$

the equation is exact.

Integrating $M(y,t)$ with respect to y, we get

$$F(y,t) = \int \frac{t}{y}\,dy = t \ln|y| + g(t) = t\ln y + g(t).$$

(From $N(y,t) = 1 + \ln y$ we conclude that $y > 0$.) Therefore,

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial t}\left[t\ln y + g(t)\right] = \ln y + g'(t) = 1 + \ln y$$

$$\Rightarrow \qquad g'(t) = 1 \qquad \Rightarrow \qquad g(t) = t$$

$$\Rightarrow \qquad F(y,t) = t\ln y + t,$$

and a general solution is given by $t\ln y + t = C$ (or, explicitly, $t = C/(\ln y + 1)$).

16. Computing

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(ye^{xy} - y^{-1}\right) = e^{xy} + xye^{xy} + y^{-2},$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(xe^{xy} + xy^{-2}\right) = e^{xy} + xye^{xy} + y^{-2},$$

we see that the equation is exact. Therefore,

$$F(x,y) = \int \left(ye^{xy} - y^{-1}\right)dx = e^{xy} - xy^{-1} + g(y).$$

So,

$$\frac{\partial F}{\partial y} = xe^{xy} + xy^{-2} + g'(y) = N(x, y) \qquad \Rightarrow \qquad g'(y) = 0.$$

Thus, $g(y) = 0$, and the answer is $e^{xy} - xy^{-1} = C$.

18. Since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y^2 + \sin(x + y),$$

the equation is exact. We find

$$F(x, y) = \int \left[2x + y^2 - \cos(x + y)\right] dx = x^2 + xy^2 - \sin(x + y) + g(y),$$

$$\frac{\partial F}{\partial y} = 2xy - \cos(x + y) + g'(y) = 2xy - \cos(x + y) - e^y$$

$$\Rightarrow \qquad g'(y) = -e^y \qquad \Rightarrow \qquad g(y) = -e^y.$$

Therefore,

$$F(x, y) = x^2 + xy^2 - \sin(x + y) - e^y = C$$

gives a general solution.

20. We find

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[y\cos(xy)\right] = \cos(xy) - xy\sin(xy),$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[x\cos(xy)\right] = \cos(xy) - xy\sin(xy).$$

Therefore, the equation is exact and

$$F(x, y) = \int \left(x\cos(xy) - y^{-1/3}\right) dy = \sin(xy) - \frac{3}{2}\, y^{2/3} + h(x)$$

$$\frac{\partial F}{\partial x} = y\cos(xy) + h'(x) = \frac{2}{\sqrt{1 - x^2}} + y\cos(xy)$$

$$\Rightarrow \qquad h'(x) = \frac{2}{\sqrt{1 - x^2}} \qquad \Rightarrow \qquad h(x) = 2\arcsin x,$$

and a general solution is given by

$$\sin(xy) - \frac{3}{2}\, y^{2/3} + 2\arcsin x = C.$$

22. In Problem 16, we found that a general solution to this equation is

$$e^{xy} - xy^{-1} = C.$$

Substituting the initial condition, $y(1) = 1$, yields $e - 1 = C$. So, the answer is

$$e^{xy} - xy^{-1} = e - 1.$$

24. First, we check the given equation for exactness.

$$\frac{dM}{dx} = e^t = \frac{\partial N}{\partial t}.$$

So, it is exact. We find

$$\begin{aligned} F(t,x) &= \int \left(e^t - 1\right) dx = x\left(e^t - 1\right) + g(t), \\ \frac{\partial F}{\partial t} &= xe^t + g'(t) = xe^t + 1 \qquad \Rightarrow \qquad g(t) = \int dt = t \\ \Rightarrow \qquad & x\left(e^t - 1\right) + t = C \end{aligned}$$

is a general solution. With $x(1) = 1$, we get

$$(1)\,(e-1) + 1 = C \qquad \Rightarrow \qquad C = e,$$

and the solution is given by

$$x = \frac{e-t}{e^t - 1}.$$

26. Taking partial derivatives M_y and N_x, we find that the equation is exact. So,

$$\begin{aligned} F(x,y) &= \int (\tan y - 2)\,dx = x(\tan y - 2) + g(y), \\ \frac{\partial F}{\partial y} &= x\sec^2 y + g'(y) = x\sec^2 y + y^{-1} \\ \Rightarrow \qquad & g'(y) = y^{-1} \qquad \Rightarrow \qquad g(y) = \ln|y|, \end{aligned}$$

and

$$x(\tan y - 2) + \ln|y| = C$$

is a general solution. Substituting $y(0) = 1$ yields $C = 0$. Therefore, the answer is

$$x(\tan y - 2) + \ln y = 0.$$

(We removed the absolute value sign in the logarithmic function because $y(0) > 0$.)

28. **(a)** Computing

$$\frac{\partial M}{\partial y} = \cos(xy) - xy\sin(xy),$$

which must be equal to $\partial N/\partial x$, we find that

$$\begin{aligned} N(x,y) &= \int [\cos(xy) - xy\sin(xy)]\,dx \\ &= \int [x\cos(xy)]'_x\,dx = x\cos(xy) + g(y). \end{aligned}$$

(b) Since

$$\frac{\partial M}{\partial y} = (1 + xy)e^{xy} - 4x^3 = \frac{\partial N}{\partial x}\,,$$

we conclude that

$$N(x, y) = \int \left[(1 + xy)e^{xy} - 4x^3\right] dx = xe^{xy} - x^4 + g(y).$$

30. (a) Differentiating, we find that

$$\begin{aligned}\frac{\partial M}{\partial y} &= 5x^2 + 12x^3y + 8xy\,,\\ \frac{\partial N}{\partial x} &= 6x^2 + 12x^3y + 6xy\,.\end{aligned}$$

Since $M_y \neq N_x$, the equation is not exact.

(b) Multiplying given equation by $x^n y^m$ and taking partial derivatives of new coefficients yields

$$\begin{aligned}&\frac{d}{dy}\left(5x^{n+2}y^{m+1} + 6x^{n+3}y^{m+2} + 4x^{n+1}y^{m+2}\right)\\ &\qquad = 5(m+1)x^{n+2}y^m + 6(m+2)x^{n+3}y^{m+1} + 4(m+2)x^{n+1}y^{m+1}\\ &\frac{d}{dx}\left(2x^{n+3}y^m + 3x^{n+4}y^{m+1} + 3x^{n+2}y^{m+1}\right)\\ &\qquad = 2(n+3)x^{n+2}y^m + 3(n+4)x^{n+3}y^{m+1} + 3(n+2)x^{n+1}y^{m+1}.\end{aligned}$$

In order that these polynomials are equal, we must have equal coefficients at similar monomials. Thus, n and m must satisfy the system

$$\begin{cases} 5(m+1) = 2(n+3)\\ 6(m+2) = 3(n+4)\\ 4(m+2) = 3(n+2).\end{cases}$$

Solving, we obtain $n = 2$ and $m = 1$. Therefore, multiplying the given equation by x^2y yields an exact equation.

(c) We find

$$\begin{aligned}F(x, y) &= \int \left(5x^4y^2 + 6x^5y^3 + 4x^3y^3\right) dx\\ &= x^5y^2 + x^6y^3 + x^4y^3 + g(y).\end{aligned}$$

Therefore,

$$\frac{\partial F}{\partial y} = 2x^5y + 3x^6y^2 + 3x^4y^2 + g'(y)$$

$$= \quad 2x^5y + 3x^6y^2 + 3x^4y^2 \qquad \Rightarrow \qquad g(y) = 0,$$

and a general solution to the given equation is

$$x^5y^2 + x^6y^3 + x^4y^3 = C.$$

32. **(a)** The slope of the orthogonal curves, say $m_\perp$, must be $-1/m$, where m is the slope of the original curves. Therefore, we have

$$m_\perp = \frac{F_y(x,y)}{F_x(x,y)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{F_y(x,y)}{F_x(x,y)} \quad \Rightarrow \quad F_y(x,y)\,dx - F_x(x,y)\,dy = 0.$$

(b) Let $F(x,y) = x^2 + y^2$. Then we have $F_x(x,y) = 2x$ and $F_y(x,y) = 2y$. Substituting these expressions into the result of part (a) gives us

$$2y\,dx - 2x\,dy = 0 \qquad \Rightarrow \qquad y\,dx - x\,dy = 0.$$

To find the orthogonal trajectories, we must solve this differential equation. To this end, note that this equation is separable and thus

$$\int \frac{1}{x}\,dx = \int \frac{1}{y}\,dy \qquad \Rightarrow \qquad \ln|x| = \ln|y| + C$$
$$\Rightarrow \qquad e^{\ln|x|-C} = e^{\ln|y|} \qquad \Rightarrow \qquad y = kx, \quad \text{where } k = \pm e^{-C}.$$

Therefore, the orthogonal trajectories are lines through the origin.

(c) Let $F(x,y) = xy$. Then we have $F_x(x,y) = y$ and $F_y(x,y) = x$. Substituting these expressions into the result of part (a) yields

$$x\,dx - y\,dy = 0.$$

To find the orthogonal trajectories, we must solve this differential equation. To this end, note that this equation is separable and thus

$$\int x\,dx = \int y\,dy \qquad \Rightarrow \qquad \frac{x^2}{2} = \frac{y^2}{2} + C \qquad \Rightarrow \qquad x^2 - y^2 = k\,,$$

where $k := 2C$. Therefore, the orthogonal trajectories are hyperbolas.

34. To use the method described in Problem 32, we rewrite the equation $x^2 + y^2 = kx$ in the form $x + x^{-1}y^2 = k$. Thus, $F(x,y) = x + x^{-1}y^2$,

$$\frac{\partial F}{\partial x} = 1 - x^{-2}y^2, \qquad \frac{\partial F}{\partial y} = 2x^{-1}y.$$

Substituting these derivatives in the equation given in Problem 32(b), we get the required. Multiplying the equation by $x^n y^m$, we obtain

$$2x^{n-1}y^{m+1}dx + \left(x^{n-2}y^{m+2} - x^n y^m\right) dy = 0.$$

Therefore,

$$\frac{\partial M}{\partial y} = 2(m+1)x^{n-1}y^m\,,$$
$$\frac{\partial N}{\partial x} = (n-2)x^{n-3}y^{m+2} - nx^{n-1}y^m\,.$$

Thus, to have an exact equation, n and m must satisfy

$$\begin{cases} n-2=0 \\ 2(m+1) = -n\,. \end{cases}$$

Solving, we obtain $n=2$, $m=-2$. With this choice, the equation becomes

$$2xy^{-1}dx + \left(1 - x^2y^{-2}\right) dy = 0,$$

and so

$$G(x,y) = \int M(x,y)dx = \int 2xy^{-1}dx = x^2y^{-1} + g(y),$$
$$\frac{\partial G}{\partial y} = -x^2y^{-2} + g'(y) = N(x,y) = 1 - x^2y^{-2}.$$

Therefore, $g(y) = y$, and the family of orthogonal trajectories is given by $x^2y^{-1} + y = C$. Writing this equation in the form $x^2 + y^2 - Cy = 0$, we see that, given C, the trajectory is the circle centered at $(0, C/2)$ and of radius $C/2$.

Several given curves and their orthogonal trajectories are shown in Fig. **2–E** at the end of the chapter.

36. The first equation in (4) follows from (9) and the Fundamental Theorem of Calculus.

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left[\int_{x_0}^{x} M(t,y)dt + g(y)\right] = M(t,y)|_{t=x} = M(x,y).$$

For the second equation in (4),

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int_{x_0}^{x} M(t,y)dt + g(y)\right]$$

$$= \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)dt + g'(y)$$

$$= \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)dt + \left[N(x,y) - \frac{\partial}{\partial y} \int_{x_0}^{x} M(t,y)dt \right] = N(x,y).$$

EXERCISES 2.5: Special Integrating Factors

2. Here, $M(x,y) = 2y^3 + 2y^2$ and $N(x,y) = 3y^2x + 2xy$. Computing

$$\frac{\partial M}{\partial y} = 6y^2 + 4y \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 3y^2 + 2y\,,$$

we conclude that this equation is not exact. Note that these derivatives, as well as M itself, depend on y only. Then, clearly, so does the expression $(\partial N/\partial x - \partial M/\partial y)/M$, and the given equation has an integrating factor depending on y alone. Also, since

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{(6y^2+4y) - (3y^2+2y)}{3y^2x + 2xy} = \frac{3y^2+2y}{x(3y^2+2y)} = \frac{1}{x}\,,$$

the equation has an integrating factor depending on x only.

Writing the equation in the form

$$\frac{dx}{dy} = -\frac{3y^2x + 2xy}{2y^3 + y^2} = -\frac{xy(3y+2)}{2y^2(y+1)} = -\frac{y(3y+2)}{2y^2(y+1)}\,x\,,$$

we conclude that it is separable and linear with x as the dependent variable.

4. This equation is not separable, because of the factor $(y^2 + 2xy)$. It is not linear in either variable because of the terms y^2 and x^2. To see if it is exact, we compute $M_y(x,y)$ and $N_x(x,y)$, and find that

$$M_y(x,y) = 2y + 2x \neq -2x = N_x(x,y).$$

Therefore, the equation is not exact. To see if we can find an integrating factor of the form $\mu(x)$, we compute

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{2y+4x}{-x^2}\,,$$

which is not a function of x alone. To see if we can find an integrating factor of the form $\mu(y)$, we compute

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{-4x-2y}{y^2+2xy} = \frac{-2(2x+y)}{y(y+2x)} = \frac{-2}{y}\,.$$

Thus the equation has an integrating factor that is a function of y alone.

6. In this problem, $M(x,y) = 2y^2x - y$ and $N(x,y) = x$. Therefore,

$$\frac{\partial M}{\partial y} = 4yx - 1 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 1 \qquad \Rightarrow \qquad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 4yx\,.$$

The equation is not exact, because $\partial M/\partial y \neq \partial N/\partial x$, but it has an integrating factor, which depends on y because

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{2 - 4yx}{2y^2x - y} = \frac{-2(2yx - 1)}{y(2yx - 1)} = \frac{-2}{y}\,.$$

Isolating dy/dx, we obtain

$$\frac{dy}{dx} = \frac{y - 2y^2x}{x} = \frac{y}{x} - 2y^2\,.$$

The right-hand side cannot be written as $p(x)q(y)$, so the equation is not separable. Also, it is not linear with y as the dependent variable (because of $2y^2$ term). Taking the reciprocals, we conclude that it is not linear with the dependent variable x.

8. The equation $(3x^2 + y)\,dx + (x^2y - x)\,dy = 0$ is not separable or linear. To see if it is exact, we compute

$$\frac{\partial M}{\partial y} = 1 \neq 2xy - 1 = \frac{\partial N}{\partial x}\,.$$

Thus, the equation is not exact. To see if we can find an integrating factor, we compute

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{-2(xy - 1)}{x(xy - 1)} = \frac{-2}{x}\,.$$

Thus, an integrating factor $\mu(x)$ is given by

$$\mu(x) = \exp\left(\int \frac{-2}{x}\,dx\right) = \exp\left(-2\ln|x|\right) = x^{-2}.$$

To solve the equation, we multiply it by the integrating factor x^{-2} and get

$$(3 + yx^{-2})\,dx + (y - x^{-1})\,dy = 0.$$

This new equation is now exact. Thus, we find $F(x,y)$ integrating $M(x,y) = 3 + yx^{-2}$ with respect to x.

$$\begin{aligned}
&F(x,y) = \int \left(3 + yx^{-2}\right) dx = 3x - yx^{-1} + g(y)\\
\Rightarrow \qquad &F_y(x,y) = -x^{-1} + g'(y) = N(x,y) = y - x^{-1}\\
\Rightarrow \qquad &g'(y) = y \qquad \Rightarrow \qquad g(y) = \frac{y^2}{2}\,.
\end{aligned}$$

Therefore,

$$F(x,y) = 3x - yx^{-1} + \frac{y^2}{2},$$

and an implicit solution to the given equation is

$$\frac{y^2}{2} - \frac{y}{x} + 3x = C.$$

Since $\mu(x) = x^{-2}$ we must check if the solution $x \equiv 0$ was lost. The function $x \equiv 0$ is a solution to the original equation, but is not given by the above implicit solution for any choice of C. Hence,

$$\frac{y^2}{2} - \frac{y}{x} + 3x = C \qquad \text{and} \qquad x \equiv 0$$

give a general solution.

10. We compute the partial derivatives of $M(x,y) = 2y^2 + 2y + 4x^2$ and $N(x,y) = 2xy + x$.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2y^2 + 2y + 4x^2\right) = 4y + 2, \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2xy + x) = 2y + 1.$$

Although the equation is not exact $(\partial M/\partial y \neq \partial N/\partial x)$, the quotient

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{(4y+2) - (2y+1)}{2xy + x} = \frac{2y+1}{x(2y+1)} = \frac{1}{x}$$

depends on x only, and so the equation has an integrating factor, which can be found by applying formula (8). Namely,

$$\mu(x) = \exp\left(\int \frac{1}{x}\,dx\right) = \exp\left(\ln|x|\right) = |x|.$$

Note that if μ is an integrating factor, then $-\mu$ is an integrating factor as well. This observation allows us to take $\mu(x) = x$. Multiplying the given differential equation by x yields an exact equation

$$\left(2y^2 + 2y + 4x^2\right) x\,dx + x^2\,(2y+1)\,dy = 0.$$

Therefore,

$$\begin{aligned}
& F(x,y) = \int x^2(2y+1)\,dy = x^2\left(y^2 + y\right) + h(x) \\
\Rightarrow \quad & \frac{\partial F}{\partial x} = 2x\left(y^2 + y\right) + h'(x) = \left(2y^2 + 2y + 4x^2\right) x \\
\Rightarrow \quad & h'(x) = 4x^3 \qquad \Rightarrow \qquad h(x) = \int 4x^3 dx = x^4 \\
\Rightarrow \quad & F(x,y) = x^2\left(y^2 + y\right) + x^4 = x^2y^2 + x^2y + x^4,
\end{aligned}$$

and $x^2y^2 + x^2y + x^4 = C$ is a general solution.

12. Here, $M(x,y) = 2xy^3 + 1$, $N(x,y) = 3x^2y^2 - y^{-1}$. Since

$$\frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x}\,,$$

the equation is exact. So, we find that

$$\begin{aligned}
F(x,y) &= \int \left(2xy^3+1\right)dx = x^2y^3 + x + g(y),\\
\frac{\partial F}{\partial y} &= 3x^2y^2 + g'(y) = 3x^2y^2 - y^{-1}\\
\Rightarrow \qquad & g'(y) = -y^{-1} \qquad \Rightarrow \qquad g(y) = -\ln|y|,
\end{aligned}$$

and the given equation has a general solution

$$x^2y^3 + x - \ln|y| = C.$$

14. Multiplying the given equation by x^ny^m yields

$$\left(12x^ny^m + 5x^{n+1}y^{m+1}\right)dx + \left(6x^{n+1}y^{m-1} + 3x^{n+2}y^m\right)dy = 0.$$

Therefore,

$$\begin{aligned}
\frac{\partial M}{\partial y} &= 12mx^ny^{m-1} + 5(m+1)x^{n+1}y^m,\\
\frac{\partial N}{\partial x} &= 6(n+1)x^ny^{m-1} + 3(n+2)x^{n+1}y^m.
\end{aligned}$$

Matching the coefficients, we get a system

$$\begin{cases} 12m = 6(n+1) \\ 5(m+1) = 3(n+2) \end{cases}$$

to determine n and m. This system has the solution $n = 3$, $m = 2$. Thus, the given equation multiplied by x^3y^2, that is,

$$\left(12x^3y^2 + 5x^4y^3\right)dx + \left(6x^4y + 3x^5y^2\right)dy = 0,$$

is exact. We compute

$$\begin{aligned}
F(x,y) &= \int \left(12x^3y^2 + 5x^4y^3\right)dx = 3x^4y^2 + x^5y^3 + g(y),\\
\frac{\partial F}{\partial y} &= 6x^4y + 3x^5y^2 + g'(y) = 6x^4y + 3x^5y^2\\
\Rightarrow \qquad & g'(y) = 0 \qquad \Rightarrow \qquad g(y) = 0,
\end{aligned}$$

and so $3x^4y^2 + x^5y^3 = C$ is a general solution to the given equation.

16. (a) An equation $Mdx + Ndy = 0$ has an integrating factor $\mu(x+y)$ if and only if the equation

$$\mu(x+y)M(x,y)dx + \mu(x+y)N(x,y)dy = 0$$

is exact. According to Theorem 2, Section 2.4, this means that

$$\frac{\partial}{\partial y}\left[\mu(x+y)M(x,y)\right] = \frac{\partial}{\partial x}\left[\mu(x+y)N(x,y)\right].$$

Applying the product and chain rules yields

$$\mu'(x+y)M(x,y) + \mu(x+y)\frac{\partial M(x,y)}{\partial y} = \mu'(x+y)N(x,y) + \mu(x+y)\frac{\partial N(x,y)}{\partial x}.$$

Collecting similar terms yields

$$\mu'(x+y)\left[M(x,y) - N(x,y)\right] = \mu(x+y)\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$$

$$\Leftrightarrow \qquad \frac{\partial N/\partial x - \partial M/\partial y}{M-N} = \frac{\mu'(x+y)}{\mu(x+y)}. \tag{2.4}$$

The right-hand side of (2.4) depends on $x+y$ only so the left-hand side does.

To find an integrating factor, we let $s = x+y$ and denote

$$G(s) = \frac{\partial N/\partial x - \partial M/\partial y}{M-N}.$$

Then (2.4) implies that

$$\frac{\mu'(s)}{\mu(s)} = G(s) \qquad \Rightarrow \qquad \ln|\mu(s)| = \int G(s)\,ds$$

$$\Rightarrow \quad |\mu(s)| = \exp\left[\int G(s)\,ds\right] \quad \Rightarrow \quad \mu(s) = \pm\exp\left[\int G(s)\,ds\right]. \tag{2.5}$$

In this formula, we can choose either sign and any integration constant.

(b) We compute

$$\frac{\partial N/\partial x - \partial M/\partial y}{M-N} = \frac{(1+y)-(1+x)}{(3+y+xy)-(3+x+xy)} = 1.$$

Applying formula (2.5), we obtain

$$\mu(s) = \exp\left[\int (1)ds\right] = e^s \qquad \Rightarrow \qquad \mu(x+y) = e^{x+y},$$

Multiplying the given equation by $\mu(x+y)$, we get an exact equation

$$e^{x+y}(3+y+xy)dx + e^{x+y}(3+x+xy)dy = 0$$

and follow the procedure of solving exact equations, Section 2.4.

$$\begin{aligned} F(x,y) &= \int e^{x+y}(3+y+xy)\,dx = e^y\left[(3+y)\int e^x dx + y\int xe^x dx\right] \\ &= e^y\left[(3+y)e^x + y(x-1)e^x\right] + h(y) = e^{x+y}(3+xy) + h(y)\,. \end{aligned}$$

Taking the partial derivative of F with respect to y, we find $h(y)$.

$$\frac{\partial F}{\partial y} = e^{x+y}(3+xy+x) + h'(y) = e^{x+y}N(x,y) = e^{x+y}(3+x+xy)$$
$$\Rightarrow \qquad h'(y) = 0 \qquad \Rightarrow \qquad h(y) = 0\,.$$

Thus, a general solution is

$$e^{x+y}(3+xy) = C\,.$$

18. The given condition, $xM(x,y) + yN(x,y) \equiv 0$, is equivalent to $yN(x,y) \equiv -xM(x,y)$. In particular, substituting $x = 0$, we obtain

$$yN(0,y) \equiv -(0)M(0,y) \equiv 0.$$

This implies that $x \equiv 0$ is a solution to the given equation.

To obtain other solutions, we multiply the equation by $x^{-1}y$. This gives

$$\begin{aligned} x^{-1}yM(x,y)dx + x^{-1}yN(x,y)dy &= x^{-1}yM(x,y)dx - x^{-1}xM(x,y)dy \\ &= xM(x,y)\left(x^{-2}ydx - x^{-1}dy\right) = -xM(x,y)d\left(x^{-1}y\right) = 0\,. \end{aligned}$$

Therefore, $x^{-1}y = C$ or $y = Cx$.

Thus, a general solution is

$$y = Cx \quad \text{and} \quad x \equiv 0.$$

20. For the equation

$$e^{\int P(x)dx}\left[P(x)y - Q(x)\right]dx + e^{\int P(x)dx}dy = 0,$$

we compute

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(e^{\int P(x)dx}\left[P(x)y - Q(x)\right]\right) = e^{\int P(x)dx}P(x), \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(e^{\int P(x)dx}\right) = e^{\int P(x)dx}\frac{d}{dx}\left(\int P(x)dx\right) = e^{\int P(x)dx}P(x). \end{aligned}$$

Therefore, $\partial M/\partial y = \partial N/\partial x$, and the equation is exact.

EXERCISES 2.6: Substitutions and Transformations

2. We can write the equation in the form

$$\frac{dy}{dx} = (y - 4x - 1)^2 = [(y - 4x) - 1]^2 = G(y - 4x),$$

where $G(t) = (t - 1)^2$. Thus, it is of the form $dy/dx = G(ax + by)$.

4. In this equation, the variables are x and t. Its coefficients, $t + x + 2$ and $3t - x - 6$, are linear functions of x and t. Therefore, the given equation is an equation with linear coefficients.

6. The given differential equation is not homogeneous due to the e^{-2x} terms. Since it can be written in the form $dy/dx + P(x)y = Q(x)y^n$, namely,

$$\frac{dy}{dx} - y = e^{2x}y^3.$$

it is a Bernoulli equation. The differential equation does not have linear coefficients, and it is not of the form $y' = G(ax + by)$ either.

8. Here, the variables are y and θ. Writing

$$\frac{dy}{d\theta} = -\frac{y^3 - \theta y^2}{2\theta^2 y} = -\frac{(y/\theta)^3 - (y/\theta)^2}{2(y/\theta)},$$

we see that the right-hand side is a function of y/θ alone. Hence, the equation is homogeneous.

10. First, we write the equation in the form

$$\frac{dy}{dx} = \frac{-3x^2 + y^2}{xy - x^3y^{-1}} = \frac{y^3 - 3x^2y}{xy^2 - x^3} = \frac{(y/x)^3 - 3(y/x)}{(y/x)^2 - 1}.$$

Therefore, it is homogeneous, and we we make a substitution $y/x = u$ or $y = xu$. Then $y' = u + xu'$, and the equation becomes

$$u + x\frac{du}{dx} = \frac{u^3 - 3u}{u^2 - 1}.$$

Separating variables and integrating yields

$$x\frac{du}{dx} = \frac{u^3 - 3u}{u^2 - 1} - u = -\frac{2u}{u^2 - 1} \qquad \Rightarrow \qquad \frac{u^2 - 1}{u}\,du = -\frac{2}{x}\,dx$$

$$\Rightarrow \quad \int \frac{u^2-1}{u}\,du = -\int \frac{2}{x}\,dx \quad \Rightarrow \quad \int \left(u - \frac{1}{u}\right) du = -2\int \frac{dx}{x}$$
$$\Rightarrow \quad \frac{1}{2}u^2 - \ln|u| = -2\ln|x| + C_1 \quad \Rightarrow \quad u^2 - \ln\left(u^2\right) + \ln(x^4) = C.$$

Substituting back y/x for u and simplifying, we finally get

$$\left(\frac{y}{x}\right)^2 - \ln\left(\frac{y^2}{x^2}\right) + \ln(x^4) = C \quad \Rightarrow \quad \frac{y^2}{x^2} - \ln\left(\frac{y^2}{x^6}\right) = C.$$

12. From

$$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy} = -\frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right),$$

making the substitution $v = y/x$, we obtain

$$v + x\frac{dv}{dx} = -\frac{1}{2}\left(\frac{1}{v} + v\right) = -\frac{1+v^2}{2v} \quad \Rightarrow \quad x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v = -\frac{1+3v^2}{2v}$$
$$\Rightarrow \quad \frac{2v\,dv}{1+3v^2} = -\frac{dx}{x} \quad \Rightarrow \quad \int \frac{2v\,dv}{1+3v^2} = -\int \frac{dx}{x}$$
$$\Rightarrow \quad \frac{1}{3}\ln\left(1+3v^2\right) = -\ln|x| + C_2 \quad \Rightarrow \quad 1+3v^2 = C_1|x|^{-3},$$

where $C_1 = e^{3C_2}$ is any positive constant. Making the back substitution, we finally get

$$1 + 3\left(\frac{y}{x}\right)^2 = \frac{C_1}{|x|^3} \quad \Rightarrow \quad 3\left(\frac{y}{x}\right)^2 = \frac{C_1}{|x|^3} - 1 = \frac{C_1 - |x|^3}{|x|^3}$$
$$\Rightarrow \quad 3|x|y^2 = C_1 - |x|^3 \quad \Rightarrow \quad 3|x|y^2 + |x|^3 = C_1 \quad \Rightarrow \quad 3xy^2 + x^3 = C\,,$$

where $C = \pm C_1$ is any nonzero constant.

14. Substituting $v = y/\theta$ yields

$$v + \theta\frac{dv}{d\theta} = \sec v + v \quad \Rightarrow \quad \theta\frac{dv}{d\theta} = \sec v$$
$$\Rightarrow \quad \cos v\,dv = \frac{d\theta}{\theta} \quad \Rightarrow \quad \int \cos v\,dv = \int \frac{d\theta}{\theta}$$
$$\Rightarrow \quad \sin v = \ln|\theta| + C \quad \Rightarrow \quad y = \theta\arcsin\left(\ln|\theta| + C\right).$$

16. We rewrite the equation in the form

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\frac{y}{x} + 1\right)$$

and substitute $v = y/x$ to get

$$v + x\frac{dv}{dx} = v\left(\ln v + 1\right) \quad \Rightarrow \quad x\frac{dv}{dx} = v\ln v \quad \Rightarrow \quad \int \frac{dv}{v\ln v} = \int \frac{dx}{x}$$

$$\Rightarrow \qquad \ln|\ln v| = \ln|x| + C_1 \qquad \Rightarrow \qquad \ln v = \pm e^{C_1} x = Cx \qquad \Rightarrow \qquad v = e^{Cx},$$

where $C \neq 0$ is any constant. Note that, separating variables, we lost a solution, $v \equiv 1$, which can be included in the above formula by letting $C = 0$. Thus we have $v = e^{Cx}$. where C is any constant. Substituting back $y = xv$ yields a general solution

$$y = xe^{Cx}$$

to the given equation.

18. With $z = x + y + 2$ and $z' = 1 + y'$, we have

$$\begin{aligned}
&\frac{dz}{dx} = z^2 + 1 \qquad \Rightarrow \qquad \frac{dz}{z^2+1} = dx \qquad \Rightarrow \qquad \int \frac{dz}{z^2+1} = \int dx \\
&\Rightarrow \qquad \arctan z = x + C \qquad \Rightarrow \qquad x + y + 2 = z = \tan(x + C) \\
&\Rightarrow \qquad y = \tan(x + C) - x - 2.
\end{aligned}$$

20. Substitution $z = x - y$ yields

$$\begin{aligned}
&1 - \frac{dz}{dx} = \sin z \qquad \Rightarrow \qquad \frac{dz}{dx} = 1 - \sin z \qquad \Rightarrow \qquad \frac{dz}{1 - \sin z} = dx \\
&\Rightarrow \qquad \int \frac{dz}{1 - \sin z} = \int dx = x + C.
\end{aligned}$$

The left-hand side integral can be found as follows.

$$\begin{aligned}
\int \frac{dz}{1 - \sin z} &= \int \frac{(1 + \sin z)dz}{1 - \sin^2 z} = \int \frac{(1 + \sin z)dz}{\cos^2 z} \\
&= \int \sec^2 z + \int \tan z \sec z \, dz = \tan z + \sec z.
\end{aligned}$$

Thus, a general solution is given implicitly by

$$\tan(x - y) + \sec(x - y) = x + C.$$

22. Dividing the equation by y^3 yields

$$y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}.$$

We now make a substitution $v = y^{-2}$ so that $v' = -2y^{-3}y'$, and get

$$\frac{dv}{dx} + 2v = -2e^{2x}.$$

This is a linear equation. So,

$$\mu(x) = \exp\left(\int 2dx\right) = e^{2x},$$
$$v(x) = e^{-2x}\int \left(-2e^{2x}\right)e^{2x}dx = -(1/2)e^{-2x}\left(e^{4x}+C\right) = -\frac{e^{2x}+Ce^{-2x}}{2}.$$

Therefore,

$$\frac{1}{y^2} = -\frac{e^{2x}+Ce^{-2x}}{2} \qquad \Rightarrow \qquad y = \pm\sqrt{-\frac{2}{e^{2x}+Ce^{-2x}}}.$$

Dividing the equation by y^3, we lost a constant solution $y \equiv 0$.

24. We divide this Bernoulli equation by $y^{1/2}$ and make a substitution $v = y^{1/2}$.

$$y^{-1/2}\frac{dy}{dx} + \frac{1}{x-2}y^{1/2} = 5(x-2)$$
$$\Rightarrow \qquad 2\frac{dv}{dx} + \frac{1}{x-2}v = 5(x-2) \qquad \Rightarrow \qquad \frac{dv}{dx} + \frac{1}{2(x-2)}v = \frac{5(x-2)}{2}.$$

An integrating factor for this linear equation is

$$\mu(x) = \exp\left[\int \frac{dx}{2(x-2)}\right] = \sqrt{|x-2|}.$$

Therefore,

$$\begin{aligned} v(x) &= \frac{1}{\sqrt{|x-2|}}\int \frac{5(x-2)\sqrt{|x-2|}}{2}\,dx \\ &= \frac{1}{\sqrt{|x-2|}}\left(|x-2|^{5/2}+C\right) = (x-2)^2 + C|x-2|^{-1/2}. \end{aligned}$$

Since $y = v^2$, we finally get

$$y = \left[(x-2)^2 + C|x-2|^{-1/2}\right]^2.$$

In addition, $y \equiv 0$ is a (lost) solution.

26. Multiplying the equation by y^2, we get

$$y^2\frac{dy}{dx} + y^3 = e^x.$$

With $v = y^3$, $v' = 3y^2y'$, the equation becomes

$$\frac{1}{3}\frac{dv}{dx} + v = e^x \qquad \Rightarrow \qquad \frac{dv}{dx} + 3v = 3e^x \qquad \Rightarrow \qquad \frac{d}{dx}\left(e^{3x}v\right) = 3e^{4x}$$

$$\Rightarrow \qquad v = e^{-3x} \int 3e^{4x}dx = \frac{3e^x}{4} + Ce^{-3x}.$$

Therefore,

$$y = \sqrt[3]{v} = \sqrt[3]{\frac{3e^x}{4} + Ce^{-3x}}\,.$$

28. First, we note that $y \equiv 0$ is a solution, which will be lost when we divide the equation by y^3 and make a substitution $v = y^{-2}$ to get a linear equation

$$y^{-3}\frac{dy}{dx} + y^{-2} + x = 0 \qquad \Rightarrow \qquad \frac{dv}{dx} - 2v = 2x.$$

An integrating factor for this equation is

$$\mu(x) = \exp\left[\int (-2)dx\right] = e^{-2x}.$$

Thus,

$$\begin{aligned} v &= e^{2x}\int 2xe^{-2x}dx = e^{2x}\left(-xe^{-2x} + \int e^{-2x}dx\right) \\ &= e^{2x}\left(-xe^{-2x} - \frac{e^{-2x}}{2} + C\right) = -x - \frac{1}{2} + Ce^{2x}. \end{aligned}$$

So,

$$y = \pm v^{-1/2} = \pm\frac{1}{\sqrt{-x - \frac{1}{2} + Ce^{2x}}}\,.$$

30. We make a substitution $x = u + h, \quad y = v + k$, where h and k satisfy system (14) in the text:

$$\begin{aligned} -4h - k - 1 &= 0 \\ h + k + 3 &= 0. \end{aligned}$$

Solving yields $h = \frac{2}{3}$, $k = -\frac{11}{3}$. Thus, $x = u + \frac{2}{3}$, $y = v - \frac{11}{3}$. Since $dx = du$, $dy = dv$, the substitution leads to

$$(-4u - v)du + (u + v)dv = 0 \qquad \Rightarrow \qquad \frac{dv}{du} = \frac{4u + v}{u + v}\,.$$

Therefore with $z = \frac{v}{u}$, $v' = z + uz'$ we have

$$\begin{aligned} z + uz' &= \frac{4 + z}{1 + z} \text{ or } u\frac{dz}{du} = \frac{4 + z}{1 + z} - z = \frac{4 - z^2}{1 + z} \\ \Rightarrow \qquad \frac{1 + z}{4 - z^2}dz &= \frac{du}{u} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \qquad -\frac{3}{4}\log|2-z| - \frac{1}{4}\log|z+2| = \log|u| + C_1 \\
&\Rightarrow \qquad |u||2-z|^{3/4}|z+2|^{1/4} = e^{-C_1} = C_2 \\
&\Rightarrow \qquad \left|x-\frac{2}{3}\right|\left|2-\frac{y+\frac{11}{3}}{x-\frac{2}{3}}\right|^{2/3}\left|\frac{y+\frac{11}{3}}{x-\frac{2}{3}}+2\right|^{1/4} = C_2
\end{aligned}$$

where C_1 and C_2 are constants. In dividing by $4-z^2$, we lost the constant solutions $z \equiv 2$ and $z \equiv -2$, but these are included in our answer if we let $C_2 = 0$.

32. We make a substitution $x = u + h$, $\quad y = v + k$, where h and k satisfy system (14) in the text:

$$2h - k + 4 = 0$$

$$h - 2k - 2 = 0.$$

Solving yields $h = -\frac{10}{3}$, $k = -\frac{8}{3}$. Thus, $x = u - \frac{10}{3}$, $y = v - \frac{8}{3}$. Since $dx = du$, $dy = dv$, the substitution leads to

$$(2u - v)du + (u - 2v)dv = 0 \qquad \Rightarrow \qquad \frac{dv}{du} = \frac{-v + 2u}{2v - u}\,.$$

Therefore with $z = \frac{v}{u}$, $v' = z + uz'$ we have

$$\begin{aligned}
&z + uz' = \frac{-z+2}{2z-1} \text{ or } u\frac{dz}{du} = \frac{-z+2}{2z-1} - z = \frac{-2z^2+2}{2z-1} \\
&\Rightarrow \qquad \frac{2z-1}{z^2-1}dz = -2\frac{du}{u} \\
&\Rightarrow \qquad \frac{1}{2}\log|1-z| + \frac{3}{2}\log|z+1| = -2\log|u| + C_1 \\
&\Rightarrow \qquad |1-z|^{\frac{1}{2}}|z+1|^{\frac{3}{2}}u^2 = \left|1-\frac{y+\frac{8}{3}}{x+\frac{10}{3}}\right|^{\frac{1}{2}}\left|1+\frac{y+\frac{8}{3}}{x+\frac{10}{3}}\right|^{\frac{3}{2}}\left(x+\frac{10}{3}\right)^2 = e^{c_1} = C_2,
\end{aligned}$$

where C_1 and C_2 are constants. In dividing by $z^2 - 1$, we lost the constant solutions $z \equiv 1$ and $z \equiv -1$, but these are included in our answer if we let $C_2 = 0$.

34. In Problem 2, we found that the given equation is of the form $dy/dx = G(y - 4x)$ with $G(u) = (u-1)^2$. Thus we make a substitution $u = y - 4x$ to get

$$\begin{aligned}
&\frac{dy}{dx} = (y - 4x - 1)^2 \qquad \Rightarrow \qquad 4 + \frac{du}{dx} = (u-1)^2 \\
&\Rightarrow \qquad \frac{du}{dx} = (u-1)^2 - 4 = (u-3)(u+1) \qquad \Rightarrow \qquad \int \frac{du}{(u-3)(u+1)} = \int dx\,.
\end{aligned}$$

To integrate the left-hand side, we use partial fractions decomposition,

$$\frac{1}{(u-3)(u+1)} = \frac{1}{4}\left(\frac{1}{u-3} - \frac{1}{u+1}\right).$$

Thus,

$$\begin{aligned}
&\frac{1}{4}\left(\ln|u-3| - \ln|u+1|\right) = x + C_1 \qquad \Rightarrow \qquad \ln\left|\frac{u-3}{u+1}\right| = 4x + C_2\\
\Rightarrow \qquad &\frac{u-3}{u+1} = Ce^{4x} \qquad \Rightarrow \qquad u = \frac{Ce^{4x}+3}{1-Ce^{4x}}\\
\Rightarrow \qquad &y = 4x + \frac{Ce^{4x}+3}{1-Ce^{4x}}, \qquad (2.6)
\end{aligned}$$

where $C \neq 0$ is an arbitrary constant. Separating variables, we lost the constant solutions $u \equiv 3$ and $u \equiv -1$, that is, $y = 4x+3$ and $y = 4x-1$. While $y = 4x+3$ can be obtained from (2.6) by setting $C = 0$, the solution $y = 4x-1$ is not included in (2.6). Therefore, a general solution to the given equation is

$$y = 4x + \frac{Ce^{4x}+3}{1-Ce^{4x}} \qquad \text{and} \qquad y = 4x-1.$$

36. This equation has linear coefficients. Thus we make a substitution $t = u+h$ and $x = v+k$ with h and k satisfying

$$\begin{aligned} h + k + 2 &= 0,\\ 3h - k - 6 &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} h &= 1,\\ k &= -3. \end{aligned}$$

As $dt = du$ and $dx = dv$, the substitution yields

$$(u+v)dv + (3u-v)du = 0 \qquad \Rightarrow \qquad \frac{du}{dv} = -\frac{u+v}{3u-v} = -\frac{(u/v)+1}{3(u/v)-1}.$$

With $z = u/v$, we have $u = vz$, $u' = z + vz'$, and the equation becomes

$$\begin{aligned}
&z + v\frac{dz}{dv} = -\frac{z+1}{3z-1} \qquad \Rightarrow \qquad v\frac{dz}{dv} = -\frac{3z^2+1}{3z-1}\\
\Rightarrow \qquad &\frac{3z-1}{3z^2+1}\,dz = -\frac{1}{v}\,dv \qquad \Rightarrow \qquad \int \frac{3z-1}{3z^2+1}\,dz = -\int \frac{1}{v}\,dv\\
\Rightarrow \qquad &\int \frac{3z\,dz}{3z^2+1} - \int \frac{dz}{3z^2+1} = -\ln|v| + C_1\\
\Rightarrow \qquad &\frac{1}{2}\ln\left(3z^2+1\right) - \frac{1}{\sqrt{3}}\arctan\left(z\sqrt{3}\right) = -\ln|v| + C_1\\
\Rightarrow \qquad &\ln\left[(3z^2+1)v^2\right] - \frac{2}{\sqrt{3}}\arctan\left(z\sqrt{3}\right) = C_2.
\end{aligned}$$

Making back substitution, after some algebra we get

$$\ln\left[(x+3)^2 + 3(t-1)^2\right] + \frac{2}{\sqrt{3}}\arctan\left[\frac{x+3}{\sqrt{3}(t-1)}\right] = C.$$

38. In Problem 6, we have written the equation in the form

$$\frac{dy}{dx} - y = e^{2x}y^3 \qquad \Rightarrow \qquad y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}.$$

Making a substitution $u = y^{-2}$ (and so $u' = -2y^{-3}y'$) in this Bernoulli equation, we get

$$\frac{du}{dx} + 2u = -2e^{2x} \qquad \Rightarrow \qquad \mu(x) = \exp\left(\int 2dx\right) = e^{2x}$$

$$\Rightarrow \quad \frac{d\left(e^{2x}u\right)}{dx} = -2e^{2x}e^{2x} = -2e^{4x} \qquad \Rightarrow \qquad e^{2x}u = \int\left(-2e^{4x}\right)dx = -\frac{1}{2}\,e^{4x} + C$$

$$\Rightarrow \quad u = -\frac{1}{2}\,e^{2x} + Ce^{-2x} \qquad \Rightarrow \qquad y^{-2} = -\frac{1}{2}\,e^{2x} + Ce^{-2x}\,.$$

Also, we lost the constant solution $y \equiv 0$ when divided the equation by y^3.

40. Since the equation is homogeneous, we make a substitution $u = y/\theta$. Thus, we get

$$\frac{dy}{d\theta} = -\frac{(y/\theta)^3 - (y/\theta)^2}{2(y/\theta)} \qquad \Rightarrow \qquad u + \theta\,\frac{du}{d\theta} = -\frac{u^3 - u^2}{2u} = -\frac{u^2 - u}{2}$$

$$\Rightarrow \qquad \theta\,\frac{du}{d\theta} = -\frac{u^2 + u}{2} \qquad \Rightarrow \qquad \frac{2du}{u(u+1)} = -\frac{d\theta}{\theta}$$

$$\Rightarrow \qquad \int\frac{2du}{u(u+1)} = -\int\frac{d\theta}{\theta} \qquad \Rightarrow \qquad \ln\frac{u^2}{(u+1)^2} = -\ln|\theta| + C_1\,,$$

which gives

$$\frac{u^2}{(u+1)^2} = \frac{C}{\theta}, \qquad C \neq 0\,.$$

Back substitution $u = y/\theta$ yields

$$\frac{y^2}{(\theta+y)^2} = \frac{C}{\theta} \qquad \Rightarrow \qquad \theta y^2 = C(\theta+y)^2\,, \qquad C \neq 0.$$

When $C = 0$, the above formula gives $\theta \equiv 0$ or $y \equiv 0$, which were lost in separating variables. Also, we lost another solution, $u + 1 \equiv 0$ or $y = -\theta$. Thus, the answer is

$$\theta y^2 = C(\theta+y)^2 \quad \text{and} \quad y = -\theta,$$

where C is an arbitrary constant.

42. Suggested substitution, $y = vx^2$ (so that $y' = 2xv + x^2v'$), yields

$$2xv + x^2\frac{dv}{dx} = 2vx + \cos v \qquad \Rightarrow \qquad x^2\frac{dv}{dx} = \cos v.$$

Solving this separable equation, we obtain

$$\frac{dv}{\cos v} = \frac{dx}{x^2} \qquad \Rightarrow \qquad \ln|\sec v + \tan v| = -x^{-1} + C_1$$

$$\Rightarrow \qquad \sec v + \tan v = \pm e^{C_1} e^{-1/x} = Ce^{-1/x}$$
$$\Rightarrow \qquad \sec\left(\frac{y}{x^2}\right) + \tan\left(\frac{y}{x^2}\right) = Ce^{-1/x},$$

where $C = \pm e^{C_1}$ is an arbitrary nonzero constant. With $C = 0$, this formula also includes lost solutions

$$y = \left[\frac{\pi}{2} + (2k+1)\pi\right] x^2, \quad k = 0, \pm 1, \pm 2, \dots .$$

So, together with the other set of lost solutions,

$$y = \left(\frac{\pi}{2} + 2k\pi\right) x^2, \quad k = 0, \pm 1, \pm 2, \dots ,$$

we get a general solution to the given equation.

44. From

$$\frac{dy}{dx} = -\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2},$$

using that $a_2 = ka_1$ and $b_2 = kb_1$, we obtain

$$\frac{dy}{dx} = -\frac{a_1 x + b_1 y + c_1}{ka_1 x + kb_1 y + c_2} = -\frac{a_1 x + b_1 y + c_1}{k(a_1 x + b_1 y) + c_2} = G\left(a_1 x + b_1 y\right),$$

where

$$G(t) = -\frac{t + c_1}{kt + c_2}.$$

48. Apply formula (8), page 50 of Section 2.3, to the transformed Bernoulli equation below equation (10) of Section 2.6, page 74. (If $n = 1$, the Bernoulli equation is linear and formula (8) can be applied directly.)

REVIEW PROBLEMS

2. $y = -8x^2 - 4x - 1 + Ce^{4x}$

4. $\dfrac{x^3}{6} - \dfrac{4x^2}{5} + \dfrac{3x}{4} - Cx^{-3}$

6. $y^{-2} = 2\ln|1 - x^2| + C$ and $y \equiv 0$

8. $y = (Cx^2 - 2x^3)^{-1}$ and $y \equiv 0$

10. $x + y + 2y^{1/2} + \arctan(x + y) = C$

12. $2ye^{2x} + y^3 e^x = C$

14. $x = \dfrac{t^2(t-1)}{2} + t(t-1) + 3(t-1)\ln|t-1| + C(t-1)$

16. $y = \cos x \ln|\cos x| + C\cos x$

18. $y = 1 - 2x + \sqrt{2}\tan\left(\sqrt{2}\,x + C\right)$

20. $y = \left(C\theta^{-3} - \dfrac{12\theta^2}{5}\right)^{1/3}$

22. $(3y - 2x + 9)(y + x - 2)^4 = C$

24. $2\sqrt{xy} + \sin x - \cos y = C$

26. $y = Ce^{-x^2/2}$

28. $(y+3)^2 + 2(y+3)(x+2) - (x+2)^2 = C$

30. $y = Ce^{4x} - x - \dfrac{1}{4}$

32. $y^2 = x^2\ln(x^2) + 16x^2$

34. $y = x^2\sin x + \dfrac{2x^2}{\pi^2}$

36. $\sin(2x+y) - \dfrac{x^3}{3} + e^y = \sin 2 + \dfrac{2}{3}$

38. $y = \left[2 - \left(\dfrac{1}{4}\right)\arctan\left(\dfrac{x}{2}\right)\right]^2$

40. $y = \dfrac{8}{1 - 3e^{-4x} - 4x}$

TABLES

n	x_n	y_n	n	x_n	y_n
1	0.1	1.475	6	0.6	1.353368921
2	0.2	1.4500625	7	0.7	1.330518988
3	0.3	1.425311875	8	0.8	1.308391369
4	0.4	1.400869707	9	0.9	1.287062756
5	0.5	1.376852388	10	1.0	1.266596983

Table 2–A: Euler's approximations to $y' = x - y$, $y(0) = 0$, on $[0, 1]$ with $h = 0.1$.

FIGURES

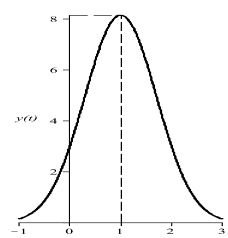

Figure 2–A: The graph of the solution in Problem 28.

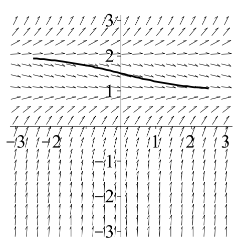

Figure 2–B: The direction field and solution curve in Problem 32.

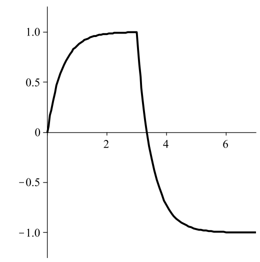

Figure 2–C: The graph of the solution in Problem 32.

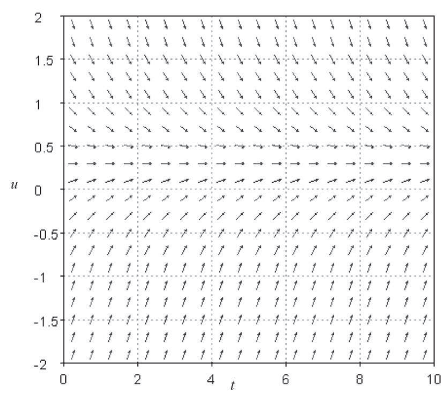

Figure 2–D: The direction field for Problem 40 ($\alpha = \frac{1}{4}, \beta = \frac{1}{2}$).

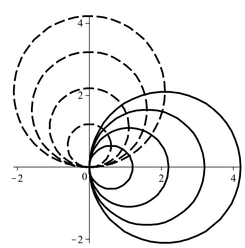

Figure 2–E: Curves and their orthogonal trajectories in Problem 34.