Elementary Number Theory 7th edition Rosen Solutions Manual

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CHAPTER 1

The Integers

1.1. Numbers, Sequences, and Sums

1.1.1. a. The set of integers greater than 3 is well-ordered. Every subset of this set is also a subset of the set of positive integers, and hence must have a least element.

b. The set of even positive integers is well-ordered. Every subset of this set is also a subset of the set of positive integers, and hence must have a least element.

c. The set of positive rational numbers is not well-ordered. This set does not have a least element. If a/b were the least positive rational number then $a/(b+a)$ would be a smaller positive rational number, which is a contradiction.

d. The set of positive rational numbers of the form $a/2$ is well-ordered. Consider a subset of numbers of this form. The set of numerators of the numbers in this subset is a subset of the set of positive integers, so it must have a least element b. Then $b/2$ is the least element of the subset.

e. The set of nonnegative rational numbers is not well-ordered. The set of positive rational numbers is a subset with no least element, as shown in part (c).

1.1.2. Let S be the set of all positive integers of the form $a - bk$. S is not empty since $a - b(-1) = a + b$ is a positive integer. Then the well-ordering principle implies that S has a least element, which is the number we're looking for.

1.1.3. Suppose that x and y are rational numbers. Then $x = a/b$ and $y = c/d$, where a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$. Then $xy = (a/b) \cdot (c/d) = ac/bd$ and $x + y = a/b + c/d = (ad + bc)/bd$ where $bd \neq 0$. Since both $x + y$ and xy are ratios of integers, they are both rational.

1.1.4. a. Suppose that x is rational and y is irrational. Then there exist integers a and b such that $x = \frac{a}{b}$ where a and b are integers with $b \neq 0$. Suppose that $x + y$ is rational. Then there exist integers c and d with $d \neq 0$ such that $x + y = \frac{c}{d}$. This implies that $y = (x + y) - x = (a/b) - (c/d) = (ad - bc)/bd$, which means that y is rational, a contradiction. Hence $x + y$ is irrational.

b. This is false. A counterexample is given by $\sqrt{2} + (-\sqrt{2}) = 0$.

c. This is false. A counterexample is given by $0 \cdot \sqrt{2} = 0$.

d. This is false. A counterexample is given by $\sqrt{2} \cdot \sqrt{2} = 2$.

1.1.5. Suppose that $\sqrt{3}$ were rational. Then there would exist positive integers a and b with $\sqrt{3} = a/b$. Consequently, the set $S = \{k\sqrt{3} \mid k \text{ and } k\sqrt{3} \text{ are positive integers}\}$ is nonempty since $a = b\sqrt{3}$. Therefore, by the well-ordering property, S has a smallest element, say $s = t\sqrt{3}$. We have $s\sqrt{3} - s = s\sqrt{3} - t\sqrt{3} = (s - t)\sqrt{3}$. Since $s\sqrt{3} = 3t$ and s are both integers, $s\sqrt{3} - s = (s - t)\sqrt{3}$ must also be an integer. Furthermore, it is positive, since $s\sqrt{3} - s = s(\sqrt{3} - 1)$ and $\sqrt{3} > 1$. It is less than s since $s = t\sqrt{3}$, $s\sqrt{3} = 3t$, and $\sqrt{3} < 3$. This contradicts the choice of s as the smallest positive integer in S. It follows that $\sqrt{3}$ is irrational.

1.1.6. Let S be a set of negative integers. Then the set $T = \{-s : s \in S\}$ is a set of positive integers. By the well-ordering principle, T has a least element t_0. We prove that $-t_0$ is a greatest element of S. First note that since $t_0 \in S$, then $t_0 = -s_0$ for some $s_0 \in S$. Then $-t_0 = s_0 \in S$. Second, if $s \in S$, then $-s \in T$, so $t_0 \leq -s$. Multiplying by -1 yields $s \leq -t_0$. Since the choice of s was arbitrary, we see that $-t_0$ is greater than or equal to every element of S.

1.1.7. a. Since $0 \leq 1/4 < 1$, we have $[1/4] = 0$.

b. Since $-1 \leq -3/4 < 0$, we have $[-3/4] = -1$.

c. Since $3 \leq 22/7 < 4$, we have $[22/7] = 3$.

d. Since $-2 \leq -2 < -1$, we have $[-2] = -2$.

e. We compute $[1/2 + [1/2]] = [1/2 + 0] = [1/2] = 0$.

f. We compute $[-3 + [-1/2]] = [-3 - 1] = [-4] = -4$.

1.1.8. a. Since $-1 \leq -1/4 < 0$, we have $[-1/4] = -1$.

b. Since $-4 \leq -22/7 < -3$, we have $[-22/7] = -4$.

c. Since $1 \leq 5/4 < 2$, we have $[5/4] = 1$.

d. We compute $[[1/2]] = [0] = 0$.

e. We compute $[[3/2] + [-3/2]] = [1 + (-2)] = [-1] = -1$.

f. We compute $[3 - [1/2]] = [3 - 0] = [3] = 3$.

1.1.9. a. Since $[8/5] = 1$, we have $\{8/5\} = 8/5 - [8/5] = 8/5 - 1 = 3/5$.

b. Since $[1/7] = 0$, we have $\{1/7\} = 1/7 - [1/7] = 1/7 - 0 = 1/7$.

c. Since $[-11/4] = -3$, we have $\{-11/4\} = -11/4 - [-11/4] = -11/4 - (-3) = 1/4$.

d. Since $[7] = 7$, we have $\{7\} = 7 - [7] = 7 - 7 = 0$.

1.1.10. a. Since $[-8/5] = -2$, we have $\{-8/5\} = -8/5 - [-8/5] = -8/5 - (-2) = 2/5$.

b. Since $[22/7] = 3$, we have $\{22/7\} = 22/7 - [22/7] = 22/7 - 3 = 1/7$.

c. Since $[-1] = -1$, we have $\{-1\} = -1 - [-1] = -1 - 1 = 0$.

d. Since $[-1/3] = -1$, we have $\{-1/3\} = -1/3 - [-1/3] = -1/3 - (-1) = 2/3$.

1.1.11. If x is an integer, then $[x] + [-x] = x - x = 0$. Otherwise, $x = z + r$, where z is an integer and r is a real number with $0 < r < 1$. In this case, $[x] + [-x] = [z + r] + [-z - r] = z + (-z - 1) = -1$.

1.1.12. Let $x = [x] + r$ where $0 \leq r < 1$. We consider two cases. First suppose that $r < \frac{1}{2}$. Then $x + \frac{1}{2} = [x] + (r + \frac{1}{2}) < [x] + 1$ since $r + \frac{1}{2} < 1$. It follows that $[x + \frac{1}{2}] = [x]$. Also $2x = 2[x] + 2r < 2[x] + 1$ since $2r < 1$. Hence $[2x] = 2[x]$. It follows that $[x] + [x + \frac{1}{2}] = [2x]$. Next suppose that $\frac{1}{2} \leq r < 1$. Then $[x] + 1 \leq x + (r + \frac{1}{2}) < [x] + 2$, so that $[x + \frac{1}{2}] = [x] + 1$. Also $2[x] + 1 \leq 2[x] + 2r = 2([x] + r) = 2x < 2[x] + 2$ so that $[2x] = 2[x] + 1$. It follows that $[x] + [x + \frac{1}{2}] = [x] + [x] + 1 = 2[x] + 1 = [2x]$.

1.1.13. We have $[x] \leq x$ and $[y] \leq y$. Adding these two inequalities gives $[x] + [y] \leq x + y$. Hence $[x + y] \geq [[x] + [y]] = [x] + [y]$.

1.1.14. Let $x = a+r$ and $y = b+s$, where a and b are integers and r and s are real numbers such that $0 \le r, s < 1$. By Exercise 12, $[2x] + [2y] = [x] + [x + \frac{1}{2}] + [y] + [y + \frac{1}{2}]$. We now need to show that $[x + \frac{1}{2}] + [y + \frac{1}{2}] \ge [x + y]$. Suppose $0 \le r, s < \frac{1}{2}$. Then $[x + \frac{1}{2}] + [y + \frac{1}{2}] = a + b + [r + \frac{1}{2}] + [s + \frac{1}{2}] = a + b$, and $[x + y] = a+b+[r+s] = a+b$, as desired. Suppose that $\frac{1}{2} \le r, s < 1$. Then $[x+\frac{1}{2}]+[y+\frac{1}{2}] = a+b+[r+\frac{1}{2}]+[s+\frac{1}{2}] = a + b + 2$, and $[x + y] = a + b + [r + s] = a + b + 1$, as desired. Suppose that $0 \le r < \frac{1}{2} \le s < 1$. Then $[x + \frac{1}{2}] + [y + \frac{1}{2}] = a + b + 1$, and $[x + y] \le a + b + 1$.

1.1.15. Let $x = a + r$ and $y = b + s$, where a and b are integers and r and s are real numbers such that $0 \le r, s < 1$. Then $[xy] = [ab + as + br + sr] = ab + [as + br + sr]$, whereas $[x][y] = ab$. Thus we have $[xy] \ge [x][y]$ when x and y are both positive. If x and y are both negative, then $[xy] \le [x][y]$. If one of x and y is positive and the other negative, then the inequality could go either direction. For examples take $x = -1.5, y = 5$ and $x = -1, y = 5.5$. In the first case we have $[-1.5 \cdot 5] = [-7.5] = -8 > [-1.5][5] = -2 \cdot 5 = -10$. In the second case we have $[-1 \cdot 5.5] = [-5.5] = -6 < [-1][5.5] = -1 \cdot 5 = -5$.

1.1.16. If x is an integer then $-[-x] = -(-x) = x$, which certainly is the least integer greater than or equal to x. Let $x = a + r$, where a is an integer and $0 < r < 1$. Then $-[-x] = -[-a - r] = -(-a + [-r]) = a - [-r] = a + 1$, as desired.

1.1.17. Let $x = [x] + r$. Since $0 \le r < 1$, $x + \frac{1}{2} = [x] + r + \frac{1}{2}$. If $r < \frac{1}{2}$, then $[x]$ is the integer nearest to x and $[x+\frac{1}{2}] = [x]$ since $[x] \le x+\frac{1}{2} = [x]+r+\frac{1}{2} < [x]+1$. If $r \ge \frac{1}{2}$, then $[x]+1$ is the integer nearest to x (choosing this integer if x is midway between $[x]$ and $[x+1]$) and $[x+\frac{1}{2}] = [x]+1$ since $[x]+1 \le x+r+\frac{1}{2} < [x]+2$.

1.1.18. Let $y = x + n$. Then $[y] = [x] + n$, since n is an integer. Therefore the problem is equivalent to proving that $[y/m] = [[y]/m]$ which is done in Example 1.37.

1.1.19. Let $x = k + \epsilon$ where k is an integer and $0 \le \epsilon < 1$. Further, let $k = a^2 + b$, where a is the largest integer such that $a^2 \le k$. Then $a^2 \le k = a^2 + b \le x = a^2 + b + \epsilon < (a + 1)^2$. Then $[\sqrt{x}] = a$ and $[\sqrt{[x]}] = [\sqrt{k}] = a$ also, proving the theorem.

1.1.20. Let $x = k + \epsilon$ where k is an integer and $0 \le \epsilon < 1$. Choose w from $0, 1, 2, \ldots, m - 1$ such that $w/m \le \epsilon < (w + 1)/m$. Then $w \le m\epsilon < w + 1$. Then $[mx] = [mk + m\epsilon] = mk + [m\epsilon] = mk + w$. On the other hand, the same inequality gives us $(w + j)/m \le \epsilon + j/m < (w + 1 + j)/m$, for any integer $j = 0, 1, 2, \ldots, m - 1$. Note that this implies $[\epsilon + j/m] = [(w + j)/m]$ which is either 0 or 1 for j in this range. Indeed, it equals 1 precisely when $w+j \ge m$, which happens for exactly w values of j in this range. Now we compute $\sum_{j=0}^{m-1}[x + j/m] = \sum_{j=0}^{m-1}[k + \epsilon + j/m] = \sum_{j=0}^{m-1} k + [\epsilon + j/m] = mk + \sum_{j=0}^{m-1}[(w + j)/m] = mk + \sum_{j=m-w}^{m-1} 1 = mk + w$ which is the same as the value above.

1.1.21. a. Since the difference between any two consecutive terms of this sequence is 8, we may compute the nth term by adding 8 to the first term $n - 1$ times. That is, $a_n = 3 + (n - 1)8 = 8n - 5$.

b. For each n, we have $a_n - a_{n-1} = 2^{n-1}$, so we may compute the nth term of this sequence by adding all the powers of 2, up to the $(n - 1)$th, to the first term. Hence $a_n = 5 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 5 + 2^n - 2 = 2^n + 3$.

c. The nth term of this sequence appears to be zero, unless n is a perfect square, in which case the term is 1. If n is not a perfect square, then $[\sqrt{n}] < \sqrt{n}$, where $[x]$ represents the greatest integer function. If n is a perfect square, then $[\sqrt{n}] = \sqrt{n}$. Therefore, $[[\sqrt{n}]/\sqrt{n}]$ equals 1 if n is a perfect square and 0 otherwise, as desired.

d. This is a Fibonacci-like sequence, with $a_n = a_{n-1} + a_{n-2}$, for $n \ge 3$, and $a_1 = 1$, and $a_2 = 3$.

1.1.22. a. Each term given is 3 times the preceding term, so we conjecture that the nth term is the first term multiplied by 3, $n - 1$ times. So $a_n = 2 \cdot 3^{n-1}$.

b. In this sequence, $a_n = 0$ if n is a multiple of 3, and equals 1 otherwise. Let $[x]$ represent the greatest integer function. Since $[n/3] < n/3$ when n is not a multiple of 3 and $[n/3] = n/3$ when n is a multiple of 3, we have that $a_n = 1 - [[n/3]/(n/3)]$.

c. If we look at the difference of successive terms, we have the sequence $1, 1, 2, 2, 3, 3, \ldots$. So if n is odd, say $n = 2k+1$, then a_n is obtained by adding $1+1+2+2+3+3+\cdots+k+k = 2t_k$ to the first term, which is 1. (Here t_k stands for the kth triangular number.) So if n is odd, then $a_n = 1 + 2t_k$ where $k = (n-1)/2$. If n is even, say $n = 2k$, then $a_n = a_{2k+1} - k = 1 - k + 2t_k$.

d. This is a Fibonacci-like sequence, with $a_n = a_{n-1} + 2a_{n-2}$, for $n \geq 3$, and $a_1 = 3$, and $a_2 = 5$.

1.1.23. Three possible answers are $a_n = 2^{n-1}$, $a_n = (n^2 - n + 2)/2$, and $a_n = a_{n-1} + 2a_{n-2}$.

1.1.24. Three possible answers are $a_n = a_{n-1}a_{n-2}$, $a_n = a_{n-1} + 2n - 3$, and $a_n =$ the number of letters in the nth word of the sentence "If our answer is correct we will join the Antidisestablishmentarianism Society and boldly state that 'If our answer is correct we will join the Antidisestablishmentarianism Society and boldly state....' "

1.1.25. This set is exactly the sequence $a_n = n - 100$, and hence is countable.

1.1.26. The function $f(n) = 5n$ is a one-to-one correspondence between this set and the set of integers, which is known to be countable.

1.1.27. One way to show this is to imitate the proof that the set of rational numbers is countable, replacing a/b with $a + b\sqrt{2}$. Another way is to consider the function $f(a + b\sqrt{2}) = 2^a 3^b$ which is a one-to-one map of this set into the rational numbers, which is known to be countable.

1.1.28. Let A and B be two countable sets. If one or both of the sets are finite, say A is finite, then the listing $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots$, where any b_i which is also in A is deleted from the list, demonstrates the countability of $A \cup B$. If both sets are infinite, then each can be represented as a sequence: $A = \{a_1, a_2, \ldots\}$, and $B = \{b_1, b_2, \ldots\}$. Consider the listing $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$ and form a new sequence c_i as follows. Let $c_1 = a_1$. Given that c_n is determined, let c_{n+1} be the next element in the listing which is different from each c_i with $i = 1, 2, \ldots, n$. Then this sequence is exactly the elements of $A \cup B$, which is therefore countable.

1.1.29. Suppose $\{A_i\}$ is a countable collection of countable sets. Then each A_i can be represented by a sequence, as follows:

$$\begin{array}{rcllll} A_1 & = & a_{11} & a_{12} & a_{13} & \ldots \\ A_2 & = & a_{21} & a_{22} & a_{23} & \ldots \\ A_3 & = & a_{31} & a_{32} & a_{33} & \ldots \\ & \vdots & & & & \end{array}$$

Consider the listing $a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots$, in which we first list the elements with subscripts adding to 2, then the elements with subscripts adding to 3 and so on. Further, we order the elements with subscripts adding to k in order of the first subscript. Form a new sequence c_i as follows. Let $c_1 = a_1$. Given that c_n is determined, let c_{n+1} be the next element in the listing which is different from each c_i with $i = 1, 2, \ldots, n$. Then this sequence is exactly the elements of $\bigcup_{i=1}^{\infty} A_i$, which is therefore countable.

1.1.30. a. Note that $\sqrt{2} \approx 1.4 = 7/5$, so we might guess that $\sqrt{2} - 7/5 \approx 0$. If we multiply through by 5 we expect that $5\sqrt{2} - 7$ should be small, and its value is approximately 0.071 which is much less than $1/8 = 0.125$. So we may take $a = 5 \leq 8$ and $b = 7$.

b. As in part (a), note that $\sqrt[3]{2} = 1.2599\ldots \approx 1.25 = 5/4$, so we investigate $4\sqrt[3]{2} - 5 = 0.039\ldots \leq 1/8$. So we may take $a = 4 \leq 8$ and $b = 5$.

c. Since we know that $\pi \approx 22/7$ we investigate $|7\pi - 22| = 0.0088\ldots \leq 1/8$. So we may take $a = 7 \leq 8$ and $b = 22$.

d. Since $e \approx 2.75 = 11/4$ we investigate $|4e - 11| = 0.126\ldots$, which is too large. A closer approximation to e is 2.718. We consider the decimal expansions of the multiples of $1/7$ and find that $5/7 = .714\ldots$, so $e \approx 19/7$. Therefore we investigate $|7e - 19| = 0.027 \leq 1/8$. So we may take $a = 7 \leq 8$ and $b = 19$.

1.1.31. a. Note that $\sqrt{3} = 1.73 \approx 7/4$, so we might guess that $\sqrt{3} - 7/4 \approx 0$. If we multiply through by 4 we find that $|4\sqrt{3} - 7| = 0.07\ldots < 1/10$. So we may take $a = 4 \leq 10$ and $b = 7$.

b. It is helpful to keep the decimal expansions of the multiples of $1/7$ in mind in these exercises. Here $\sqrt[3]{3} = 1.442\ldots$ and $3/7 = 0.428\ldots$ so that we have $\sqrt[3]{3} \approx 10/7$. Then as in part (a), we investigate $|7\sqrt[3]{3} - 10| = 0.095\ldots < 1/10$. So we may take $a = 7 \leq 10$ and $b = 10$.

c. Since $\pi^2 = 9.869\ldots$ and $6/7 = 0.857\ldots$, we have that $\pi^2 \approx 69/7$, so we compute $|7\pi^2 - 69| = 0.087\ldots < 1/10$. So we may take $a = 7 \leq 10$ and $b = 69$.

d. Since $e^3 = 20.0855\ldots$ we may take $a = 1$ and $b = 20$ to get $|1e^3 - 20| = 0.855\ldots < 1/10$.

1.1.32. For $j = 0, 1, 2, \ldots, n+1$, consider the $n+2$ numbers $\{j\alpha\}$, which all lie in the interval $0 \leq \{j\alpha\} < 1$. We can partition this interval into the $n+1$ subintervals $(k-1)/(n+1) \leq x < k/(n+1)$ for $k = 1, \ldots, n+1$. Since we have $n+2$ numbers and only $n+1$ intervals, by the pigeonhole principle, some interval must contain at least two of the numbers. So there exist integers r and s such that $1 \leq r < s \leq n+1$ and $|\{r\alpha\} - \{s\alpha\}| \leq 1/(n+1)$. Let $a = s - r$ and $b = [s\alpha] - [r\alpha]$. Since $1 \leq r < s \leq n+1$, we have $1 \leq a \leq n$. Also, $|a\alpha - b| = |(s-r)\alpha - ([s\alpha] - [r\alpha])| = |(s\alpha - [s\alpha]) - (r\alpha - [r\alpha]a)| = |\{s\alpha\} - \{r\alpha\}| < 1/(n+1)$. Therefore, a and b have the desired properties.

1.1.33. The number α must lie in some interval of the form $r/k \leq \alpha < (r+1)/k$. If we divide this interval into equal halves, then α must lie in one of the halves, so either $r/k \leq \alpha < (2r+1)/2k$ or $(2r+1)/2k \leq \alpha < (r+1)/k$. In the first case we have $|\alpha - r/k| < 1/2k$, so we take $u = r$. In the second case we have $|\alpha - (r+1)/k| < 1/2k$, so we take $u = r+1$.

1.1.34. Suppose that there are only finitely many positive integers $q_1, q_2, \ldots, q_n$ with corresponding integers $p_1, p_2, \ldots, p_n$ such that $|\alpha - p_i/q_i| < 1/q_i^2$. Since α is irrational, $|\alpha - p_i/q_i|$ is positive for every i, and so is $|q_i\alpha - p_i|$ so we may choose an integer N so large that $|q_i\alpha - p_i| > 1/N$ for all i. By Dirichlet's Approximation Theorem, there exist integers r and s with $1 \leq s \leq N$ such that $|s\alpha - r| < 1/N < 1/s$, so that $|\alpha - r/s| < 1/s^2$, and s is not one of the q_i. Therefore, we have another solution to the inequality. So no finite list of solutions can be complete, and we conclude that there must be an infinite number of solutions.

1.1.35. First we have $|\sqrt{2} - 1/1| = 0.414\ldots < 1/1^2$. Second, Exercise 30, part (a), gives us $|\sqrt{2} - 7/5| < 1/50 < 1/5^2$. Third, observing that $3/7 = 0.428\ldots$ leads us to try $|\sqrt{2} - 10/7| = 0.014\ldots < 1/7^2 = 0.0204\ldots$. Fourth, observing that $5/12 = 0.4166\ldots$ leads us to try $|\sqrt{2} - 17/12| = 0.00245\ldots < 1/12^2 = 0.00694\ldots$.

1.1.36. First we have $|\sqrt[3]{5} - 1/1| = 0.7099\ldots < 1/1^2$. Second, $|\sqrt[3]{5} - 5/3| = 0.04\ldots < 1/3^2$. Third, since $\sqrt[3]{5} = 1.7099\ldots$, we try $|\sqrt[3]{5} - 17/10| = 0.0099\ldots < 1/10^2$. Likewise, we get a fourth rational number with $|\sqrt[3]{5} - 171/100| = 0.000024\ldots < 1/100^2$. Fifth, consideration of multiples of $1/7$ leads to $|\sqrt[3]{5} - 12/7| = 0.0043\ldots < 1/7^2$.

1.1.37. We may assume that b and q are positive. Note that if $q > b$, we have $|p/q - a/b| = |pb - aq|/qb \geq 1/qb > 1/q^2$. Therefore, solutions to the inequality must have $1 \leq q \leq b$. For a given q, there can be only finitely many p such that the distance between the rational numbers a/b and p/q is less than $1/q^2$ (indeed there is at most one.) Therefore there are only finitely many p/q satisfying the inequality.

1.1.38. a. Since $n2$ is an integer for all n, so is $[n2]$, so the first ten terms of the spectrum sequence are 2, 4, 6, 8, 10, 12, 14, 16, 18, 20.

b. The sequence for $n\sqrt{2}$, rounded, is 1.414, 2.828, 4.242, 5.656, 7.071, 8.485, 9.899, 11.314, 12.728, 14.142. When we apply the floor function to these numbers we get 1, 2, 4, 5, 7, 8, 9, 11, 12, 14 for the spectrum sequence.

c. The sequence for $n(2+\sqrt{2})$, rounded, is 3.414, 6.828, 10.24, 13.66, 17.07, 20.48, 23.90, 27.31, 30.73, 34.14. When we apply the floor function to these numbers we get 3, 6, 10, 13, 17, 20, 23, 27, 30, 34, for the spectrum sequence.

d. The sequence for ne, rounded is 2.718, 5.436, 8.155, 10.87, 13.59, 16.31, 19.03, 21.75, 24.46, 27.18. When we apply the floor function to these numbers we get 2, 5, 8, 10, 13, 16, 19, 21, 24, 27, for the spectrum sequence.

e. The sequence for $n(1+\sqrt{5})/2$, rounded, is 1.618, 3.236, 4.854, 6.472, 8.090, 9.708, 11.33, 12.94, 14.56, 16.18. When we apply the floor function to these numbers we get 1, 3, 4, 6, 8, 9, 11, 12, 14, 16 for the spectrum sequence.

1.1.39. a. Since $n3$ is an integer for all n, so is $[n3]$, so the first ten terms of the spectrum sequence are 3, 6, 9, 12, 15, 18, 21, 24, 27, 30.

b. The sequence for $n\sqrt{3}$, rounded, is 1.732, 3.464, 5.196, 6.928, 8.660, 10.39, 12.12, 13.86, 15.59, 17.32. When we apply the floor function to these numbers we get 1, 3, 5, 6, 8, 10, 12, 13, 15, 17 for the spectrum sequence.

c. The sequence for $n(3+\sqrt{3})/2$, rounded, is 2.366, 4.732, 7.098, 9.464, 11.83, 14.20, 16.56, 18.93, 21.29, 23.66. When we apply the floor function to these numbers we get 2, 4, 7, 9, 11, 14, 16, 18, 21, 23 for the spectrum sequence.

d. The sequence for $n\pi$, rounded is 3.142, 6.283, 9.425, 12.57, 15.71, 18.85, 21.99, 25.13, 28.27, 31.42. When we apply the floor function to these numbers we get 3, 6, 9, 12, 15, 18, 21, 25, 28, 31, for the spectrum sequence.

1.1.40. Since $\alpha \neq \beta$, their decimal expansions must be different. If they differ in digits that are to the left of the decimal point, then $[\alpha] \neq [\beta]$, so certainly the spectrum sequences are different. Otherwise, suppose that they differ in the kth position to the right of the decimal. Then $[10^k\alpha] \neq [10^k\beta]$, and so the spectrum sequences will again differ.

1.1.41. Assume that $1/\alpha + 1/\beta = 1$. Note first that for all integers n and m, $m\alpha \neq n\beta$, for otherwise, we solve the equations $m\alpha = n\beta$ and $1/\alpha + 1/\beta = 1$ and get rational solutions for α and β, a contradiction. Therefore the sequences $m\alpha$ and $n\beta$ are disjoint.

For an integer k, define $N(k)$ to be the number of elements of the sequences $m\alpha$ and $n\beta$ which are less than k. Now $m\alpha < k$ if and only if $m < k/\alpha$, so there are exactly $[k/\alpha]$ members of the sequence $m\alpha$ less than k. Likewise, there are exactly $[k/\beta]$ members of the sequence $n\beta$ less than k. So we have $N(k) = [k/\alpha] + [k/\beta]$. By definition of the greatest integer function, we have $k/\alpha - 1 < [k/\alpha] < k/\alpha$ and $k/\beta - 1 < [k/\beta] < k/\beta$, where the inequalities are strict because the numbers are irrational. If we add these inequalities we get $k/\alpha + k/\beta - 2 < N(k) < k/\alpha + k/\beta$ which simplifies to $k - 2 < N(k) < k$. Since $N(k)$ is an integer, we conclude that $N(k) = k - 1$. This shows that there is exactly one member of the union of the sequences $m\alpha$ and $n\beta$ in each interval of the form $k - 1 \leq x < k$, and therefore, when we apply the floor function to each member, exactly one will take on the value k.

Conversely, suppose that α and β are irrational numbers such that $1/\alpha + 1/\beta \neq 1$. If $1/\alpha + 1/\gamma = 1$ then we know from the first part of the theorem that the spectrum sequences for α and γ partition the positive integers. By Exercise 40, we know that the spectrum sequences for β and γ are different, so the sequences for α and β can not partition the positive integers.

1.1.42. The first two Ulam numbers are 1 and 2. Since $3 = 1 + 2$, it is the third Ulam number and since $4 = 1 + 3$, it is the fourth Ulam number. Note that 5 is not an Ulam number since $5 = 1 + 4 = 2 + 3$. The fifth Ulam number is 6 since $6 = 4 + 2$ and no other two Ulam numbers have 6 as their sum. We have $7 = 4 + 3 = 6 + 1$, so 7 is not an Ulam number. The sixth Ulam number is $8 = 6 + 2$. Note that $9 = 8 + 1 = 6+3$ and $10 = 8+2 = 4+6$ so neither 9 nor 10 is an Ulam number. The seventh Ulam number is 11 since $11 = 8 + 3$ is the unique way to write 11 as the sum of two distinct Ulam numbers. Next note that $12 = 8 + 4 = 1 + 11$ so that 12 is not an Ulam number. Note that $13 = 11 + 2$ is the unique way to write 13 as the eighth Ulam number. We see that $14 = 13+1 = 11+3$ and $15 = 2+13 = 4+11$, so that neither 14 nor 15 are Ulam numbers. We note that $16 = 3 + 13$ is the unique way to write 16 as the sum of two Ulam numbers, so that the ninth Ulam number is 16. Note that $17 = 1 + 16 = 4 + 13$ so that 17 is not an Ulam number. Note that $18 = 2 + 16$ is the unique way to write 18 as the sum of two Ulam numbers so that 18 is the tenth Ulam number. In summary, the first ten Ulam numbers are: 1, 2, 3, 4, 6, 8, 11, 13, 16, 18.

1.1.43. Assume that there are only finitely many Ulam numbers. Let the two largest Ulam numbers be u_{n-1} and u_n. Then the integer $u_n + u_{n-1}$ is an Ulam number larger than u_n. It is the unique sum of two Ulam numbers since $u_i + u_j < u_n + u_{n-1}$ if $j < n$ or $j = n$ and $i < n - 1$.

1.1.44. Suppose that e is rational so that $e = a/b$ where a and b are integers and $b \neq 0$. Let $k \geq b$ be an integer and set $c = k!(e - 1 - 1/1! - 1/2! - 1/3! - \cdots - 1/k!)$. Since every denominator in the expression divides evenly into $k!$, we see that c is an integer. Since $e = 1 + 1/1! + 1/2! + \cdots$, we have $0 < c = k!(1/(k+1)! + 1/(k+2)! + \cdots) = 1/(k+1) + 1/(k+1)(k+2) + \cdots < 1/(k+1) + 1/(k+1)^2 + \cdots$. This last geometric series is equal to $1/k$, so we have that $0 < c < 1/k$, which is impossible since c is an integer. Therefore e must be irrational.

1.1.45. To get a contradiction, suppose that the set of real numbers is countable. Then the subset of real numbers strictly between 0 and 1 is also countable. Then there is a one-to-one correspondence $f : \mathbb{Z}^+ \to (0,1)$. Each real number $b \in (0,1)$ has a decimal representation of the form $b = 0.b_1b_2b_3\ldots$, where b_i is the ith digit after the decimal point. For each $k = 1, 2, 3, \ldots$, Let $f(k) = a_k \in (0,1)$. Then each a_k has a decimal representation of the form $a_k = a_{k1}a_{k2}a_{k3}\ldots$. Form the real number $c = c_1c_2c_3\ldots$ as follows: If $a_{kk} = 5$, then let $c_k = 4$. If $a_{kk} \neq 5$, then let $c_k = 5$. Then $c \neq a_k$ for every k because it differs in the kth decimal place. Therefore $f(k) \neq c$ for all k, and so f is not a one-to-one correspondence. This gives us our contradiction, and so we conclude that the real numbers are uncountable.

1.1.46. The first term, a_0, is 0. For a_1, attempting to calculate $a_1 = a_0 - 1$ yields a negative value, so instead $a_1 = a_0 + 1 = 1$. For a_2 and subsequently a_3, $a_{n-1} - n$ would likewise yield nonpositive results, so $a_2 = 1 + 2 = 3$ and $a_3 = 3 + 3 = 6$. The a_4 term is the first for which $a_{n-1} - n$ is positive and $a_4 = 6 - 4 = 2$. Then $a_5 = 2 + 5 = 7$, since subtracting would result in a negative result. The first time that the non-repetition condition plays a role is a_6, since computing $a_5 - 6 = 7 - 6 = 1$ already appears as a_1. So instead we calculate $a_6 = a_5 + 6 = 13$. Similarly, $a_6 - 7 = 6$ would equal a_3, so instead $a_7 = 13 + 7 = 20$. Continuing, we obtain $a_8 = 12$, $a_9 = 21$, and $a_{10} = 11$.

1.1.47. Two proofs are provided here. The first uses the well-ordering property, while the second looks ahead to Section 3 and uses mathematical induction.

Well-ordering: Suppose, for the sake of contradiction, that the statement is false. Then there is an index n for which $a_n > \frac{n(n+1)}{2}$. Note that $a_0 = 0 \leq 0(0+1)/2$ and the inequality holds for $n = 0$. Let $S = \left\{ n \in \mathbf{Z}^+ \mid a_n > \frac{n(n+1)}{2} \right\}$ be the set of indices for which the inequality fails. By hypothesis, S is a nonempty set of positive integers, and thus by the well-ordering property, contains a smallest member, say k. Since k is positive, $k - 1$ is nonnegative and so an index at which the sequence is defined. Moreover k is smallest for which the inequality fails, and so a_{k-1} satisfies $a_{k-1} \leq \frac{(k-1)k}{2}$. But now, either $a_k = a_{k-1} - k$ or $a_k = a_{k-1} + k$. In either case, $a_k \leq a_{k-1} + k \leq \frac{(k-1)k}{2} + k = \frac{(k-1)k+2k}{2} = \frac{k(k+1)}{2}$. In other words, a_k does satisfy the inequality after all, which is the needed contradiction.

Induction: We use mathematical induction. The basis step is $n = 0$, and indeed $a_0 = 0 \leq 0 = \frac{0(0+1)}{2}$. The inductive hypothesis is that k is a nonnegative integer for which $a_k \leq \frac{k(k+1)}{2}$. The value of a_{k+1} is either $a_k-(k+1)$ or $a_k+(k+1)$. In either case, we know that $a_{k+1}\le a_k+(k+1)\le \frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}$, where the second inequality is the inductive hypothesis.

1.1.48. Continuing the computations begun in Exercise 46, one sees that the statement is false since, for example, 42 appears as both a_{20} and a_{24}. (The number 42 is the value with the earliest second appearance; 43 is earlier in its first appearance and later in its second at a_{18} and a_{26}.)

1.2. Sums and Products

1.2.1. a. We have $\displaystyle\sum_{j=1}^{5} j^2 = 1^2+2^2+3^2+4^2+5^2=55.$

b. We have $\displaystyle\sum_{j=1}^{5}(-3)=(-3)+(-3)+(-3)+(-3)+(-3)=-15.$

c. We have $\displaystyle\sum_{j=1}^{5}1/(j+1)=1/2+1/3+1/4+1/5+1/6=29/20.$

1.2.2. a. We have $\displaystyle\sum_{j=0}^{4}3=3+3+3+3+3=15.$

b. We have $\displaystyle\sum_{j=0}^{4}(j-3)=(-3)+(-2)+(-1)+0+1=-5.$

c. We have $\displaystyle\sum_{j=0}^{4}(j+1)/(j+2)=1/2+2/3+3/4+4/5+5/6=71/20.$

1.2.3. a. We use the formula from Example 1.16 as follows. We evaluate the sum $\displaystyle\sum_{j=0}^{8}2^j=2^9-1=511$ as in Example 1.18. Then we have $\displaystyle\sum_{j=1}^{8}2^j=\sum_{j=0}^{8}2^j-2^0=510.$

b. We could proceed as in part (a), or we may do the following: $\displaystyle\sum_{j=1}^{8}5(-3)^j=\sum_{j=0}^{7}5(-3)^{j+1}$ $\displaystyle=\sum_{j=0}^{7}-15(-3)^j$. We may apply the formula in Example 1.16 to this last sum, with $a=-15, n=7$ and $r=-3$, to get the sum equal to $\dfrac{-15(-3)^8-(-15)}{-3-1}=24600.$

c. We manipulate the sum as in part (b), so we can apply the formula from Example 1.16: $\displaystyle\sum_{j=1}^{8}3(-1/2)^j=$ $\displaystyle\sum_{j=0}^{7}3(-1/2)^{j+1}=\sum_{j=0}^{7}(-3/2)(-1/2)^j=\frac{(-3/2)(-1/2)^8-(-3/2)}{-1/2-1}=-\frac{255}{256}.$

1.2.4. a. We have $\displaystyle\sum_{j=0}^{10}8\cdot 3^j=\frac{8\cdot 3^{11}-8}{3-1}=708584$, using the formula from Example 1.16 with $a=8, n=10$ and $r=3$.

b. We have $\displaystyle\sum_{j=0}^{10}(-2)^{j+1}=\sum_{j=0}^{10}(-2)(-2)^j=\frac{(-2)\cdot(-2)^{11}-(-2)}{(-2)-1}=-1366$, using the formula from Example 1.16 with $a=-2, n=10$ and $r=-2$.

c. We have $\sum_{j=0}^{10}(1/3)^j = \frac{(1/3)^{11}-1}{(1/3)-1} = \frac{88573}{59049}$, using the formula from Example 1.16 with $a = 1, n = 10$ and $r = (1/3)$.

1.2.5. The sum $\sum_{k=1}^{n}[\sqrt{k}]$ counts 1 for every value of k with $\sqrt{k} \geq 1$. There are n such values of k in the range $k = 1, 2, 3, \ldots, n$. It counts another 1 for every value of k with $\sqrt{k} \geq 2$. There are $n-3$ such values in the range. The sum counts another 1 for each value of k with $\sqrt{k} \geq 3$. There are $n-8$ such values in the range. In general, for $m = 1, 2, 3, \ldots, [\sqrt{n}]$ the sum counts a 1 for each value of k with $\sqrt{k} \geq m$, and there are $n - (m^2-1)$ values in the range. Therefore $\sum_{k=1}^{n}[\sqrt{k}] = \sum_{m=1}^{[\sqrt{n}]} n - (m^2-1) = [\sqrt{n}](n+1) - \sum_{m=1}^{[\sqrt{n}]} m^2 = [\sqrt{n}](n+1) - ([\sqrt{n}]([\sqrt{n}]+1)(2[\sqrt{n}]+1))/6$.

1.2.6. We see that $t_n = \sum_{j=1}^{n} j$, and $t_{n-1} = \sum_{j=1}^{n-1} j = \sum_{j=1}^{n-1}(n-j)$. Now, $t_{n-1} + t_n = \sum_{j=1}^{n-1}(n-j+j) + n = n(n-1) + n = n^2$.

1.2.7. The total number of dots in the n by $n+1$ rectangle, namely $n(n+1)$ is $2t_n$ since the rectangle is made from two triangular arrays. Dividing both sides by 2 gives the desired formula.

1.2.8. By the closed formula for the nth triangular number, we have $3t_n + t_{n-1} = 3(n(n+1)/2) + (n-1)(n-1+1)/2 = 3n(n+1)/2 + n(n-1)/2 = (3n^2+3n+n^2-n)/2 = (4n^2+2n)/2 = 2n(2n+1)/2 = t_{2n}$, as desired.

1.2.9. By the closed formula for the nth triangular number, we have $t_{n+1}^2 - t_n^2 = ((n+1)(n+1+1)/2)^2 - (n(n+1)/2)^2 = (n+1)^2((n+2)^2/4 - n^2/4) = (n+1)^2(n^2+4n+4-n^2)/4 = (n+1)^2(4n+4)/4 = (n+1)^3$, as desired.

1.2.10. It is clear that $p_1 = 1$. Suppose we know p_{k-1}. To compute p_k we consider k nested pentagons as in the figure. Note that $p_k - p_{k-1}$ counts the number of dots on three sides of the outer pentagon. Each side consists of k dots, but two of the dots belong to two sides. Therefore $p_k - p_{k-1} = 3k-2$, which is the formula desired. Then $p_n = 3n-2+p_{n-1} = 3n-2+3(n-1)-2+p_{n-2} = 3n-2+3(n-1)-2+3(n-2)-2+p_{n-3} = \cdots = 3n-2+3(n-1)-2+\cdots+3(1)-2 = \sum_{k=1}^{n}(3k-2)$. Evaluating this sums gives us $p_n = \sum_{k=1}^{n}(3k-2) = 3\sum_{k=1}^{n} - 2\sum_{k=1}^{n} 1 = 3t_n - 2n = 3n(n+1)/2 - 2n = (3n^2+3n-4n)/2 = (3n^2-n)/2$.

1.2.11. By Exercise 10, we have $p_n = (3n^2-n)/2$. On the other hand, $t_{n-1} + n^2 = (n-1)n/2 + n^2 = (3n^2-n)/2$, which is the same as above.

1.2.12. a. Consider a regular hexagon which we border successively by hexagons with $3, 4, 5, \ldots$ on each side. Define the *hexagonal number* h_k to be the number of dots contained in the k nested hexagons.

b. First note that $h_1 = 1$. To get a recursive relationship we consider $h_k - h_{k-1}$, which counts the dots added to the $(k-1)$st hexagon to obtain the kth hexagon. To do this, we must add 4 sides of k dots each, but 3 of the dots belong to two sides. Therefore $h_k - h_{k-1} = 4k-3$. A closed formula is then given by adding these differences together: $h_k = \sum_{i=1}^{k}(4i-3) = 4t_k - 3k = 4k(k+1)/2 - 3k = 2k^2 - k$.

1.2.13. a. Consider a regular heptagon which we border successively by heptagons with $3, 4, 5, \ldots$ on each side. Define the *heptagonal numbers* $s_1, s_2, s_3, \ldots, s_k, \ldots$ to be the number of dots contained in the k nested heptagons.

b. First note that $s_1 = 1$. To get a recursive relationship we consider $s_k - s_{k-1}$, which counts the dots added to the $(k-1)$st heptagon to obtain the kth heptagon. To do this, we must add 5 sides of k dots each, but 4 of the dots belong to two sides. Therefore $s_k - s_{k-1} = 5k-4$. A closed formula is then given by adding these differences together: $s_k = \sum_{i=1}^{k}(5i-4) = 5t_k - 4k = 5k(k+1)/2 - 4k = (5k^2-3k)/2$.

1.2.14. By Exercise 12 we have $h_n = 2n^2 - n$. Also, $t_{2n-1} = (2n-1)(2n-1+1)/2 = n(2n-1) = 2n^2 - n = h_n$.

1.2.15. By Exercise 10 we have $p_n = (3n^2 - n)/2$. Also, $t_{3n-1}/3 = (1/3)(3n-1)(3n)/2 = (3n-1)(n)/2 = (3n^2 - n)/2 = p_n$.

1.2.16. First consider the difference $T_k - T_{k-1}$. This counts the number of dots on one face of the kth tetrahedron. But this is simply the kth nested triangle used to define the triangular numbers. Therefore, $T_k - T_{k-1} = t_k$. Hence, since $T_1 = t_1 = 1$, it follows that $T_n = \sum_{k=1}^{n} t_k$.

1.2.17. We continue with the formula from Exercise 16. $T_n = \sum_{k=1}^{n} t_k = \sum_{k=1}^{n} k(k+1)/2$. Exploiting the same technique as in Example 1.20, we consider $(k+1)^3 - k^3 = 3k^2 + 3k + 1 = 3(k^2 + k) + 1$ and solve for $k^2 + k$ to get $k^2 + k = ((k+1)^3 - k^3)/3 - (1/3)$. Then $T_n = (1/2)\sum_{k=1}^{n} k(k+1) = (1/6)\sum_{k=1}^{n}((k+1)^3 - k^3) - (1/6)\sum_{k=1}^{n} 1$. The first sum is telescoping and the second sum is trivial, so we have $T_n = (1/6)((n+1)^3 - 1^3) - (n/6) = (n^3 + 3n^2 + 2n)/6$.

1.2.18. Using the fact $n! = n \cdot (n-1)!$, we find that $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$, $8! = 40320$, $9! = 362880$, and $10! = 3628800$.

1.2.19. Each of these four quantities are products of 100 integers. The largest product is 100^{100}, since it is the product of 100 factors of 100. The second largest is 100! which is the product of the integers $1, 2, \ldots, 100$, and each of these terms is less or equal to 100. The third largest is $(50!)^2$ which is the product of $1^2, 2^2, \ldots, 50^2$, and each of these factors j^2 is less than $j(50+j)$, whose product is 100!. The smallest is 2^{100} which is the product of 100 2's.

1.2.20. a. $\displaystyle\prod_{i=1}^{n} ka_i = k^n \prod_{i=1}^{n} a_i.$

b. $\displaystyle\prod_{i=1}^{n} ia_i = (a_1)(2a_2)\cdots(na_n) = (1 \cdot 2 \cdots n)(a_1 a_2 \cdots a_n) = n! \prod_{i=1}^{n} a_i.$

c. $\displaystyle\prod_{i=1}^{n} a_i^k = \left(\prod_{i=1}^{n} a_i\right)^k.$

1.2.21. $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{k(k+1)}\right) = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$. Let $a_j = 1/(j+1)$. Notice that this is a telescoping sum, as in Example 1.20. Therefore, we have $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{k(k+1)}\right) = \sum_{j=1}^{n}(a_{j-1} - a_j) = a_0 - a_n = 1 - 1/(n+1) = n/(n+1)$.

1.2.22. $\displaystyle\sum_{k=2}^{n}\frac{1}{k^2-1} = \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{2}\sum_{k=2}^{n}\left(\left(\frac{1}{k-1} - \frac{1}{k}\right) + \left(\frac{1}{k} - \frac{1}{k+1}\right)\right) =$
$\displaystyle\frac{1}{2}\sum_{k=2}^{n}(\frac{1}{k-1} - \frac{1}{k}) + \frac{1}{2}\sum_{k=2}^{n}(\frac{1}{k} - \frac{1}{k+1}) = \frac{1}{2}(1 - \frac{1}{n}) + \frac{1}{2}(\frac{1}{2} - \frac{1}{n+1}) = \frac{3}{4} - \frac{2n+1}{2n(n+1)}.$

1.2.23. We sum both sides of the identity $(k+1)^3 - k^3 = 3k^2 + 3k + 1$ from $k = 1$ to $k = n$. $\sum_{k=1}^{n}((k+1)^3 - k^3) = (n+1)^3 - 1$, since the sum is telescoping. $\sum_{k=1}^{n}(3k^2 + 3k + 1) = 3(\sum_{k=1}^{n} k^2) + 3(\sum_{k=1}^{n} k) + \sum_{k=1}^{n} 1 = 3(\sum_{k=1}^{n} k^2) + 3n(n+1)/2 + n$. As these two expressions are equal, solving for $\sum_{k=1}^{n} k^2$, we find that $\sum_{k=1}^{n} k^2 = (n/6)(2n+1)(n+1)$.

1.2.24. We sum both sides of the identity $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$ from $k = 1$ to $k = n$. Using Exercise 23 we find that $\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$.

1.2.25. a. $10! = (7!)(8 \cdot 9 \cdot 10) = (7!)(720) = (7!)(6!)$

b. $10! = (7!)(6!) = (7!)(5!) \cdot 6 = (7!)(5!)(3!)$

c. $16! = (14!)(15 \cdot 16) = (14!)(240) = (14!)(5!)(2!)$

d. $9! = (7!)(8 \cdot 9) = (7!)(6 \cdot 6 \cdot 2) = (7!)(3!)(3!)(2!)$

1.2.26. Since $c = a_1!a_2! \cdots a_n!$ and $b = (a_1!a_2! \cdots a_n!) - 1$, it follows that $c! = c \cdot (c-1)! = c \cdot b! = a_1!a_2! \cdots a_n! \cdot b!$.

1.2.27. Assume that $x \le y$. Then $z! = x! + y! \le y! + y! = 2(y!)$. Since $z > y$ we have $z! \ge (y+1)y!$. This implies that $y + 1 \le 2$. Hence the only solution with x, y, and z positive integers is $x = y = 1$ and $z = 2$.

1.2.28. a. $\prod_{j=2}^{n}(1 - \frac{1}{j}) = (1 - 1/2)(1 - 1/3) \cdots (1 - 1/n) = \frac{1}{2}\frac{2}{3}\frac{3}{4} \cdots \frac{n-1}{n} = \frac{1}{n}$

b. $\prod_{j=2}^{n}(1 - \frac{1}{j^2}) = \prod_{j=2}^{n}(1 - 1/j)\prod_{j=2}^{n}(1 + 1/j) = \left(\frac{1}{n}\right)\left(\frac{3}{2}\frac{4}{3}\frac{5}{4} \cdots \frac{n+1}{n}\right) = \frac{n+1}{2n}$

1.2.29. According to the definition, the double factorial of a positive odd integer n is the product of the positive odd integers at most n, and similarly for positive evens. The necessary computations are shown below.

$1!! = 1$	$= 1$	$6!! = 6 \cdot 4 \cdot 2$	$= 48$
$2!! = 2$	$= 2$	$7!! = 7 \cdot 5 \cdot 3 \cdot 1$	$= 105$
$3!! = 3 \cdot 1$	$= 3$	$8!! = 8 \cdot 6 \cdot 4 \cdot 2$	$= 384$
$4!! = 4 \cdot 2$	$= 8$	$9!! = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$	$= 945$
$5!! = 5 \cdot 3 \cdot 1$	$= 15$	$10!! = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2$	$= 3840$

1.2.30. Note that for $n = 0$, the formula is $1!!2^0(1!) = 1 \cdot 1 \cdot 1 = 1!$. For n positive, expanding the double factorial and factorial on the left-hand side, we see that $(2n+1)!!2^n(n!) = (2n+1)(2n-1)(2n-3) \cdots 1(2^n)n(n-1)(n-2) \cdots 1 = (2n+1)(2n-1)(2n-3) \cdots 1 \cdot (2n)(2n-2)(2n-4) \cdots 2 = (2n+1)(2n)(2n-1)(2n-2)(2n-3)(2n-4) \cdots 2 \cdot 1 = (2n+1)!$, where the second equality comes from combining one of the n factors of 2 from 2^n with each of the n factors in the expansion of $n!$.

1.2.31. The claim is that $n!!(n-1)!! = n!$ for all nonnegative integers n. In the cases $n = 0$ or $n = 1$, the left-hand side is $0!!(-1)!! = 1 \cdot 1 = 0!$ and $1!!0!! = 1 \cdot 1 = 1!$. For $n \ge 2$ and even, observe that $n!!(n-1)!! = [n(n-2)(n-4) \cdots 6 \cdot 4 \cdot 2] \cdot [(n-1)(n-3)(n-5) \cdots 5 \cdot 3 \cdot 1]$. Rearranging the factors, we see this is the same as $n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!$. The case for n odd is similar.

1.3. Mathematical Induction

1.3.1. For $n = 1$ we have $1 < 2^1 = 2$. This is the basis case. Now assume $n < 2^n$. We then have $n + 1 < 2^n + 1 < 2^n + 2^n = 2^{n+1}$. This completes the inductive step and the proof by mathematical induction.

1.3.2. We have $2 = 2$, $2 + 4 = 6$, $2 + 4 + 6 = 12$, $2 + 4 + 6 + 8 = 20$, and $2 + 4 + 6 + 8 + 10 = 30$. We conjecture that $\sum_{j=1}^{n} 2j = n(n+1)$ since this formula holds for small values of n. To prove this by mathematical induction we have $\sum_{j=1}^{1} 2j = 2 = 2 \cdot (1+1)$ so the result is true for 1. Now assume that the formula holds for n. Then $\sum_{j=1}^{n+1} 2j = (\sum_{j=1}^{n} 2j) + 2(n+1) = n(n+1) + 2(n+1) = (n+1)(n+2)$. This completes the proof.

1.3.3. For the basis step, observe that $\sum_{k=1}^{1} \frac{1}{k^2} = 1 \le 2 - \frac{1}{1} = 1$. For the inductive step, assume that n is a positive integer such that $\sum_{k=1}^{n} \frac{1}{k^2} \le 2 - \frac{1}{n}$. Separating the final term from the sum and applying the inductive hypothesis, we have $\sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \le 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$. Combining the fractions, the last expression equals $2 - \frac{n^2+n+1}{n(n+1)^2} = 2 - \frac{1}{n+1}\frac{n^2+n+1}{n^2+n} < 2 - \frac{1}{n+1}$, where the inequality holds because $n^2 + n + 1 > n^2 + n$ and thus $-\frac{n^2+n+1}{n^2+n} < -1$. Hence, $\sum_{k=1}^{n+1} \frac{1}{k^2} \le 2 - \frac{1}{n+1}$, as needed.

1.3.4. For the basis step, we have $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{2}$. For the inductive step, we assume that $\sum_{k+1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$. Then, $\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$, as desired.

1.3.5. We see that $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and so on. We conjecture that $\mathbf{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. To prove this by mathematical induction we first note that the basis step follows since $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Next, we assume that $\mathbf{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Then $\mathbf{A}^{n+1} = \mathbf{A}^n\mathbf{A} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$.

1.3.6. The basis step holds since $1 = 1 \cdot (1+1)/2$. For the inductive step assume that $\sum_{j=1}^n j = n(n+1)/2$. It follows that

$$\sum_{j=1}^{n+1} j = \sum_{j=1}^{n} j + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)(\frac{n}{2}+1) = \frac{(n+1)(n+2)}{2}.$$

This finishes the inductive proof.

1.3.7. For the basis step, we have $\sum_{j=1}^1 j^2 = 1 = 1(1+1)(2\cdot 1+1)/6$. For the inductive step, we assume that $\sum_{j=1}^n j^2 = n(n+1)(2n+1)/6$. Then, $\sum_{j=1}^{n+1} j^2 = \sum_{j=1}^n j^2 + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2 = (n+1)\left(n(2n+1)/6 + n + 1\right) = (n+1)(2n^2+7n+6)/6 = (n+1)(n+2)[2(n+1)+1]/6$, as desired.

1.3.8. For the basis step, we have $\sum_{j=1}^1 j^3 = 1$, and $(1(1+1)/2)^2 = 1$ also. For the inductive step, we assume that $\sum_{j=1}^n j^3 = (n(n+1)/2)^2$. Then, $\sum_{j=1}^{n+1} j^3 = \sum_{j=1}^n j^3 + (n+1)^3 = (n(n+1)/2)^2 + n^3 + 3n^2 + 3n + 1 = ((n+1)(n+2)/2)^2$, as desired.

1.3.9. For the basis step, we have $\sum_{j=1}^1 j(j+1) = 2 = 1(2)(3)/3$. Assume it is true for n. Then $\sum_{j=1}^{n+1} j(j+1) = n(n+1)(n+2)/3 + (n+1)(n+2) = (n+1)(n+2)(n/3+1) = (n+1)(n+2)(n+3)/3$.

1.3.10. For the basis step, we have $\sum_{j=1}^1 (-1)^{j-1} j^2 = 1 = (-1)^{1-1} 1(1+1)/2$. For the inductive step, we assume that $\sum_{j=1}^n (-1)^{j-1} j^2 = (-1)^{n-1} n(n+1)/2$. Then, $\sum_{j=1}^{n+1} (-1)^{j-1} j^2 = \sum_{j=1}^n (-1)^{j-1} j^2 + (-1)^n (n+1)^2 = (-1)^{n-1} n(n+1)/2 + (-1)^n (n+1)^2 = (-1)^n \frac{1}{2}(n+1)[2(n+1) - n] = (-1)^{(n+1)-1}(n+1)(n+2)/2$, as desired.

1.3.11. We have $\prod_{j=1}^n 2^j = 2^{\sum_{j=1}^n j} = 2^{n(n+1)/2}$ since $\sum_{j=1}^n j = \frac{n(n+1)}{2}$.

1.3.12. We use mathematical induction. For $n = 1$ we have $\sum_{j=1}^1 j \cdot j! = 1 \cdot 1! = 1 = (1+1)! - 1 = 1$. Now assume that $\sum_{j=1}^n j \cdot j! = (n+1)! - 1$. Then $\sum_{j=1}^{n+1} j \cdot j! = (n+1)! - 1 + (n+1)\cdot(n+1)! = (n+1)!(1 + n + 1) - 1 = (n+2)! - 1$. This completes the proof.

1.3.13. We will prove this using mathematical induction. We see that $12 = 4 \cdot 3$. Now assume that postage of n cents can be formed, with $n = 4a + 5b$, where a and b are nonnegative integers. To form $n+1$ cents postage, if $a > 0$ we can replace a 4-cent stamp with a 5-cent stamp; that is, $n + 1 = 4(a-1) + 5(b+1)$. If no 4-cent stamps are present, then all 5-cent stamps were used. It follows that there must be at least three 5-cent stamps and these can be replaced by four 4-cent stamps; that is, $n + 1 = 4(a+4) + 5(b-3)$.

1.3.14. We prove this using mathematical induction. We see that $54 = 4 \cdot 10 + 2 \cdot 7$. Now assume that postage of n cents can be formed, with $n = 10a + 7b$, where a and b are positive integers. To form $n+1$ cents postage, if $a > 1$ we can replace 2 ten-cent stamps with 3 seven-cent stamps, that is, $n+1 = 10(a-2) + 7(b+3)$. If $a < 2$, then notice that $b \geq 7$. We can replace 7 seven-cent stamps with 5 ten-cent stamps, that is, $n + 1 = 10(a+5) + 7(b-7)$.

1.3.15. We use mathematical induction. The inequality is true for $n = 0$ since $H_{2^0} = H_1 = 1 \geq 1 = 1 + 0/2$. Now assume that the inequality is true for n, that is, $H_{2^n} \geq 1 + n/2$. Then $H_{2^{n+1}} = \sum_{j=1}^{2^n} 1/j + \sum_{j=2^n+1}^{2^{n+1}} 1/j \geq H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} 1/2^{n+1} \geq 1 + n/2 + 2^n \cdot 1/2^{n+1} = 1 + n/2 + 1/2 = 1 + (n+1)/2$. This completes the inductive proof.

1.3.16. For the basis step, we have $H_{2^0} = H_1 = 1 \le 1 + 0 = 1$. For the inductive step, we assume that $H_{2^n} \le 1 + n$. Then,

$$H_{2^{n+1}} = H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} < 1 + n + 2^n \frac{1}{2^n} = 1 + (n+1),$$

as desired.

1.3.17. For the basis step, we have $(2 \cdot 1)! = 2 < 2^{2 \cdot 1}(1!)^2 = 4$. For the inductive step, we assume that $(2n)! < 2^{2n}(n!)^2$. Then $[2(n+1)]! = (2n)!(2n+1)(2n+2) < 2^{2n}(n!)^2(2n+1)(2n+2) < 2^{2n}(n!)^2(2n+2)^2 = 2^{2(n+1)}[(n+1)!]^2$, as desired.

1.3.18. We will use the second principle of mathematical induction to prove this. For the basis step, we have $x - y$ is a factor of $x^1 - y^1$. For the inductive step, we assume that $x - y$ is a factor of $x^n - y^n$ and $x^{n-1} - y^{n-1}$. Then, $x^{n+1} - y^{n+1} = (x^n - y^n)(x + y) + xy(x^{n-1} - y^{n-1})$. Since $x - y$ is a factor of both $(x^n - y^n)(x + y)$ and $xy(x^{n-1} - y^{n-1})$, it is a factor of $x^{n+1} - y^{n+1}$.

1.3.19. Let A be such a set. Define B as $B = \{x - k + 1 \mid x \in A \text{ and } x \ge k\}$. Since $x \ge k$, B is a set of positive integers. Since $k \in A$ and $k \ge k$, $k - k + 1 = 1$ is in B. Since $n + 1$ is in A whenever n is, $n + 1 - k + 1$ is in B whenever $n - k + 1$ is. Thus B satisfies the hypothesis for mathematical induction, i.e. B is the set of positive integers. Mapping B back to A in the natural manner, we find that A contains the set of integers greater than or equal to k.

1.3.20. The basis step holds since $2^4 = 16 < 4! = 24$. Now assume that $2^n < n!$. Then $2^{n+1} = 2 \cdot 2^n < 2 \cdot n! < (n+1) \cdot n! = (n+1)!$.

1.3.21. For the basis step, we have $4^2 = 16 < 24 = 4!$. For the inductive step, we assume that $n^2 < n!$. Then, $(n+1)^2 = n^2 + 2n + 1 < n! + 2n + 1 < n! + 3n < n! + n! = 2n! < (n+1)n! = (n+1)!$, as desired.

1.3.22. The basis step is clear when $n = 0$. For the inductive step, we assume that $1 + hn \le (1 + h)^n$. Then, $(1+h)^{n+1} = (1+h)^n(1+h) \ge (1+hn)(1+h) = 1 + nh + h + nh^2 \ge 1 + h(n+1)$ since nh^2 is positive. This last inequality proves the induction hypothesis.

1.3.23. We use the second principle of mathematical induction. For the basis step, if the puzzle has only one piece, then it is assembled with exactly 0 moves. For the induction step, assume that all puzzles with $k \le n$ pieces require $k - 1$ moves to assemble. Suppose it takes m moves to assemble a puzzle with $n + 1$ pieces. Then the m move consists of joining two blocks of size a and b, respectively, with $a + b = n + 1$. But by the induction hypothesis, it requires exactly $a - 1$ and $b - 1$ moves to assemble each of these blocks. Thus, $m = (a - 1) + (b - 1) + 1 = a + b + 1 = n + 1$. This completes the induction.

1.3.24. The $n = 2$ case does not follow from the $n = 1$ case, since, when $n = 2$, the set of horses labelled 1 to $n - 1$ (which is just the set containing horse 1) does not have any common elements with the set of horses labelled from 2 to n (which is just the set containing horse 2.)

1.3.25. Suppose that $f(n)$ is defined recursively by specifying the value of $f(1)$ and a rule for finding $f(n+1)$ from $f(n)$. We will prove by mathematical induction that such a function is well-defined. First, note that $f(1)$ is well-defined since this value is explicitly stated. Now assume that $f(n)$ is well-defined. Then $f(n+1)$ also is well-defined since a rule is given for determining this value from $f(n)$.

1.3.26. The function is $f(n) = 2^n$. For the basis step, we have $f(1) = 2 = 2^1$. For the inductive step, we assume that $f(n) = 2^n$. Then, $f(n+1) = 2f(n) = 2 \cdot 2^n = 2^{n+1}$, as desired.

1.3.27. We have $g(1) = 2$, $g(2) = 2^{g(1)} = 4$, $g(3) = 2^{g(2)} = 2^4 = 16$, and $g(4) = 2^{g(3)} = 2^{16} = 65536$.

1.3.28. The basis step is given. For the inductive step, we assume that the value of f at the first n positive integers are uniquely determined. Then $f(n+1)$ is uniquely determined from the rule. Therefore, by mathematical induction, $f(n)$ is determined for every positive integer n.

1.3.29. We use the second principle of mathematical induction. The basis step consists of verifying the formula for $n = 1$ and $n = 2$. For $n = 1$ we have $f(1) = 1 = 2^1 + (-1)^1$ and for $n = 2$ we have $f(2) = 5 = 2^2 + (-1)^2$. Now assume that $f(k) = 2^k + (-1)^k$ for all positive integers k with $k < n$ where $n > 2$. By the induction hypothesis it follows that $f(n) = f(n-1) + 2f(n-2) = (2^{n-1} + (-1)^{n-1}) + 2(2^{n-2} + (-1)^{n-2}) = (2^{n-1} + 2^{n-1}) + (-1)^{n-2}(-1+2) = 2^n + (-1)^n$. This finishes the proof.

1.3.30. Since $2^5 = 32 > 25 = 5^2$, the basis step holds. Assume that $2^n > n^2$. Note that for $n > 4$, $2n^2 = n^2 + n^2 > n^2 + 3n = n^2 + 2n + n > n^2 + 2n + 1 = (n+1)^2$. Then we have $(n+1)^2 < 2n^2 < 2 \cdot 2^n = 2^{n+1}$, which completes the induction.

1.3.31. We use the second principle of mathematical induction. We see that $a_0 = 1 \leq 3^0 = 1$, $a_1 = 3 \leq 3^i = 3$, and $a_2 = 9 \leq 3^2 = 9$. These are the basis cases. Now assume that $a_k \leq 3^k$ for all integers k with $0 \leq k < n$. It follows that $a_n = a_{n-1} + a_{n-2} + a_{n-3} \leq 3^{n-1} + 3^{n-2} + 3^{n-3} = 3^{n-3}(1+3+9) = 13 \cdot 3^{n-3} < 27 \cdot 3^{n-3} = 3^n$. The induction argument is complete.

1.3.32. a. For the basis step notice that for 1 ring only, $1 = 2^1 - 1$ moves are needed. For the inductive step we assume that it takes $2^n - 1$ steps to transfer n rings. To make the inductive step, first transfer n of $n+1$ rings to the third peg. This takes $2^n - 1$ steps. Now transfer the bottom ring to the second peg. This is one step. Then transfer the n rings on the third peg to the second peg. This is $2^n - 1$ more steps. Altogether, this takes $2^n - 1 + 1 + 2^n - 1 = 2^{n+1} - 1$ steps.

b. The world will last, according to this legend, $2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$ seconds $= 3.07445 \cdot 10^{17}$ minutes $= 5.12409 \cdot 10^{15}$ hours $= 2.13503 \cdot 10^{14}$ days $= 5.84942 \cdot 10^{11}$ years, that is more than 580 billion years.

1.3.33. Let P_n be the statement for n. Then P_2 is true, since we have $((a_1 + a_2)/2)^2 - a_1a_2 = ((a_1 - a_2)/2)^2 \geq 0$. Assume P_n is true. Then by P_2, for $2n$ positive real numbers $a_1, \ldots, a_{2n}$ we have $a_1 + \cdots + a_{2n} \geq 2(\sqrt{a_1a_2} + \sqrt{a_3a_4} + \cdots + \sqrt{a_{2n-1}a_{2n}})$. Apply P_n to this last expression to get $a_1 + \cdots + a_{2n} \geq 2n(a_1a_2\cdots a_{2n})^{1/2n}$ which establishes P_n for $n = 2^k$ for all k. Again, assume P_n is true. Let $g = (a_1a_2\cdots a_{n-1})^{1/(n-1)}$. Applying P_n, we have $a_1 + a_2 + \cdots + a_{n-1} + g \geq n(a_1a_2\cdots a_{n-1}g)^{1/n} = n(g^{n-1}g)^{1/n} = ng$. Therefore, $a_1 + a_2 + \cdots + a_{n-1} \geq (n-1)g$ which establishes P_{n-1}. Thus P_{2^k} is true and P_n implies P_{n-1}. This establishes P_n for all n.

1.3.34. There are four 2×2 chess boards with one square missing. Each can be covered with exactly one L-shaped piece. This is the basis step. Now assume that any $2^n \times 2^n$ chess board can be covered with L-shaped pieces. Consider a $2^{n+1} \times 2^{n+1}$ chess board with one square missing. Split this into four $2^n \times 2^n$ chess boards three of which contain every square and the fourth has one square missing. By the inductive hypothesis we can cover the fourth $2^n \times 2^n$ chess board because it is missing one square. Now use one L-shaped piece to cover the three squares in the other three chess boards that touch at the center of the larger $2^{n+1} \times 2^{n+1}$ chess board. What is left to cover is all the rest of the squares in each of the three $2^n \times 2^n$ chess boards. The inductive hypothesis says that we can cover all the remaining squares in each of these chess boards. This completes the proof.

1.3.35. Note that since $0 < p < q$ we have $0 < p/q < 1$. The proposition is trivially true if $p = 1$. We proceed by strong induction on p. Let p and q be given and assume the proposition is true for all rational numbers between 0 and 1 with numerators less than p. To apply the algorithm, we find the unit fraction $1/s$ such that $1/(s-1) > p/q > 1/s$. When we subtract, the remaining fraction is $p/q - 1/s = (ps-q)/qs$. On the other hand, if we multiply the first inequality by $q(s-1)$ we have $q > p(s-1)$ which leads to $p > ps - q$, which shows that the numerator of p/q is strictly greater than the numerator of the remainder $(ps-q)/qs$ after one step of the algorithm. By the induction hypothesis, this remainder is expressible as a sum of unit fractions, $1/u_1 + \cdots + 1/u_k$. Therefore $p/q = 1/s + 1/u_1 + \cdots + 1/u_k$ which completes the induction step.

1.3.36. a. Since $1/2 < 2/3$, we subtract to get $2/3 = 1/2 + 1/6$.

b. Since $1/2 < 5/8$, we subtract to get $5/8 = 1/2 + 1/8$.

c. Since $1/2 < 11/17$ we subtract to get $11/17 = 1/2 + 5/34$. The largest unit fraction less than $5/34$ is $1/7$ so we subtract to get $11/17 = 1/2 + 1/7 + 1/238$.

d. The largest unit fraction less than $44/101$ is $1/3$ so we subtract and get $44/101 = 1/3 + 31/303$. The largest unit fraction less than $31/303$ is $1/10$, so we subtract to get $44/101 = 1/3 + 1/10 + 7/3030$. The largest unit fraction less than $7/3030$ is $1/433$, so we subtract to get $44/101 = 1/3 + 1/10 + 1/433 + 1/1311990$. (Note that this is the result of the "greedy algorithm." Other representations are possible, such as $44/101 = 1/3 + 1/10 + 1/440 + 1/26664$.)

1.4. The Fibonacci Numbers

1.4.1. a. We have $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Hence $f_3 = f_2 + f_1 = 1 + 1 = 2$, $f_4 = f_3 + f_2 = 2 + 1 = 3$, $f_5 = 3 + 2 = 5$, $f_6 = 5 + 3 = 8$, $f_7 = 8 + 5 = 13$, $f_8 = 13 + 8 = 21$, $f_9 = 21 + 13 = 34$, and $f_{10} = 34 + 21 = 55$.

b. We continue beyond part (a) finding that $f_{11} = f_{10} + f_9 = 55 + 34 = 89$, $f_{12} = 89 + 55 = 144$, and $f_{13} = 144 + 89 = 233$.

c. We continue beyond part (b) finding that $f_{14} = f_{13} + f_{12} = 233 + 144 = 377$, and $f_{15} = 377 + 233 = 610$.

d. We continue beyond part (c) finding that $f_{16} = 610 + 377 = 987$, $f_{17} = 987 + 610 = 1597$, and $f_{18} = 1597 + 987 = 2584$.

e. We continue beyond part (d) finding that $f_{19} = 2584 + 1597 = 4181$, $f_{20} = 4181 + 2584 = 6765$.

f. We continue beyond part (e) finding that $f_{21} = 6765 + 4181 = 10946$, $f_{22} = 10946 + 6765 = 17711$, $f_{23} = 17711 + 10946 = 28657$, $f_{24} = 28657 + 17711 = 46368$, and $f_{25} = 46368 + 28657 = 75025$.

1.4.2. a. We continue from Exercise 1 part (a), finding that $f_{11} = 55 + 34 = 89$ and $f_{12} = 89 + 55 = 144$.

b. We continue from Exercise 1 part (c), finding that $f_{16} = 610 + 377 = 987$.

c. We computed $f_{24} = 46368$ in Exercise 1 part (f).

d. We continue from Exercise 1 part (f), finding that $f_{26} = 75025 + 46368 = 121393$, $f_{27} = 121393 + 75025 = 196418$, $f_{28} = 196418 + 121393 = 317811$, $f_{29} = 317811 + 196418 = 514229$, and $f_{30} = 514229 + 317811 = 832040$.

e. We continue from part (d), finding $f_{31} = 832040 + 514229 = 1346269$, and $f_{32} = 1346269 + 832040 = 2178309$.

f. We continue from part (e), finding $f_{33} = 2178309 + 1346269 = 3524578$, $f_{34} = 3524578 + 2178309 = 5702887$, $f_{35} = 5702887 + 3524578 = 9227465$ and $f_{36} = 9227465 + 5702887 = 14930352$.

1.4.3. Note that by the Fibonacci identity, whenever n is a positive integer, $f_{n+2} - f_n = f_{n+1}$. Then we have $2f_{n+2} - f_n = f_{n+2} + (f_{n+2} - f_n) = f_{n+2} + f_{n+1} = f_{n+3}$. If we add f_n to both sides of this equation, we have the desired identity.

1.4.4. Assuming n is a positive integer, we have compute $2f_{n+1} + f_n = f_{n+1} + (f_{n+1} + f_n) = f_{n+1} + f_{n+2} = f_{n+3}$. If we subtract f_n from both sides of this equation, we have the desired identity.

1.4.5. For $n = 1$ we have $f_{2\cdot 1} = 1 = 1^2 + 2\cdot 1\cdot 0 = f_1^2 + 2f_0 f_1$, and for $n = 2$, we have $f_{2\cdot 2} = 3 = 1^2 + 2\cdot 1\cdot 1 = f_2^2 + 2f_1 f_2$. So the basis step holds for strong induction. Assume, then that $f_{2n-4} = f_{n-2}^2 + 2f_{n-3}f_{n-2}$ and $f_{2n-2} = f_{n-1}^2 + 2f_{n-2}f_{n-1}$. Now compute $f_{2n} = f_{2n-1} + f_{2n-2} = 2f_{2n-2} + f_{2n-3} = 3f_{2n-2} - f_{2n-4}$. Now we may substitute in our induction hypotheses to set this last expression equal to $3f_{n-1}^2 + 6f_{n-2}f_{n-1} - f_{n-2}^2 - 2f_{n-3}f_{n-2} = 3f_{n-1}^2 + 6(f_n - f_{n-1})f_{n-1} - (f_n - f_{n-1})^2 - 2(f_{n-1} - f_{n-2})(f_n - f_{n-1}) = -2f_{n-1}^2 + 6f_n f_{n-1} - f_n^2 + 2f_n(f_n - f_{n-1}) - 2f_{n-1}(f_n - f_{n-1}) = f_n^2 + 2f_{n-1}f_n$ which completes the induction step.

1.4.6. For n a positive integer greater than 1, we have $f_{n+2} = f_{n+1} + f_n = (f_n + f_{n-1}) + f_n = (f_n + (f_n - f_{n-2})) + f_n = 3f_n - f_{n-2}$. Adding f_{n-2} to both sides yields the desired identity.

1.4.7. Note that $f_1 = 1 = f_2$, $f_1 + f_3 = 3 = f_4$, and $f_1 + f_3 + f_5 = 8 = f_6$ so we conjecture that $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$. We prove this by induction. The basis step is checked above. Assume that our formula is true for n, and consider $f_1 + f_3 + f_5 + \cdots + f_{2n-1} + f_{2n+1} = f_{2n} + f_{2n+1} = f_{2n+2}$, which is the induction step. Therefore the formula is correct.

1.4.8. Note that $f_2 = 1 = f_3 - 1$, $f_2 + f_4 = 4 = f_5 - 1$, and $f_2 + f_4 + f_6 = 12 = f_7 - 1$, so we conjecture that $f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$. We prove this by induction. The basis step is checked above. Assume that our formula is true for n, and consider $f_2 + f_4 + f_6 + \cdots + f_{2n} + f_{2n+2} = f_{2n+1} - 1 + f_{2n+2} = f_{2n+3} - 1$, which is the induction step. Therefore the formula is correct. Another solution is to subtract the formula in Exercise 7 from the formula in Example 1.28, as follows: $\sum_{i=1}^{n} f_{2i} = \sum_{i=1}^{2n} f_i - \sum_{i=1}^{n} f_{2i-1} = (f_{2n+2} - 1) - f_{2n} = f_{2n+1} - 1$.

1.4.9. First suppose $n = 2k$ is even. Then $f_n - f_{n-1} + \cdots + (-1)^{n+1}f_1 = (f_{2k} + f_{2k-1} + \cdots + f_1) - 2(f_{2k-1} + f_{2k-3} + \cdots + f_1) = (f_{2k+2} - 1) - 2(f_{2k})$ by the formulas in Example 1.28 and Exercise 7. This last equals $(f_{2k+2} - f_{2k}) - f_{2k} - 1 = f_{2k+1} - f_{2k} - 1 = f_{2k-1} - 1 = f_{n-1} - 1$. Now suppose $n = 2k+1$ is odd. Then $f_n - f_{n-1} + \cdots + (-1)^{n+1} = f_{2k+1} - (f_{2k} - f_{2k-1} + \cdots - (-1)^{n+1}f_1) = f_{2k+1} - (f_{2k-1} - 1)$ by the formula just proved for the even case. This last equals $(f_{2k+1} - f_{2k-1}) + 1 = f_{2k} + 1 = f_{n-1} + 1$. We can unite the formulas for the odd and even cases by writing the formula as $f_{n-1} - (-1)^n$.

1.4.10. For $n = 1$ we have $f_3 = 2 = f_2^2 + f_1^2 = 1^2 + 1^2$. And when $n = 2$ we have $f_5 = 5 = 2^2 + 1^2 = f_3^2 + f_2^2$, so the basis steps hold for mathematical induction. Now assume, for the strong form of induction, that the identity holds for all values of n up to $n = k$. Then $f_{2k-3} = f_{k-1}^2 + f_{k-2}^2$ and $f_{2k-1} = f_k^2 + f_{k-1}^2$. Now we calculate $f_{2k+1} = f_{2k} + f_{2k-1} = f_{2k-1} + f_{2k-2} + f_{2k-1} = 2f_{2k-1} + (f_{2k-1} - f_{2k-3}) = 3f_{2k-1} - f_{2k-3}$. Now substituting in the induction hypothesis, makes this last expression equal to $3(f_k^2 + f_{k-1}^2) - f_{k-1}^2 - f_{k-2}^2 = 3f_k^2 + 2f_{k-2}^2 - (f_k - f_{k-1})^2 = 2f_k^2 + f_{k-1}^2 + 2f_k f_{k-1} = 2f_k^2 + (f_{k+1} - f_k)^2 + 2f_k(f_{k+1} - f_k) = f_{k+1}^2 + f_k^2$, which completes the induction step.

1.4.11. We can construct an induction proof similar to the ones in Exercises 5 and 10, or we may proceed as follows. By Exercise 5, we have $f_{2n} = f_n^2 + 2f_{n-1}f_n = f_n(f_n + f_{n-1} + f_{n-1}) = (f_{n+1} - f_{n-1})(f_{n+1} + f_{n-1}) = f_{n+1}^2 - f_{n-1}^2$, which is the desired identity.

1.4.12. Let $S_n = f_n + f_{n-1} + f_{n-2} + 2f_{n-3} + \cdots + 2^{n-4}f_2 + 2^{n-3}f_1$. We proceed by induction. If $n = 3$ we have $S_3 = f_3 + f_2 + f_1 = 2 + 1 + 1 = 4 = 2^{3-1}$, and when $n = 4$ we have $S_4 = f_4 + f_3 + f_2 + 2f_1 = 3 + 2 + 1 + 2\cdot 1 = 8 = 2^{4-1}$, so the basis steps hold. Now assume the identity holds for all values less or equal to n and consider $S_{n+1} = f_{n+1} + f_n + f_{n-1} + 2f_{n-2} + 4f_{n-3} + \cdots + 2^{n-4}f_3 + 2^{n-3}f_2 + 2^{n-2}f_1$. We use the Fibonacci identity to expand every term except the last two to get $S_{n+1} = (f_n + f_{n-1}) + (f_{n-1} + f_{n-2}) + (f_{n-2} + f_{n-3}) + 2(f_{n-3} + f_{n-4}) + 4(f_{n-4} + f_{n-5}) + \cdots + 2^{n-4}(f_2 + f_1) + 2^{n-3}f_2 + 2^{n-2}f_1$. Next we regroup, taking the first term from each set of parentheses, plus the second last term together in one group, the last term from each set of parentheses together in another group, and leaving the last term by itself to get $S_{n+1} = (f_n + f_{n-1} + f_{n-2} + 2f_{n-3} + 4f_{n-4} + \cdots + 2^{n-4}f_2 + 2^{n-3}f_2) + (f_{n-1} + f_{n-2} + f_{n-3} + 2f_{n-4} + 4f_{n-5} + \cdots + 2^{n-4}f_1) + 2^{n-2}f_1$. The first group is seen to be equal to S_n when we realize that the last $f_2 = f_1$. The second group is equal to S_{n-1}, so we have $S_{n+1} = S_n + S_{n-1} + 2^{n-1} = 2^{n-2} + 2^{n-2} + 2^{n-1} = 2^n$ by the induction hypothesis. Therefore, by mathematical induction, the proposition is proved.

1.4.13. We proceed by mathematical induction. For the basis step, $\sum_{j=1}^{1} f_j^2 = f_1^2 = f_1 f_2$. To make the inductive step we assume that $\sum_{j=1}^{n} f_j^2 = f_n f_{n+1}$. Then $\sum_{j=1}^{n+1} f_j^2 = \sum_{j=1}^{n} f_j^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 = f_{n+1} f_{n+2}$.

1.4.14. We use mathematical induction. We will use the recursive definition $f_n = f_{n-1} + f_{n-2}$, with $f_0 = 0$ and $f_1 = 1$. For $n = 1$ we have $f_2 f_0 - f_1^2 = 1 \cdot 0 - 1^2 = -1 = (-1)^1$. Hence the basis step holds. Now assume that $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$. Then $f_{n+2} f_n - f_{n+1}^2 = (f_{n+1} + f_n) f_n - f_{n+1}(f_n + f_{n-1}) = f_n^2 - f_{n+1} f_{n-1} = -(-1)^n = (-1)^{n+1}$. This completes the proof.

1.4.15. By Exercise 13, we have $f_{n+1} f_n - f_{n-1} f_{n-2} = (f_1^2 + \cdots + f_n^2) - (f_1^2 + \cdots f_{n-2}^2) = f_n^2 + f_{n-1}^2$. The identity in Exercise 10 shows that this is equal to f_{2n-1} when n is a positive integer, and in particular when n is greater than 2.

1.4.16. Since $f_1 f_2 = 1 \cdot 1 = 1^2 = f_2^2$, the basis step holds. By the induction hypothesis we have $f_1 f_2 + \cdots + f_{2n-1} f_{2n} + f_{2n} f_{2n+1} + f_{2n+1} f_{2(n+1)} = f_{2n}^2 + f_{2n} f_{2n+1} + f_{2n+1} f_{2(n+1)} = f_{2n}(f_{2n} + f_{2n+1}) + f_{2n+1} f_{2(n+1)} = f_{2n} f_{2(n+1)} + f_{2n+1} f_{2(n+1)} = (f_{2n} + f_{2n+1}) f_{2(n+1)} = f_{2(n+1)}^2$.

1.4.17. For fixed m, we proceed by induction on n. The basis step is $f_{m+1} = f_m f_2 + f_{m-1} f_1 = f_m \cdot 1 + f_{m-1} \cdot 1$ which is true. Assume the identity holds for $1, 2, \ldots, k$. Then $f_{m+k} = f_m f_{k+1} + f_{m-1} f_k$ and $f_{m+k-1} = f_m f_k + f_{m-1} f_{k-1}$. Adding these equations gives us $f_{m+k} + f_{m+k-1} = f_m(f_{k+1} + f_k) + f_{m-1}(f_k + f_{k-1})$. Applying the recursive definition yields $f_{m+k+1} = f_m f_{k+2} + f_{m-1} f_{k+1}$, which is precisely the identity.

1.4.18. We're given that $L_1 = 1$ and $L_2 = 3$. Adding each consecutive pair to generate the next Lucas number yields the sequence $1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \ldots$.

1.4.19. A few trial cases lead us to conjecture that $\sum_{i=1}^{n} L_i = L_{n+2} - 3$. We prove that this formula is correct by induction. The basis step is $L_1 = 1$ and $L_3 - 3 = 4 - 3 = 1$, which checks. Assume that the formula holds for n and compute $\sum_{i=1}^{n+1} L_i = \sum_{i=1}^{n} L_i + L_{n+1} = L_{n+2} - 3 + L_{n+1}$ by the induction hypothesis. This last equals $(L_{n+2} + L_{n+1}) - 3 = L_{n+3} - 3$, which completes the induction step.

1.4.20. A few trial cases lead us to conjecture that $\sum_{i=1}^{n} L_{2i-1} = L_{2n} - 2$. We prove that this formula is correct by induction. The basis step is $L_1 = 1 = L_2 - 2$. Assume that the formula holds for n and compute $\sum_{i=1}^{n+1} L_{2i-1} = \sum_{i=1}^{n} L_{2i-1} + L_{2n+1} = L_{2n} - 2 + L_{2n+1} = L_{2n+2} - 2$, which completes the induction step.

1.4.21. A few trial cases lead us to conjecture that $\sum_{i=1}^{n} L_{2i} = L_{2n+1} - 1$. We prove that this formula is correct by induction. The basis step is $L_2 = 3 = L_3 - 1$. Assume that the formula holds for n and compute $\sum_{i=1}^{n+1} L_{2i} = \sum_{i=1}^{n} L_{2i} + L_{2n+2} = L_{2n+1} - 1 + L_{2n+2} = L_{2n+3} - 1$, which completes the induction step.

1.4.22. We proceed by induction. The basis step is when $n = 2$, and we have $L_2^2 - L_3 L_1 = 3^2 - 4 \cdot 1 = 5 = 5(-1)^2$. Now assume the identity holds for n. Then for $n+1$ we have $L_{n+1}^2 - L_{n+2} L_n = (L_n + L_{n-1}) L_{n+1} - (L_{n+1} + L_n) L_n = L_n L_{n+1} + L_{n-1} L_{n+1} - L_{n+1} L_n - L_n^2 = -(L_n^2 - L_{n-1} L_{n+1}) = -(5(-1)^n) = 5(-1)^{n+1}$, where we apply the induction hypothesis at the penultimate step.

1.4.23. We proceed by induction. The basis step is $L_1^2 = 1 = L_1 L_2 - 2 = 1 \cdot 3 - 2$. Assume the formula holds for n and consider $\sum_{i=1}^{n+1} L_i^2 = \sum_{i=1}^{n} L_i^2 + L_{n+1}^2 = L_n L_{n+1} - 2 + L_{n+1}^2 = L_{n+1}(L_n + L_{n+1}) - 2 = L_{n+1} L_{n+2} - 2$, which completes the induction step.

1.4.24. For $n = 2$, we have $L_2 = 3 = 1 + 2 = f_1 + f_3$. For $n = 3$, we have $L_3 = 4 = 1 + 3 = f_2 + f_4$. This serves as the basis step. Now assume that the statement is true for $k = 2, 3, 4, \ldots, n$. Then $L_{n+1} = L_n + L_{n-1} = (f_{n+1} + f_{n-1}) + (f_n + f_{n-2}) = (f_{n+1} + f_n) + (f_{n-1} + f_{n-2}) = f_{n+2} + f_n$, which completes the induction.

1.4.25. For the basis step, we check that $L_1 f_1 = 1 \cdot 1 = 1 = f_2$ and $L_2 f_2 = 3 \cdot 1 = 3 = f_4$. Assume the identity is true for all positive integers up to n. Then we have $f_{n+1}L_{n+1} = (f_{n+2} - f_n)(f_{n+2} + f_n)$ by Exercise 24. This equals $f_{n+2}^2 - f_n^2 = (f_{n+1} + f_n)^2 - (f_{n-1} + f_{n-2})^2 = f_{n+1}^2 + 2f_{n+1}f_n + f_n^2 - f_{n-1}^2 - 2f_{n-1}f_{n-2} - f_{n-2}^2 = (f_{n+1}^2 - f_{n-1}^2) + (f_n^2 - f_{n-2}^2) + 2(f_{n+1}f_n - f_{n-1}f_{n-2}) = (f_{n+1} - f_{n-1})(f_{n+1} + f_{n-1}) + (f_n - f_{n-2})(f_n + f_{n-2}) + 2(f_{2n-1})$, where the last parenthetical expression is obtained from Exercise 15. This equals $f_nL_n + f_{n-1}L_{n-1} + 2f_{2n-1}$. Applying the induction hypothesis yields $f_{2n} + f_{2n-2} + 2f_{2n-1} = (f_{2n} + f_{2n-1}) + (f_{2n-1} + f_{2n-2}) = f_{2n+1} + f_{2n} = f_{2n+2}$, which completes the induction.

1.4.26. For the basis step, we check that when $n = 1$, $5f_2 = 5 \cdot 1 = 1 + 4 = L_1 + L_3$ and when $n = 2$, $5f_3 = 10 = 3 + 7 = L_2 + L_4$. Now assume the identity holds for integers less than n, and compute $5f_{n+1} = 5f_n + 5f_{n-1} = (L_{n-1} + L_{n+1}) + (L_{n-2} + L_n) = (L_{n-1} + L_{n-2}) + (L_{n+1} + L_n) = L_n + L_{n+1}$, which completes the induction step.

1.4.27. We prove this by induction on n. Fix m a positive integer. If $n = 2$, then for the basis step we need to show that $L_{m+2} = f_{m+1}L_2 + f_mL_1 = 3f_{m+1} + f_m$, for which we will use induction on m. For $m = 1$ we have $L_3 = 4 = 3 \cdot f_2 + f_1$ and for $m = 2$ we have $L_4 = 7 = 3 \cdot f_3 + f_2$, so the basis step for m holds. Now assume that the basis step for n holds for all values of m less than and equal to m. Then $L_{m+3} = L_{m+2} + L_{m+1} = 3f_{m+1} + f_m + 3f_m + f_{m-1} = 3f_{m+2} + f_{m+1}$, which completes the induction step on m and proves the basis step for n. To prove the induction step on n, we compute $L_{m+n+1} = L_{m+n} + L_{m+n-1} = (f_{m+1}L_n + f_mL_{n-1}) + (f_{m+1}L_{n-1} + f_mL_{n-2}) = f_{m+1}(L_n + L_{n-1}) + f_m(L_{n-1} + L_{n-2}) = f_{m+1}L_{n+1} + f_mL_n$, which completes the induction on n and proves the identity.

1.4.28. First check that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. We proceed by induction. The basis steps are $\alpha + \beta = (1 + \sqrt{5})/2 + (1 - \sqrt{5})/2 = 1 = L_1$ and $\alpha^2 + \beta^2 = (1 + \alpha) + (1 + \beta) = 2 + L_1 = 3 = L_2$. Assume the identity is true for all positive integers up to n. Then $L_{n+1} = L_n + L_{n-1} = \alpha^n + \beta^n + \alpha^{n-1} + \beta^{n-1} = \alpha^{n-1}(\alpha + 1) + \beta^{n-1}(\beta + 1) = \alpha^{n-1}(\alpha^2) + \beta^{n-1}(\beta^2) = \alpha^{n+1} + \beta^{n+1}$, which completes the induction.

1.4.29. We find that $50 = 34 + 13 + 3 = f_9 + f_7 + f_4$, $85 = 55 + 21 + 8 + 1 = f_{10} + f_8 + f_6 + f_2$, $110 = 89 + 21 = f_{11} + f_8$ and $200 = 144 + 55 + 1 = f_{12} + f_{10} + f_2$. In each case, we used the "greedy" algorithm, always subtracting the largest possible Fibonacci number from the remainder.

1.4.30. Suppose there is a positive integer that has no Zeckendorf representation. Then by the well-ordering property, there is a smallest such integer, n. Let f_k be the largest Fibonacci number less than or equal to n. Note that if $n = f_k$, then n has a Zeckendorf representation, contrary to our assumption. Then $n - f_k$ is a positive integer less than n, so it has a Zeckendorf representation $n - f_k = \sum_{i=1}^m f_{a_i}$. Since n has no Zeckendorf representation, it must be that one of the f_{a_i}'s is equal to or consecutive to f_k. That is, one of f_{k-1}, f_k, or f_{k+1} appears in the summation for $n - f_k$. Then $n = \sum_{i=1}^m f_{a_i} + f_k \geq f_{k-1} + f_k = f_{k+1}$. But this contradicts the choice of f_k as the largest Fibonacci number less than n. This establishes existence. To establish uniqueness of the Zeckendorf representation, suppose that there is a positive integer that has two distinct representations. Then the well-ordering property gives us a smallest such integer, n. Suppose $n = \sum_{i=1}^m f_{a_i} = \sum_{j=1}^l f_{b_l}$ are two distinct representations for n. Then no $f_{a_i} = f_{b_j}$, else we could cancel this term from each side and have a smaller integer with two distinct representations. Without loss of generality, assume that $f_{a_1} > f_{a_2} > \cdots > f_{a_m}$ and $f_{b_1} > f_{b_2} > \cdots > f_{b_l}$ and that $f_{a_1} > f_{b_1}$. If b_1 is even, we compute $n = \sum_{i=1}^l f_{b_i} \leq f_{b_1} + f_{b_1-2} + f_{b_1-4} + \cdots + f_2 = f_{b_1+1} - 1$ by Exercise 8. But this last is less than or equal to $f_{a_1} - 1 < n$, a contradiction. If b_1 is odd, we compute, now using Exercise 7, $n = \sum_{i=1}^l f_{b_i} \leq f_{b_1} + f_{b_1-2} + f_{b_1-4} + \cdots + f_3 = f_{b_1+1} - f_1 \leq f_{a_1} - 1 < n$, which is also a contradiction. This proves uniqueness.

1.4.31. We proceed by mathematical induction. The basis steps ($n = 2$ and 3) are easily seen to hold. For the inductive step, we assume that $f_n \leq \alpha^{n-1}$ and $f_{n-1} \leq \alpha_{n-2}$. Now, $f_{n+1} = f_n + f_{n-1} \leq \alpha^{n-1} + \alpha^{n-2} = \alpha^n$, since α satisfies $\alpha^n = \alpha^{n-1} + \alpha^{n-2}$.

1.4.32. We proceed by the second principle of mathematical induction on n. For the basis step, we observe that $\binom{0}{0} = f_{0+1} = 1$. For the inductive step, we assume that $\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = f_{n+1}$, and that $\binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots = f_n$. Now, $\binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \cdots = \binom{n}{0} + [\binom{n-1}{1} + \binom{n-1}{0}] + [\binom{n-2}{2} +$

$\binom{n-2}{1}] + \cdots = \binom{n}{0} + \binom{n-1}{1}\binom{n-2}{2} + \cdots + \binom{n-1}{0} + \binom{n-2}{1} + \cdots = f_{n+1} + f_n = f_{n+2}$.

1.4.33. Using Theorem 1.7 and the notation therein, we have $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$, since they are roots of $x^2 - x - 1 = 0$. Then we have $f_{2n} = (\alpha^{2n} - \beta^{2n})/\sqrt{5} = (1/\sqrt{5})((\alpha+1)^n - (\beta+1)^n) = (1/\sqrt{5})\left(\sum_{j=0}^{n}\binom{n}{j}\alpha^j - \sum_{j=0}^{n}\binom{n}{j}\beta^j\right) = (1/\sqrt{5})\sum_{j=0}^{n}\binom{n}{j}(\alpha^j - \beta^j) = \sum_{j=1}^{n}\binom{n}{j}f_j$ since the first term is zero in the penultimate sum.

1.4.34. We prove this using mathematical induction. For $n = 1$ we have

$$\mathbf{F}^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix}$$

where $f_0 = 0$. Now assume that this formula is true for n. Then

$$\mathbf{F}^{n+1} = \mathbf{F}^n\mathbf{F} = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_{n+1}+f_n & f_{n+1} \\ f_n + f_{n-1} & f_n \end{pmatrix} = \begin{pmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{pmatrix}.$$

1.4.35. On one hand, $\det(\mathbf{F}^n) = \det(\mathbf{F})^n = (-1)^n$. On the other hand,

$$\det\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = f_{n+1}f_{n-1} - f_n^2.$$

1.4.36. We proceed by induction. Clearly the basis step holds. For the inductive step, we assume that $g_n = af_{n-2} + bf_{n-1}$. Then, $g_{n+1} = g_n + g_{n-1} = af_{n-2} + bf_{n-1} + af_{n-3} + bf_{n-2} = af_{n-1} + bf_n$.

1.4.37. We use the relationship $f_n = f_{n+2} - f_{n+1}$ to extend the definition to include negative indices. Thus, $f_0 = 0, f_{-1} = 1, f_{-2} = -1, f_{-3} = 2, f_{-4} = -3, f_{-5} = 5, f_{-6} = -8, f_{-7} = 13, f_{-8} = -21, f_{-9} = 34, f_{-10} = -55$.

1.4.38. We conjecture that $f_{-n} = (-1)^{n+1}f_n$. The basis step is given in Exercise 37. Assume the conjecture is true for n. Then $f_{-(n+1)} = f_{-(n-1)} - f_{-n} = (-1)^n f_{n-1} - (-1)^{n+1}f_n = (-1)^n(f_{n-1} + f_n) = (-1)^{n+2}f_{n+1}$, which completes the induction step.

1.4.39. The square has area 64 square units, while the rectangle has area 65 square units. This corresponds to the identity in Exercise 14, which tells us that $f_7f_5 - f_6^2 = 1$. Notice that the slope of the hypotenuse of the triangular piece is $3/8$, while the slope of the top of the trapezoidal piece is $2/5$. We have $2/5 - 3/8 = 1/40$. Thus, the "diagonal" of the rectangle is really a very skinny parallelogram of area 1, hidden visually by the fact that the two slopes are nearly equal.

1.4.40. First check that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$ as in the solution to Exercise 28. We compute $a_1 = (1/\sqrt{5})(\alpha - \beta) = (1/\sqrt{5})\left((1+\sqrt{5}/2 - (1-\sqrt{5}/2)\right) = (1/\sqrt{5})\left(2\sqrt{5}/2\right) = 1$ and $a_2 = (1/\sqrt{5})(\alpha^2 - \beta^2) = (1/\sqrt{5})(\alpha + 1 - \beta - 1) = (1/\sqrt{5})(\alpha - \beta) = 1$. Finally, we check that $a_{n-1} + a_{n-2} = (1/\sqrt{5})(\alpha^{n-1} - \beta^{n-1}) + (1/\sqrt{5})(\alpha^{n-2} - \beta^{n-2}) = (1/\sqrt{5})(\alpha^{n-1} + \alpha^{n-2} - \beta^{n-1} - \beta^{n-2}) = (1/\sqrt{5})(\alpha^{n-2}(\alpha+1) - \beta^{n-2}(\beta+1)) = (1/\sqrt{5})(\alpha^{n-2}\alpha^2 - \beta^{n-2}\beta^2) = (1/\sqrt{5})(\alpha^n - \beta^n) = a_n$. Since these a_n satisfy the defining relationships of the Fibonacci numbers, we can conclude that $a_n = f_n$ for $n = 1, 2, \ldots$.

1.4.41. We solve the equation $r^2 - r - 1 = 0$ to discover the roots $r_1 = (1+\sqrt{5})/2$ and $r_2 = (1-\sqrt{5})/2$. Then according to the theory in the paragraph above, $f_n = C_1r_1^n + C_2r_2^n$. For $n = 0$ we have $0 = C_1r_1^0 + C_2r_2^0 = C_1 + C_2$. For $n = 1$ we have $1 = C_1r_1 + C_2r_2 = C_1(1+\sqrt{5})/2 + C_2(1-\sqrt{5})/2$. Solving these two equations simultaneously yields $C_1 = 1/\sqrt{5}$ and $C_2 = -1/\sqrt{5}$. So the explicit formula is $f_n = (1/\sqrt{5})r_1^n - (1/\sqrt{5})r_2^n = (r_1^n - r_2^n)/\sqrt{5}$.

1.4.42. First note that $G(x) - xG(x) - x^2G(x) = \sum_{k=0}^{\infty} f_kx^k - \sum_{k=0}^{\infty} f_kx^{k+1} - \sum_{k=0}^{\infty} f_kx^{k+2} = \sum_{k=0}^{\infty} f_kx^k - \sum_{k=1}^{\infty} f_{k-1}x^k - \sum_{k=2}^{\infty} f_{k-2}x^k = f_0x^0 + f_1x - f_0x + \sum_{k=2}^{\infty}(f_k - f_{k-1} - f_{k-2})x^k = 0 + x - 0 + \sum_{k=2}^{\infty} 0x^k = x$. Solving this for $G(x)$ yields $G(x) = x/(1 - x - x^2)$. Let α and β be defined as in Exercise 40. Then the denominator of $G(x)$ factors as $-(x+\beta)(x+\alpha)$. Expand $G(x)$ into partial fractions to get

$G(x) = (1/\sqrt{5})\left(\beta/(x+\beta) - \alpha/(x+\alpha)\right)$. Since $1/\alpha = -\beta$ we can write the above as $G(x) = (1/\sqrt{5})$ $\left(1/(1-x\alpha) - 1/(1-x\beta)\right)$. But these last two fractions represent the sums of geometric series, so we have $G(x) = (1/\sqrt{5})\left((1+\alpha x+(\alpha x)^2+\cdots) - (1+\beta x+(\beta x)^2+\cdots)\right) = (1/\sqrt{5})(0+(\alpha-\beta)x+(\alpha^2-\beta^2)x^2+\cdots)$. Thus the coefficient on the nth power of x is given by $(1/\sqrt{5})(\alpha^n-\beta^n) = f_n$, for all $n \geq 0$.

1.4.43. We seek to solve the recurrence relation $L_n = L_{n-1} + L_{n-1}$ subject to the initial conditions $L_1 = 1$ and $L_2 = 3$. We solve the equation $r^2 - r - 1 = 0$ to discover the roots $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then according to the theory in the paragraph above Exercise 41, $L_n = C_1\alpha^n + C_2\beta^n$. For $n = 1$ we have $L_1 = 1 = C_1\alpha + C_2\beta$. For $n = 2$ we have $3 = C_1\alpha^2 + C_2\beta^2$. Solving these two equations simultaneously yields $C_1 = 1$ and $C_2 = 1$. So the explicit formula is $L_n = \alpha^n + \beta^n$.

1.4.44. Let $H(x) = \sum_{k=0}^{\infty} L_k x^k$ be the generating function for the Lucas numbers. Note that we define $L_0 = 2$ so that $L_0 + L_1 = 3 = L_2$. Consider $H(x) - xH(x) - x^2H(x) = \sum_{k=0}^{\infty} L_k x^k - \sum_{k=0}^{\infty} L_k x^{k+1} - \sum_{k=0}^{\infty} L_k x^{k+2}$ $= \sum_{k=0}^{\infty} L_k x^k - \sum_{k=1}^{\infty} L_{k-1}x^k - \sum_{k=2}^{\infty} L_{k-2}x^k = L_0x^0 + L_1x - L_0x + \sum_{k=2}^{\infty}(L_k - L_{k-1} - L_{k-2})x^k = 2 + x - 2x + \sum_{k=2}^{\infty} 0x^k = 2 - x$. We solve for $H(x)$ and find its partial fraction expansion $H(x) = (2-x)/(1-x-x^2) = (1/(2\sqrt{5}))\left((5+\sqrt{5})/(x+\alpha) - (5-\sqrt{5})/(x+\beta)\right)$, where α and β are defined as in Exercise 40. We multiply the top and bottom of the first fraction by β and use the fact that $\alpha\beta = 1$, and similarly treat the second fraction to get the above equal to $1/(1-\alpha x) + 1/(1-\beta x)$. But these are the representations for the sums of geometric series, so we have $H(x) = (1+\alpha x+(\alpha x)^2+\cdots) + (1+\beta x+(\beta x)^2+\cdots) = 2 + (\alpha+\beta)x + (\alpha^2+\beta^2)x^2 + \cdots$. Therefore, $L_n = \alpha^n + \beta^n$ the coefficient on the nth power of x.

1.4.45. First check that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. We proceed by induction. The basis steps are $(1/\sqrt{5})(\alpha - \beta) = (1/\sqrt{5})(\sqrt{5}) = 1 = f_1$ and $(1/\sqrt{5})(\alpha^2 - \beta^2) = (1/\sqrt{5})((1+\alpha) - (1+\beta)) = (1/\sqrt{5})(\alpha - \beta) = 1 = f_2$. Assume the identity is true for all positive integers up to n. Then $f_{n+1} = f_n + f_{n-1} = (1/\sqrt{5})(\alpha^n - \beta^n) + (1/\sqrt{5})(\alpha^{n-1} - \beta^{n-1}) = (1/\sqrt{5})(\alpha^{n-1}(\alpha+1) - \beta^{n-1}(\beta+1)) = (1/\sqrt{5})(\alpha^{n-1}(\alpha^2) - \beta^{n-1}(\beta^2)) = (1/\sqrt{5})(\alpha^{n+1} - \beta^{n+1})$, which completes the induction.

1.4.46. Computing according to the recurrence relation, we obtain $J_0 = 0$, $J_1 = 1$, $J_2 = 1$, $J_3 = 3$, $J_4 = 5$, $J_5 = 11$, $J_6 = 21$, $J_7 = 43$, $J_8 = 85$, $J_9 = 171$, and $J_{10} = 341$.

1.4.47. We use the principle of mathematical induction. For the basis case $n = 0$, the identity requires $J_1 = 2J_0 + (-1)^0$, and in fact $1 = 2 \cdot 0 + 1$. Now assume that n is a nonnegative integer and $J_{n+1} = 2J_n + (-1)^n$, which we can rewrite as $2J_n = J_{n+1} - (-1)^n$. Then applying the recurrence relation followed by the inductive hypothesis, we have that $J_{n+2} = J_{n+1} + 2J_n = J_{n+1} + J_{n+1} - (-1)^n = 2J_{n+1} + (-1)^{n+1}$, completing the induction.

1.4.48. Following the preamble to Exercise 41, we consider the equation $r^2 - r - 2 = 0$. Factoring, this is $(r-2)(r+1) = 0$ and so the equation has two distinct roots, $r_1 = 2$ and $r_2 = -1$. Consequently, $J_n = C_1 2^n + C_2(-1)^n$ for some constants C_1 and C_2. We use the initial conditions to determine the constants. For $n = 0$, the initial conditions specify that $J_0 = 0$, and substituting into $J_n = C_1 2^n + C_2(-1)^n$ produces $0 = C_1 + C_2$. For $n = 1$, $J_1 = 1$ yields $1 = 2C_1 - C_2$. Adding the two equations, we have $1 = 3C_1$, so $C_1 = \frac{1}{3}$. It follows from the equation with $n = 0$ that $C_2 = -\frac{1}{3}$. Hence, $J_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n$.

1.4.49. a. Following the definition of the recurrence relation, we compute that $j_0 = 2$, $j_1 = 1$, $j_2 = 5$, $j_3 = 7$, $j_4 = 17$, $j_5 = 31$, $j_6 = 65$, $j_7 = 127$, $j_8 = 257$, $j_9 = 511$, and $j_{10} = 1025$.

b. We will prove the statement "both J_n and j_n are odd for all positive integers n" via mathematical induction. For the basis case, observe that J_1 and j_1 both equal 1 and are odd. Now assume that n is a positive integer for which J_n and j_n are both odd. To see that the claim is true for the case $n+1$, note that both $J_{n+1} = J_n + 2J_{n-1}$ and $j_{n+1} = j_n + 2j_{n-1}$ represent the sum of an odd number (by the inductive hypothesis) and a multiple of 2. Since the sum of an odd and even is an odd integer, we conclude both J_{n+1} and j_{n+1} are odd, completing the induction.

c. We use the second principle of mathematical induction. For $n = 0$, the initial conditions specify that $j_0 = 2$ and we calculate $2^0 + (-1)^0 = 2$; for $n = 1$, $j_1 = 1$ and $2^1 + (-1)^1 = 1$. Now we turn to the inductive step and assume $n \geq 1$ is an integer and that $j_k = 2^k + (-1)^k$ for all $k \leq n$, particularly for $k = n$ and $k = n-1$. Combining the recurrence relation with the inductive hypothesis, we have $j_{n+1} = j_n + 2j_{n-1} = 2^n + (-1)^n + 2\left(2^{n-1} + (-1)^{n-1}\right) = 2^n + (-1)(-1)^{n-1} + 2^n + 2(-1)^{n-1} = 2 \cdot 2^n + (-1+2)(-1)^{n-1} = 2^{n+1} + (-1)^{n-1} = 2^{n+1} + (-1)^{n+1}$, as required.

d. Again we use strong induction. For $n = 0$ note $J_0 + j_0 = 0 + 2$ and $2J_1 = 2 \cdot 1$, and for $n = 1$ we have $J_1 + j_1 = 1 + 1$ and $2J_2 = 2 \cdot 1$. Now assume that $n \geq 1$ is an integer and $J_k + j_k = 2J_{k+1}$ for both $k = n$ and $k = n-1$. We must show that $J_{n+1} + j_{n+1} = 2J_{n+2}$. Beginning on the left-hand side and applying the recurrence relations, we have that $J_{n+1} + j_{n+1} = J_n + 2J_{n-1} + j_n + 2j_{n-1} = J_n + j_n + 2(J_{n-1} + j_{n-1})$. Next, invoking the inductive hypothesis, we have $J_n + j_n + 2(J_{n-1} + j_{n-1}) = 2J_{n+1} + 4J_n = 2(J_{n+1} + 2J_n) = 2J_{n+2}$, with the last step being another use of the recurrence relation. This completes the induction and proves the statement.

1.4.50. a. Applying the recursive definition, we have $P(0) = P(1) = P(2) = 1$, $P(3) = P(4) = 2$, $P(5) = 3$, $P(6) = 4$, $P(7) = 5$, $P(8) = 7$, $P(9) = 9$, $P(10) = 12$, $P(11) = 16$, and $P(12) = 21$.

b. For $n \geq 5$, the recursive definition asserts $P(n) = P(n-2) + P(n-3)$. Applying the definition to $P(n-2)$, we can write that equation as $P(n) = (P(n-4) + P(n-5)) + P(n-3)$. Rearranging and applying the recursive definition in reverse, we have $P(n) = P(n-3) + P(n-4) + P(n-5) = P(n-1) + P(n-5)$.

c. First observe that part (b) implies that $P(n+6) = P(n+5) + P(n+1)$ for all $n \geq 0$. We use mathematical induction to prove the clam. For $n = 1$, the equation is $P(0) + P(1) = P(6) - 2$. The left-hand side of that equation is $1 + 1 = 2$ and the right-hand side is $4 - 2 = 2$, so the basis case is verified. Now assume that n is a positive integer and that $\sum_{j=0}^{n} P(j) = P(n+5) - 2$. Then $\sum_{j=0}^{n+1} P(j) = P(n+1) + \sum_{j=0}^{n} P(j) = P(n+1) + P(n+5) - 2 = P(n+6) - 2$, by the inductive hypothesis and the remark about part (b).

d. First observe that part (b) implies that $P(5n+6) = P(5n+5) + P(5n+1)$ and proceed with induction to prove the claim. For $n = 1$ the assertion is that $P(0) + P(5) = P(6)$. By part (a), we see that $P(0) + P(5) = 1 + 3 = 4 = P(6)$. For the inductive step, assume n is a positive integer for which $\sum_{j=0}^{n} P(5j) = P(5n+1)$. Then, using the inductive hypothesis and the observation about part (b), we have $\sum_{j=0}^{n+1} P(5j) = P(5(n+1)) + \sum_{j=0}^{n} P(5j) = P(5n+5) + P(5n+1) = P(5n+6) = P(5(n+1)+1)$, which verifies that the equation holds for $n+1$.

e. Substituting $n+3$ for n in $P(n) = P(n-2) + P(n-3)$ results in $P(n+3) = P(n+1) + P(n)$, from which we obtain $P(n) = P(n+3) - P(n+1)$. We can use this to compute $P(-1) = P(2) - P(0) = 1 - 1 = 0$, $P(-2) = P(1) - P(-1) = 1 - 0 = 1$, $P(-3) = P(0) - P(-2) = 1 - 1 = 0$. Continuing in this manner, we find $P(-4) = 0$, $P(-5) = 1$, $P(-6) = -1$, $P(-7) = 1$, $P(-8) = 0$, $P(-9) = -1$, $P(-10) = 2$, $P(-11) = -2$, and $P(-12) = 1$.

1.5. Divisibility

1.5.1. We find that $3 \mid 99$ since $99 = 3 \cdot 33$, $5 \mid 145$ since $145 = 5 \cdot 29, 7 \mid 343$ since $343 = 7 \cdot 49$, and $888 \mid 0$ since $0 = 888 \cdot 0$.

1.5.2. We see that 1001 is divisible by 7, 11, and 13.

1.5.3. a. Yes, $0 = 7 \cdot 0$.

b. Yes, $707 = 7 \cdot 101$.

c. By the division algorithm, we have $1717 = 245 \cdot 7 + 2$. Since the remainder is nonzero, we know that $7 \nmid 1717$.

d. By the division algorithm, we have $123321 = 17617 \cdot 7 + 2$. Since the remainder is nonzero, we know that $7 \nmid 123321$.

e. By the division algorithm, we have $-285714 = -40817 \cdot 7 + 5$. Since the remainder is nonzero, we know that $7 \nmid -285714$.

f. By the division algorithm, we have $-430597 = -61514 \cdot 7 + 1$. Since the remainder is nonzero, we know that $7 \nmid -430597$.

1.5.4. a. Yes, $0 = 22 \cdot 0$.

b. By the division algorithm, we have $444 = 20 \cdot 22 + 4$. Since the remainder is nonzero, we know that $22 \nmid 444$.

c. Yes, $1716 = 22 \cdot 78$.

d. Yes, $192544 = 22 \cdot 8752$.

e. Yes, $-32516 = 22 \cdot -1478$.

f. By the division algorithm, we have $-195518 = -8888 \cdot 22 + 18$. Since the remainder is nonzero, we know that $22 \nmid -195518$.

1.5.5. a. We have $100 = 5 \cdot 17 + 15$, so the quotient is 5 and the remainder is 15.

b. We have $289 = 17 \cdot 17$, so the quotient is 17 and the remainder is 0.

c. We have $-44 = -3 \cdot 17 + 7$, so the quotient is -3 and the remainder is 7.

d. We have $-100 = -6 \cdot 17 + 2$, so the quotient is -6 and the remainder is 2.

1.5.6. a. The positive integers which divide 12 are 1, 2, 3, 4, 6, and 12.

b. The positive integers which divide 22 are 1, 2, 11 and 22.

c. The positive integers which divide 37 are 1 and 37.

d. The positive integers which divide 41 are 1 and 41.

1.5.7. a. The positive integers which divide 13 are 1 and 13.

b. The positive integers which divide 21 are 1, 3, 7, and 21.

c. The positive integers which divide 36 are 1, 2, 3, 4, 6, 9, 12, 18, and 36

d. The positive integers which divide 44 are 1, 2, 4, 11, 22, and 44.

1.5.8. a. The positive integers which divide 8 are 1, 2, 4, and 8. The positive integers which divide 12 are 1, 2, 3, 4, 6, and 12. The largest integer in both sets is 4, so $(8, 12) = 4$.

b. The positive integers which divide 7 are 1 and 7. The positive integers which divide 9 are 1, 3 and 9. The largest integer in both sets is 1, so $(7, 9) = 1$.

c. The positive integers which divide 15 are 1, 3, 5, and 15. The positive integers which divide 25 are 1, 5 and 25. The largest integer in both sets is 5, so $(15, 25) = 5$.

d. The positive integers which divide 16 are 1, 2, 4, 8, and 16. The positive integers which divide 27 are 1, 3, 9, and 27. The largest integer in both sets is 1, so $(16, 27) = 1$.

1.5.9. a. The positive integers which divide 11 are 1 and 11. The positive integers which divide 22 are 1, 2 and 11. The largest integer in both sets is 11, so $(11, 22) = 11$.

b. The positive integers which divide 36 are 1, 2, 3, 4, 6, 9, 12, 18, and 36. The positive integers which divide 42 are 1, 2, 3, 6, 7, 14, 21, and 42. The largest integer in both sets is 6, so $(36, 42) = 6$.

c. The positive integers which divide 21 are 1, 3, 7, and 21. The positive integers which divide 22 are 1, 2, 11, and 22. The largest integer in both sets is 1, so $(21, 22) = 1$.

d. The positive integers which divide 16 are 1, 2, 4, 8, and 16. The positive integers which divide 64 are 1, 2, 4, 8, 16, 32, and 64. The largest integer in both sets is 16, so $(16, 64) = 16$.

1.5.10. Note that 10 is divisible by 2 and 5. Since 2, 4, 6, and 8 are divisible by 2 and since 5 is divisible by 5, none of these integers is relatively prime to 10. This leaves 1, 3, 7, and 9, which are all relatively prime to 10.

1.5.11. The only positive integers which divide 11 are 1 and 11. Therefore each of 1, 2, 3, $\ldots$, 10 is relatively prime to 11.

1.5.12. Since $(a, b) = (b, a)$ we can assume without loss of generality that $a \leq b$. We check to see that this leaves us with $(1, 1)$, $(1, 2)$, $(1, 3)$, $\ldots$, $(1, 10)$, $(2, 3)$, $(2, 5)$, $(2, 7)$, $(2, 9)$, $(3, 4)$, $(3, 5)$, $(3, 7)$, $(3, 8)$, $(3, 10)$, $(4, 5)$, $(4, 7)$, $(4, 9)$, $(5, 6)$, $(5, 7)$, $(5, 8)$, $(5, 9)$, $(6, 7)$, $(7, 8)$, $(7, 9)$, $(7, 10)$, $(8, 9)$ and $(9, 10)$.

1.5.13. Without loss of generality, we assume $a < b$. This leaves us with $(10, 11)$, $(10, 13)$, $(10, 17)$, $(10, 19)$, $(11, 12)$, $(11, 13)$, $\ldots$, $(11, 20)$, $(12, 13)$, $(12, 17)$, $(12, 19)$, $(13, 14)$, $(13, 15)$, $\ldots$, $(13, 20)$, $(14, 15)$, $(14, 17)$, $(14, 19)$, $(15, 16)$, $(15, 17)$, $(15, 19)$, $(16, 17)$, $(16, 19)$, $(17, 18)$, $(17, 19)$, $(17, 20)$, $(18, 19)$ and $(19, 20)$.

1.5.14. Suppose that $a \mid b$ and $b \mid a$. Then there are integers k and l such that $b = ka$ and $a = lb$. This implies that $b = klb$, so that $kl = 1$. Hence either $k = l = 1$ or $k = l = -1$. It follows that either $a = b$ or $a = -b$.

1.5.15. By hypothesis we know $b = ra$ and $d = sc$, for some r and s. Thus $bd = rs(ac)$ and $ac \mid bd$.

1.5.16. We have $6 \mid 2 \cdot 3$, but 6 divides neither 2 nor 3.

1.5.17. If $a \mid b$, then $b = na$ and $bc = n(ca)$, i.e. $ac \mid bc$. Now, suppose $ac \mid bc$. Thus $bc = nac$ and, as $c \neq 0$, $b = na$, i.e., $a \mid b$.

1.5.18. Suppose $a \mid b$. Then $b = na$, and $b - a = na - a = (n - 1)a$. Since a and b are positive $(n - 1)a$ is positive and $a \leq b$.

1.5.19. By definition, $a \mid b$ if and only if $b = na$ for some integer n. Then raising both sides of this equation to the kth power yields $b^k = n^k a^k$ whence $a^k \mid b^k$.

1.5.20. Suppose that x and y are even. Then $x = 2k$ and $y = 2l$ where k and l are integers. Hence $x + y = 2k + 2l = 2(k + l)$ so that $x + y$ is also even. Suppose that x and y are odd. Then $x = 2k + 1$ and $y = 2l + 1$ where k and l are integers. Hence $x + y = (2k + 1) + (2l + 1) = 2k + 2l + 2 = 2(k + l + 1)$, so that $x + y$ is even. Suppose that x is even and y is odd. Then $x = 2k$ and $y = 2l + 1$ where k and l are integers. Hence $x + y = 2k + (2l + 1) = 2(k + l) + 1$. It follows $x + y$ is odd.

1.5.21. Let a and b be odd, and c even. Then $ab = (2x + 1)(2y + 1) = 4xy + 2x + 2y + 1 = 2(2xy + x + y) + 1$, so ab is odd. On the other hand, for any integer n, we have $cn = (2z)n = 2(zn)$ which is even.

1.5.22. By the division algorithm, there exist integers s, t such that $a = bs + t, 0 < t < b$ since $b \nmid a$. If t is odd, then we are done. If t is even, then $b - t$ is odd, $|t - b| < b$, and $a = b(s + 1) + (t - b)$.

1.5.23. By the division algorithm, $a = bq + r$, with $0 \leq r < b$. Thus $-a = -bq - r = -(q + 1)b + b - r$. If $0 \leq b - r < b$ then we are done. Otherwise $b - r = b$, or $r = 0$ and $-a = -qb + 0$.

1.5.24. We have $a = qb + r = (tc + s)b + r = tcb + bs + r$.

1.5.25. a. The division algorithm covers the case when b is positive. If b is negative, then we may apply the division algorithm to a and $|b|$ to get a quotient q and remainder r such that $a = q|b| + r$ and $0 \leq r < |b|$. But since b is negative, we have $a = q(-b) + r = (-q)b + r$, as desired.

b. We have $17 = -7(-2) + 3$. Here $r = 3$.

1.5.26. This is called the *least remainder algorithm*. Suppose that a and b are positive integers. By the division algorithm there are integers s and t with $a = bs + t$ and $0 \leq t < b$. If $0 \leq t \leq \frac{b}{2}$ set $r = t, e = 1$, and $q = s$, so that $a = bq + er$ with $0 \leq r \leq \frac{b}{2}$. If $\frac{b}{2} < t < b$ set $r = b - t, e = -1$, and $q = s + 1$ so that $bq + er = b(s + 1) + (t - b) = bs + t = a$ and $0 < r = t - b < \frac{b}{2}$. Hence there are integers q, e and r such that $a = bq + er$ where $e = \pm 1$ and $0 \leq r \leq \frac{b}{2}$.

1.5.27. By the division algorithm, let $m = qn + r$, with $0 \leq r < n - 1$ and $q = [m/n]$. Then $[(m + 1)/n] = [(qn + r + 1)/n] = [q + (r + 1)/n] = q + [(r + 1)/n]$ as in Example 1.4. If $r = 0, 1, 2, \ldots, n - 2$, then $m \neq kn - 1$ for any integer k and $1/n \leq (r + 1)/n < 1$ and so $[(r + 1)/n] = 0$. In this case, we have $[(m + 1)/n] = q + 0 = [m/n]$. On the other hand, if $r = n - 1$, then $m = qn + n - 1 = n(q + 1) - 1 = nk - 1$, and $[(r + 1)/n] = 1$. In this case, we have $[(m + 1)/n] = q + 1 = [m/n] + 1$.

1.5.28. Suppose $n = 2k$. Then $n - 2[n/2] = 2k - 2[2k/2] = 0$. On the other hand, suppose $n - 2[n/2] = 0$. Then $n/2 = [n/2]$ and $n/2$ is an integer. In other words, n is even.

1.5.29. The positive integers divisible by the positive integer d are those integers of the form kd where k is a positive integer. The number of these that are less than x is the number of positive integers k with $kd \leq x$, or equivalently with $k \leq x/d$. There are $[x/d]$ such integers.

1.5.30. There are $[1000/5] = 200$ positive integers not exceeding 1000 that are divisible by 5, $[1000/25] = 40$ such integers that are divisible by 25, $[1000/125] = 8$ such integers that are divisible by 125, and $[1000/625] = 1$ such integer that is divisible by 625.

1.5.31. There are $[1000/7] - [100/7] = 142 - 14 = 128$ integers between 100 and 1000 that are divisible by 7. There are $[1000/49] - [100/49] = 20 - 2 = 18$ integers between 100 and 1000 that are divisible by 49.

1.5.32. The number of integers not exceeding 1000 that are not divisible by either 3 or 5 equals $1000 - ([1000/3] + [1000/5]) + [1000/15] = 533$.

1.5.33. Using the Principle of Inclusion-Exclusion, the answer is $1000 - ([1000/3] + [1000/5] + [1000/7]) + ([1000/15] + [1000/21] + [1000/35]) - ([1000/105) = 1000 - (333 + 200 + 142) + (66 + 47 + 28) - 9 = 457$.

1.5.34. For an integer to be divisible by 3, but not by 4, an integer must be divisible by 3, but not by 12. There are $[1000/3] = 333$ positive integers not exceeding 1000 that are divisible by 3. Of these $[1000/12] = 82$ are divisible by 12 (since anything that is divisible by 12 is automatically divisible by 3). Hence there are $333 - 83 = 250$ possible integers not exceeding 1000 that are divisible by 3, but not by 4.

1.5.35. Let w be the weight of a letter in ounces. Note that the function $-[-x]$ rounds x up to the least integer less than or equal to x. (That is, it's the equivalent of the ceiling function.) The cost of mailing a letter weighing w ounces is, then, 55 cents plus 20 cents for each ounce or part thereof more than 1, so we need to round $w - 1$ up to the next integer. So the cost is $c(w) = 55 - [1 - w]20$ cents. Suppose that $55 - [1 - w]20 = 195$ then $-[1 - w]20 = 195 - 55 = 140 = 7 \cdot 20$. Then $[1 - w] = -7$, so $-7 \leq 1 - w <$

-6, or $7 < w \leq 8$. So a letter weighing more than 7 ounces but no more than 8 ounces would cost \$1.95. Suppose that $55 - [1-w]20 = 265$ then $-[1-w]20 = 265 - 55 = 210$, which is not a multiple of 20, so no letter can cost \$2.65.

1.5.36. Note that $a^3 - a = a(a^2-1) = (a-1)a(a+1)$. By the division algorithm $a = 3k, a = 3k+1$, or $a = 3k+2$, where k is an integer. If $a = 3k$, 3 divides a, if $a = 3k+1$ then $a - 1 = 3k$, so that 3 divides $a-1$, and if $a = 3k+2$, then $a+1 = 3k+3 = 3(k+1)$, so that 3 divides $a+1$. Hence 3 divides $(a-1)a(a+1) = a^3 - a$ for every nonnegative integer a. (Note: This can also be proved using mathematical induction.)

1.5.37. Multiplying two integers of this form gives us $(4n+1)(4m+1) = 16mn + 4m + 4n + 1 = 4(4mn + m + n) + 1$. Similarly, $(4n+3)(4m+3) = 16mn + 12m + 12n + 9 = 4(4mn + 3m + 3n + 2) + 1$.

1.5.38. Suppose that n is odd. Then $n = 2t+1$ where t is an integer. It follows that $n^2 = (2t+1)^2 = 4t^2 + 4t + 1 = 4t(t+1) + 1$. Now if t is even, then $t = 2u$ where u is an integer. Hence $n^2 = 8u(2u+1)+1 = 8k+1$, where $k = u(2u+1)$ is an integer. If t is odd, then $t = 2u+1$ where u is an integer. Hence $n^2 = (8u+4)(2u+2)+1 = 8(2u+1)(u+1)+1 = 8k+1$, where $k = (2u+1)(u+1)$.

1.5.39. Every odd integer may be written in the form $4k+1$ or $4k+3$. Observe that $(4k+1)^4 = 16^2k^4 + 4(4k)^3 + 6(4k)^2 + 4(4k) + 1 = 16(16k^4 + 16k^3 + 6k^2 + k) + 1$. Proceeding further, $(4k+3)^4 = (4k)^4 + 12(4k)^3 + 54(4k)^2 + 108(4k) + 3^4 = 16(16k^4 + 48k^3 + 54k^2 + 27k + 5) + 1$.

1.5.40. The product of the integers $6k+5$ and $6l+5$ is $(6k+5)(6l+5) = 36kl + 30(k+l) + 25 = 6[6kl + 5(k+l) + 4] + 1 = 6N + 1$ where $N = 6kl + 5(k+l) + 4$. Hence this product is of the form $6N+1$.

1.5.41. Of any consecutive three integers, one is a multiple of three. Also, at least one is even. Therefore, the product is a multiple of $2 \cdot 3 = 6$.

1.5.42. The basis step is completed by noting that $1^5 - 1 = 0$ is divisible by 5. For the inductive hypothesis, assume that $n^5 - n$ is divisible by 5. This implies that there is an integer k such that $n^5 - n = 5k$. It follows that $(n+1)^5 - (n+1) = (n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) - (n+1) = (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n) = 5k + 5l = 5(k+l)$. Hence $(n+1)^5 - (n+1)$ is also divisible by 5.

1.5.43. For the basis step note that $0^3 + 1^3 + 2^3 = 9$ is a multiple of 9. Suppose that $n^3 + (n+1)^3 + (n+2)^3 = 9k$ for some integer k. Then $(n+1)^3 + (n+2)^3 + (n+3)^3 = n^3 + (n+1)^3 + (n+2)^3 + (n+3)^3 - n^3 = 9k + n^3 + 9n^2 + 27n + 27 - n^3 = 9k + 9n^2 + 27n + 27 = 9(k + n^2 + 3n + 3)$ which is a multiple of 9.

1.5.44. We prove this by mathematical induction. We will prove that f_{3n-2} is odd, f_{3n-1} is odd, and f_{3n} is even whenever n is a positive integer. For $n = 1$ we see that $f_{3\cdot1-2} = f_1 = 1$ is odd, $f_{3\cdot1-1} = f_2 = 1$ is odd, and $f_{3\cdot1} = f_3 = 2$ is even. Now assume that f_{3n-2} is odd, f_{3n-1} is odd, and f_{3n} is even where n is a positive integer. Then $f_{3(n+1)-2} = f_{3n+1} = f_{3n} + f_{3n-1}$ is odd since f_{3n} is even and f_{3n-1} is odd, $f_{3(n+1)-1} = f_{3n+2} = f_{3n+1} + f_{3n}$ is odd since f_{3n+1} is odd and f_{3n} is even, and $f_{3(n+1)} = f_{3n+3} = f_{3n+2} + f_{3n+1}$ is even since f_{3n+2} and f_{3n+1} are odd. This completes the proof.

1.5.45. We proceed by mathematical induction. The basis step is clear. Assume that only f_{4n}'s are divisible by 3 for $f_i, i \leq 4k$. Then, as $f_{4k+1} = f_{4k} + f_{4k-1}, 3 \mid f_{4k}$ and $3 \mid f_{4k+1}$ gives us the contradiction $3 \mid f_{4k-1}$. Thus $3 \nmid f_{4k+1}$. Continuing on, if $3 \mid f_{4k}$ and $3 \mid f_{4k+2}$, then $3 \mid f_{4k+1}$, which contradicts the statement just proved. If $3 \mid f_{4k}$ and $3 \mid f_{4k+3}$, then since $f_{4k+3} = 2f_{4k+1} + f_{4k}$, we again have a contradiction. But, as $f_{4k+4} = 3f_{4k+1} + 2f_{4k}$, and $3 \mid f_{4k}$ and $3 \mid 3 \cdot f_{4k+1}$, we see that $3 \mid f_{4k+4}$.

1.5.46. We proceed by induction. The basis step is clear. Suppose f_n is divisible by 4. By Exercise 44, $f_{n+1}, f_{n+2}, f_{n+4}, f_{n+5}$ are all odd. Suppose f_{n+3} is divisible by 4. Now, $f_{n+3} = 2f_{n+1} + f_n$. Since f_n and f_{n+3} are divisible by 4, so must by $2f_{n+1}$. This is a contradiction. On the other hand, $f_{n+6} = 8f_{n+1} + f_n$. Since both terms are multiples of 4, so is f_{n+6}.

1.5.47. First note that for $n > 5$, $5f_{n-4} + 3f_{n-5} = 2f_{n-4} + 3(f_{n-4} + f_{n-5}) = 2f_{n-4} + 3f_{n-3} = 2(f_{n-4} + f_{n-3}) + f_{n-3} = 2f_{n-2} + f_{n-3} = f_{n-2} + f_{n-2} + f_{n-3} = f_{n-2} + f_{n-1} = f_n$, which proves the first identity.

Now note that $f_5 = 5$ is divisible by 5. Suppose that f_{5n} is divisible by 5. By the identity above $f_{5n+5} = 5f_{5n+5-4} + 3f_{5n+5-5} = 5f_{5n+1} + 3f_{5n}$, which is divisible by 5 since $5f_{5n+1}$ is a multiple of 5 and, by the induction hypothesis, so is f_{5n}. This completes the induction.

1.5.48. We use mathematical induction on the integer m. For $m = 1$ we have $f_{n+1} = f_{n-1}f_1 + f_nf_2 = f_{n-1} + f_n$ which is true by the definition of the Fibonacci numbers. For $m = 2$ we have $f_{n+2} = f_{n-1}f_2 + f_nf_3 = f_{n-1} + 2f_n = f_{n-1} + f_n + f_n = f_{n+1} + f_n$ which is true by the definition of the Fibonacci numbers. This finishes the basis step of the proof. Now assume that $f_{n+m} = f_mf_{n+1} + f_{m-1}f_n$ holds for all integers m with $m < k$. We will show that it must also hold for $m = k$. We have $f_{n+k-2} = f_{k-2}f_{n+1} + f_{k-3}f_n$ and $f_{n+k-1} = f_{k-1}f_{n+1} + f_{k-2}f_n$. Adding these two equations gives $f_{n+k-2} - f_{n+k-1} = f_{n+1}(f_{k-2} + f_{k-1}) + f_n(f_{k-3} + f_{k-2})$. Hence $f_{n+k} = f_{n+1}f_k + f_nf_{k-1}$. Hence the identity is also true for $m = k$. We now show that $f_m \mid f_n$ if $m \mid n$. Since $m \mid n$ we have $n = km$. We prove this using mathematical induction on k. For $k = 1$ we have $n = m$ so $f_m \mid f_n$ since $f_m = f_n$. Now assume f_{mk} is divisible by f_m. Note that $f_{m(k+1)} = f_{mk+m} = f_{mk-1}f_m + f_{mk}f_{m+1}$. The first product is divisible by f_m since f_m is a factor in this term and the second product is divisible by f_m by the inductive hypothesis. Hence $f_m \mid f_{m(k+1)}$. This finishes the inductive proof.

1.5.49. Iterating the transformation T starting with 39 we find that $T(39) = 59$; $T(59) = 89$; $T(89) = 134$; $T(134) = 67$; $T(67) = 101$; $T(101) = 152$; $T(152) = 76$; $T(76) = 38$; $T(38) = 19$; $T(19) = 29$; $T(29) = 44$; $T(44) = 22$; $T(22) = 11$; $T(11) = 17$; $T(17) = 26$; $T(26) = 13$; $T(13) = 20$; $T(20) = 10$; $T(10) = 5$; $T(5) = 8$; $T(8) = 4$; $T(4) = 2$; $T(2) = 1$.

1.5.50. If $3n$ is odd, then so is n. So, $T(n) = (3n+1)/2 = 2^{2k}/2 = 2^{2k-1}$. Because $T(n)$ is a power of 2, the exponent will decrease down to one with repeated applications of T.

1.5.51. We prove this using the second principle of mathematical induction. Since $T(2) = 1$, the Collatz conjecture is true for $n = 2$. Now assume that the conjecture holds for all integers less that n. By assumption there is an integer k such that k iterations of the transformation T, starting at n, produces an integer m less than n. By the inductive hypothesis there is an integer l such that iterating T l times starting at m produces the integer 1. Hence iterating T $k + l$ times starting with n leads to 1. This finishes the proof.

1.5.52. Suppose $n = 2k$ for some k. Then $T(n) = k < 2k = n$. Suppose that $n = 4k + 1$ for some k. Then $T(T(n)) = T(6k + 2) = 3k + 1 < 4k + 1 = n$. Now suppose that $n = 8k + 3$, where k is an even number. $T(T(T(T(n)))) = 9k/2 + 1 < 8k + 3 = n$. This leaves 17 numbers to be considered, $7, 11, 15, 23, 27, 31, 39, 43, 47, 55, 59, 63, 71, 75, 79, 87, 91, 95$. These can be methodically tested. The worst of them is 27, which requires over 70 applications of T to reach 1.

1.5.53. We first show that $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an even integer. By the binomial theorem it follows that $(2+\sqrt{3})^n + (2-\sqrt{3})^n = \sum_{j=0}^{n}\binom{n}{j}2^j\sqrt{3}^{n-j} + \sum_{j=0}^{n}\binom{n}{j}2^j(-1)^{n-j}\sqrt{3}^{n-j} = 2(2^n + \binom{n}{2}3\cdot 2^{n-2} + \binom{n}{4}3^2\cdot 2^{n-4} + \cdots) = 2l$ where l is an integer. Next, note that $(2-\sqrt{3})^n < 1$. Since $(2+\sqrt{3})^n$ is not an integer, we see that $[(2+\sqrt{3})^n] = (2+\sqrt{3})^n + (2-\sqrt{3})^n - 1$. It follows that $[(2+\sqrt{3})^n]$ is odd.

1.5.54. Suppose $[a/2] + [a/3] + [a/5] = a$. By the division algorithm, there exist integers q and r such that $a = 30q + r$ with $0 \le r \le 29$. Since a is positive, we must have $q \ge 0$. Then $[a/2] + [a/3] + [a/5] = [(30q + r)/2] + [(30q + r)/3] + [(30q + r)/5] = 15q + [r/2] + 10q + [r/3] + 6q + [r/5] = 31q + [r/2] + [r/3] + [r/5] = 30q + r$. Simplifying gives us $q = r - [r/2] - [r/3] - [r/5]$. Note the following fact: If c and b are positive integers, the division algorithm gives us integers s and t with $c = sb + t$ and $0 \le t < b$. Then $[c/b] = [(sb + t)/b] = sb/b = (c - t)/b \ge (c - (b - 1))/b$. Using this inequality in our last equation gives us $q \le r - (r-1)/2 - (r-2)/3 - (r-4)/5 = r(-1/30) + 59/30 \le 59/30$ since $r \ge 0$. Thus $q = 0$ or 1, which forces $1 \le a \le 30(1) + 29 = 59$. So we need only check these 59 numbers. We compute $a - ([a/2] + [a/3] + [a/5])$ for $a = 1, 2, 3, \ldots, 59$ and find the 29 solutions: 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 47, 49, 53, and 59.

1.5.55. We prove existence of q and r by induction on a. First assume that $a \ge 0$. Assume existence in the division algorithm holds for all nonnegative integers less than a. If $a < b$, then let $q = 0$ and $r = a$, so that $a = qb + r$ and $0 \le r = a < b$. If $a \ge b$, then $a - b$ is nonnegative and by the induction hypothesis, there exist q' and r' such that $a - b = q'b + r'$, with $0 \leq r' < b$. Then $a = (q' + 1)b + r'$, so we let $q = q' + 1$ and $r = r'$. This establishes the induction step, so existence is proved for $a \geq 0$. Now suppose $a < 0$. Then $-a > 0$ so from our work above, there exist q' and r' such that $-a = q'b + r'$ and $0 \leq r' < b$. Then $a = -q'b - r'$. If $r' = 0$, we're done. If not, then $0 \leq b - r' < b$ and $a = (-q' - 1)b + b - r'$, so letting $q = -q' - 1$ and $r = b - r'$ satisfies the theorem. Uniqueness is proved just as in the text.