Foundations of Geometry 3rd edition Venema Solutions Manual

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INSTRUCTOR'S SOLUTIONS MANUAL

FOUNDATIONS OF GEOMETRY

THIRD EDITION

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Pearson

Foundations of Geometry 3rd edition Venema Solutions Manual

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Teaching from FOUNDATIONS OF GEOMETRY, third edition

The following suggestions for teaching from the third edition of FOUNDATIONS OF GEOMETRY are quite subjective. They are presented in the hope that they will be useful to some instructors, but they are not meant to be at all prescriptive. This first section contains a few general comments and a table showing topic dependencies. The second part contains comments about how to cover individual chapters from the book.

Getting Started

The main consideration in getting the course off the ground is that you want get to Chapter 4 as quickly as possible. Chapters 1 and 2 should be treated as introductory and the course should not be allowed to get bogged down in them. The comments below on individual chapters contain specific suggestions about how to achieve this.

One thing students may not understand at the outset is what it means to study the *foundations* of geometry and why understanding the foundations might be an important goal. The Van Hiele model is useful in helping to explain this. Ideally students should be at Level 3 when they enter the course and the course should bring them to Level 4. (Assuming the steps are numbered as in Appendix D.) Of course some students are not at Level 3 when they enter the course. The first three and a half chapters are designed to help bring students to that level. It is important that students reach Level 3 before they are pushed to get to Level 4.

Supplementary Reading

It is suggested that students read something about the history of geometry while working through this book. *Euclid's Window* [**Mlo01**] has a terrible reputation (see [**Lan02**]), but it is really not such a bad choice—for this particular course, at least. It is the only book I have found that is exclusively devoted to geometry and which describes the various revolutions that have taken place in the conventional understanding of the relationship between the theorems of geometry and the real world. The chapter on string theory is rather thin. The book is written in a style that students will appreciate and enjoy even if you don't.

Choose one of the books listed in the suggested readings at the end of the chapters and stick with it. Assign readings from your selected book as the topics arise.

Continuity

There are occasional proofs that involve (ϵ, δ)-arguments. If your class is not comfortable with such proofs, they can be omitted without serious consequence. No harm is done by simply accepting the continuity results as additional axioms.

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Logical Relationships Between Topics

The following chart indicates relationships between topics and is designed to assist an instructor in making decisions about what to cover and what to omit.

Section	Topic	Prerequisites	Needed for
§3.5	Continuity Axiom	Crossbar Theorem	Circular Continuity (§8.4)
§4.3	Continuity of Distance	Triangle Inequality	Dissection Theorem (7.3.2), Bolyai's Theorem (7.6.1), and Elementary Circular Continuity (8.1.12)
§4.8	Properties of Saccheri and Lambert quadrilaterals	§§4.6, 4.8	Chapter 6
§5.6	Exploring the geometry of the triangle	§§5.1–5.4	§8.6
§6.1	Theorem 6.1.12	§6.1	Bolyai's Theorem (7.6.1)
§6.7	Properties of angle of parallelism	§§6.3–6.6	Properties of defect (§6.8) and Bolyai's Theorem (7.6.1)
§7.6	Hyperbolic dissection theory and Bolyai's Theorem	§7.5, §6.1, and Continuity of defect (Theorem 6.8.5)	Not used elsewhere
§§8.1–8.2	Circles in neutral geometry	Chapter 4	§8.3, Chapter 9
§8.3	Circles in Euclidean geometry	§§8.1, 8.2, 5.1–5.4	§§8.6, 9.3, 10.7
§8.4	Circular continuity	§§8.1, 8.2	Chapter 9 (see comment below)
§8.5	Area and circumference of Euclidean circles	§§5.3, 7.2, 8.3	Not used elsewhere
§8.6	Exploring Euclidean circles	§§5.6, 8.3	Not used elsewhere
Chap. 9	Compass & straightedge constructions	Circular continuity (§8.4), which may be assumed as an axiom	Not used elsewhere
§10.1	Properties of Isometries	Chapter 4	§§10.2–10.7
§§10.3, 10.4	Classification of rigid motions	§10.2	Not used elsewhere
§10.5	Reflection Postulate	§10.1	Proof that Poincaré models satisfy neutral postulates (§11.2)
§10.6	Similarity Transformations	§10.3	Not used elsewhere
§10.7	Inversions in circles	§§5.3, 8.3	Proof that Poincaré models satisfy neutral postulates (§11.2)
Chap. 11	Proofs that the models satisfy the neutral postulates	§§10.5 and 10.7	Not used elsewhere
Chap. 12	Polygonal models and geometry of space	At least some of Chapter 6	Not used elsewhere

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GeoGebra

As indicated in the textbook, the use of GeoGebra is recommended. The software is being continuously improved, so some superficial aspects of the current version of the software may be different from what is described in the text. The version of GeoGebra that was in use when the textbook was originally written is now called "GeoGebra Classic" and it is still the recommended tool for use with this textbook. When you open GeoGebra Classic you will see a number of options; click on "Geometry" to get started.

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Comments on Individual Chapters

Chapter 1. This chapter should not be omitted because it provides the background for the rest of the course. But there is no need to spend more than one or two days on it. It is important to discuss the statements of Euclid's postulates and their place in the logical arrangement of the *Elements*. It is also a good idea to go step-by-step through one or two of the proofs from Euclid that are included and to discuss both the proof as Euclid would have understood it and also the gaps that we now see in Euclid's proofs.

Most students will not be prepared to fully appreciate the subtleties of the critique of Euclid's proofs. It might be a good idea to review Euclid's proofs again later in connection with the neutral geometry of Chapters 3 and 4. By that time the students may have gained the necessary maturity to understand why some of Euclid's proofs are considered to be incomplete.

It will be apparent to the reader that I have a rather high regard for Euclid and his work. In particular, I think it is a mistake to tell students that Euclid's proofs are "flawed" just because they do not conform to modern standards of rigor. Referring to Euclid's proofs as flawed leads students to conclude that Euclid is not worth studying when, in fact, nothing could be further from the truth. I think it is important to try, as best we can, to understand how Euclid understood the role of axioms and to recognize (in the next chapter) that the modern understanding is different.

Much of Chapter 1 can be assigned as reading.

Suggestion: After the students have tried to work Exercise 1.7.11 by hand, have them draw the diagram using GeoGebra. They can vary the shape of the triangle and see that the diagrams shown in the text never actually occur.

Chapter 2. This chapter is also important to what comes later and should not be omitted. But again you should not spend too much time on the chapter. Two days suffice for sections 2.1 through 2.3 and section 2.4 can be assigned as reading.

Incidence geometry is a kind of "laboratory" in which to try out some of the ideas in the course. It is important for the course in two ways. First, it provides a setting in which to understand what it means to say that a parallel postulate is independent of the other postulates. Second, it is a setting in which to learn to write proofs and to base those proofs on what is actually stated in the postulates (and nothing else).

The philosophy of the book is that geometry should be seen as a subject that is closely tied to actual spatial relationships in the real world. In keeping with that philosophy, the only example of an axiomatic system included in the chapter is incidence geometry. An instructor may want to include one or more examples that involve undefined terms that do not carry as much intuitive freight as the terms "point" and "line." Something like the "Scorpling Flugs" example on pages 164–168 of [**Tru01**] would be good.

How much time you spend on how to write proofs will depend on the background of the students. If the students have never written proofs before, you could spend two or

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three days discussing some of the principles explained in §2.5. If the students already have experience writing proofs, you could simply ask them to read the section as review and go directly to the proofs of the theorems from incidence geometry in §2.6. In any case, the theorems from incidence geometry should be discussed and several of the proofs should be assigned as exercises. In my experience most students make rather fundamental logical errors in their first attempts to prove these theorems. In particular, most students want to read Incidence Axiom 3 to say what Theorems 2.6.8 and 2.6.9 say. (See the notes after the solutions to Exercises 2.6.3, and 2.6.4, below.) I have not found a way to prevent students from making this error, but they seem to learn something when they make the error and it is then explained to them. As a result, the experience of writing proofs of the theorems in neutral geometry can be an important step of the development of student thinking.

It might be helpful to distinguish between the style of a written proof and the style of a proof that is communicated orally. In this chapter you will want to use class time to go line-by-line through at least one written proof, but later in the course you should not use valuable class time to present the details of written proofs. If you make this distinction clear, you will be able to require students to submit careful proofs that include complete written justifications for each step even though the proofs you present in class may follow a different style. Whatever you do in class, the students can use the proofs in the text as models for their written proofs.

Chapter 3. In this chapter the foundations of the remainder of the course are laid, so the chapter should be covered carefully. At the same time, this is not where the interesting and beautiful theorems of geometry are to be found, so you don't want to get bogged down in the chapter. Students will probably not learn to appreciate proofs in this chapter either (that will happen in the next chapter), so only a limited number of proofs should be assigned.

It is important that future teachers have some idea of the kind of thinking that goes into the selection of a system of axioms as well as some knowledge of the range of options available when a system of axioms is being chosen. For that reason, at least some of Appendices B and C should be discussed in some form. But it might be better not to slow down for that discussion at this point in the course. The two appendices can be assigned later in the course, after the students have seen how the axioms support the rest of the course in geometry, and discussed at that time.

There is another reason for preferring metric axioms that is not emphasized in the text: essentially all modern work in geometry is done in the context of a metric. From that point of view, the axioms of Mac Lane are to be preferred over any of the other systems mentioned in the text. The decision to use the more SMSG-like axioms in Chapter 3 is based on the need to make direct connections with what is actually done in high school geometry.

All the statements of the six neutral axioms and the related definitions should be covered carefully. The examples and explanatory material in the second half of Section 3.2 are somewhat optional. They add to student understanding of what is being asserted in the Ruler Postulate, but coordinate functions, for example, are not essential to the logical development of the course. The Ray Theorem and the Consistency of Betweenness Theorem are used repeatedly in Chapter 4 to fill in some of the gaps in Euclid's proofs. For that reason they should at least be discussed. But the proofs could be omitted and the results accepted as additional axioms if time pressure requires it.

Section 3.5 contains two results that are also important for the later development of the course. At a minimum, students need to know the statements of the named theorems in the

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section and all the definitions. The proofs in the section are not difficult; the problem is that they are also not interesting. Most students find the proofs in Chapter 4 to be much more satisfying. It is perfectly respectable and logically correct to simply assume the main results of the section as axioms and move on. Another possibility is to skip many of the proofs in this section temporarily and come back to them later, after the next few chapters have been studied. On the other hand, a strong argument in favor of studying this section carefully is the fact that these results are an essential part of the foundations of geometry and only by mastering them will you really come to understand the foundations. In particular, that is the only way to appreciate what is normally omitted from high school treatments of Euclidean geometry.

In §10.5 it will be important to know that the Existence of Perpendicular Bisectors can be proved without Side-Angle-Side.

Most instructors will want to include some of the proofs from Chapter 3 and omit others. It is, for example, quite acceptable to treat the Betweenness Theorem for Rays and the Linear Pair Theorem as two additional parts of the Protractor Postulate. After all, exactly that is quite often done in courses at this level—see [**WW04**], [**Kay01**], [**SMS61**], [**Moi90**], and [**UCS09**], for example. The instructor can simply point to the fact that there is a proof in the book as evidence that these axioms can be proved as consequences of the others. It is not necessary for every student of the subject to verify this.

Be careful with the exercises in Chapter 3. Some of them (such as 3.2.20, 3.2.21, 3.2.22, 3.2.26, 3.2.27, 3.3.6, and 3.3.7) may require mathematical maturity that the students do not yet have. Think carefully before assigning these exercises. On the other hand, the proofs of existence and uniqueness of midpoints and bisectors should definitely be assigned.

Chapter 4. This chapter is the heart of the course. In the preceding chapters it was important to keep moving. Now it is time to slow down and enjoy the material in this chapter. It's classic and beautiful mathematics, but understanding it is well within the grasp of students of average ability.

It is in this chapter that the students learn to write good proofs and begin to really appreciate proofs. Student ability to write proofs and to construct mathematical arguments should develop significantly in this chapter. The proofs in the first part of the chapter are straightforward arguments that are good practice for what is to come. The proofs of the equivalences with the Euclidean Parallel Postulate (EPP) require a substantial increase in the level of thinking involved. Once students reach this level they feel good about the fact that they can prove so many statements are equivalent to EPP.

Try to assign as many of the omitted proofs as possible. Students come to a deep understanding of the material by working out these proofs for themselves. If you are not able to assign all of them as homework, you could discuss some of them in class. These proofs are not difficult and students learn a lot from working through the details for themselves. It may appear that some of the hints give away too much, but most of the students do not see it that way. If they know the basic outline of the proof they can concentrate on the form of the proof.

I think it is important in these early proofs that students include a justification for every step in the argument. Learning this discipline early heads off lots of sloppiness that leads to trouble later. Each instructor will have to set a standard that is appropriate for his or her own class. The solutions provided in this manual include a justification for nearly every step.

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FOUNDATIONS OF GEOMETRY

SECOND EDITION

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PEARSON

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Amsterdam Cape Town Dubai London Madrid Milan Munich Paris Montreal Toronto
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Solutions to Exercises in Chapter 1

1.6.1 Check that the formula $A = \frac{1}{4}(a + c)(b + d)$ works for rectangles but not for parallelograms.

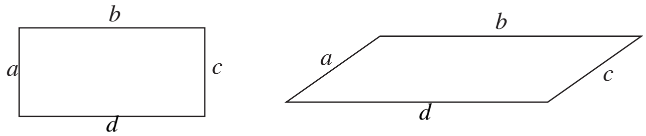

FIGURE S1.1: Exercise 1.6.1. A rectangle and a parallelogram

For rectangles and parallelograms, $a = c$ and $b = d$ and $Area = base{*}height$.
For a rectangle, the base and the height will be equal to the lengths of two adjacent sides. Therefore $A = a * d = \frac{1}{2}(a + a) * \frac{1}{2}(b + b) = \frac{1}{4}(a + c)(b + d)$
In the case of a parallelogram, the height is smaller than the length of the side so the formula does not give the correct answer.

1.6.2 The area of a circle is given by the formula $A = \pi(\frac{d}{2})^2$. According the Egyptians, A is also equal to the area of a square with sides equal to $\frac{8}{9}d$; thus $A = (\frac{8}{9})^2 d^2$. Equating and solving for π gives

$$\pi = \frac{(\frac{8}{9})^2 d^2}{\frac{1}{4}d^2} = \frac{\frac{64}{81}}{\frac{1}{4}} = \frac{256}{81} \approx 3.160494.$$

1.6.3 The sum of the measures of the two acute angles in $\triangle ABC$ is 90°, so the first shaded region is a square. We must show that the area of the shaded region in the first square (c^2) is equal to the area of the shaded region in the second square $(a^2 + b^2)$. The two large squares have the same area because they both have side length $a + b$. Also each of these squares contains four copies of triangle $\triangle ABC$ (in white). Therefore, by subtraction, the shadesd regions must have equal area and so $a^2 + b^2 = c^2$.

1.6.4 **(a)** Suppose $a = u^2 - v^2$, $b = 2uv$ and $c = u^2 + v^2$. We must show that $a^2 + b^2 = c^2$. First, $a^2 + b^2 = (u^2 - v^2)^2 + (2uv)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2 = u^4 + 2u^2v^2 + v^4$ and, second, $c^2 = (u^2 + v^2)^2 = u^4 + 2u^2v^2 + v^4 = u^4 + 2u^2v^2 + v^4$. So $a^2 + b^2 = c^2$.

(b) Let u and v be odd. We will show that a, b and c are all even. Since u and v are both odd, we know that u^2 and v^2 are also odd. Therefore $a = u^2 - v^2$ is even (the difference between two odd numbers is even). It is obvious that $b = 2uv$ is even, and $c = u^2 + v^2$ is also even since it is the sum of two odd numbers.

(c) Suppose one of u and v is even and the other is odd. We will show that a, b, and c do not have any common prime factors. Now a and c are both odd, so 2 is not a factor of a or c. Suppose $x \neq 2$ is a prime factor of b. Then either x divides u or x divides v, but not both because u and v are relatively prime. If x divides u, then it also divides u^2 but not v^2. Thus x is not a factor of a or c.

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If x divides v, then it divides v^2 but not u^2. Again x is not a factor of a or c. Therefore (a, b, c) is a primitive Pythagorean triple.

1.6.5 Let $h + x$ be the height of the entire (untruncated) pyramid. We know that

$$\frac{h + x}{x} = \frac{a}{b}$$

(by the Similar Triangles Theorem), so $x = h\dfrac{b}{a - b}$ (algebra). The volume of the truncated pyramid is the volume of the whole pyramid minus the volume of the top pyramid. Therefore

$$\begin{aligned}
V &= \frac{1}{3}(h + x)a^2 - \frac{1}{3}xb^2 \\
&= \frac{1}{3}(h + h\frac{b}{a - b})a^2 - \frac{1}{3}h(\frac{b^3}{a - b}) \\
&= \frac{h}{3}(a^2 + \frac{a^2 b}{a - b}) - \frac{h}{3}(\frac{b^3}{a - b}) \\
&= \frac{h}{3}(a^2 + \frac{a^2 b - b^3}{a - b}) \\
&= \frac{h}{3}(a^2 + \frac{(a - b)(ab + b^2)}{(a - b)}) \\
&= \frac{h}{3}(a^2 + ab + b^2).
\end{aligned}$$

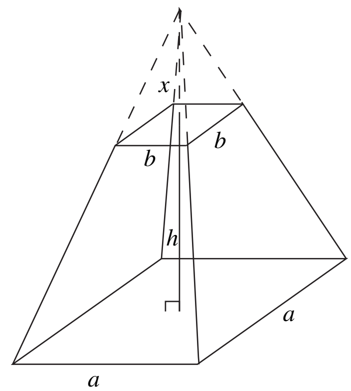

FIGURE S1.2: Exercise 1.6.5. A truncated pyramid.

1.6.6 Constructions using a compass and a straightedge. There are numerous ways in which to accomplish each of these constructions; just one is indicated in each case.

(a) The perpendicular bisector of a line segment $\overline{AB}$.
Using the compass, construct two circles, the first about A through B, the second about B through A. Then use the straightedge to construct a line through the two points created by the intersection of the two circles.

(b) A line through a point P perpendicular to a line ℓ.
Use the compass to construct a circle about P, making sure the circle is big enough so that it intersects ℓ at two points, A and B. Then construct the perpendicular bisector of segment $\overline{AB}$ as in part (a).

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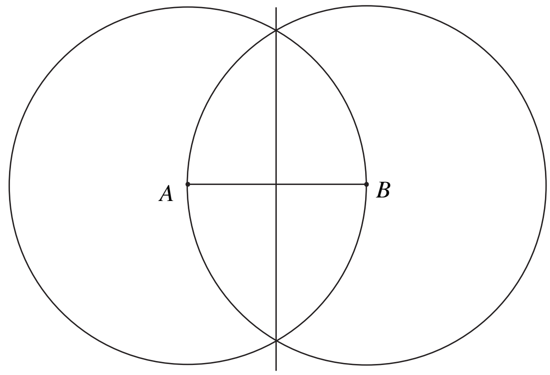

FIGURE S1.3: Exercise 1.6(a) Construction of a perpendicular bisector

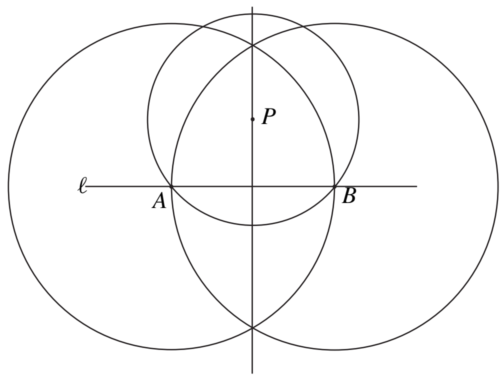

FIGURE S1.4: Exercise 1.6.6(b) Construction of a line through P, perpendicular to ℓ

(c) The angle bisector of $\angle BAC$.
Using the compass, construct a circle about A that intersects $\overline{AB}$ and $\overline{AC}$. Call those points of intersection D and E respectively. Then construct the perpendicular bisector of $\overline{DE}$. This line is the angle bisector.

1.6.7 **(a)** No. Euclid's postulates say nothing about the number of points on a line.
(b) No.
(c) No. The postulates only assert that there is a line; they do not say there is only one.

1.6.8 The proof of Proposition 29.

1.6.9 Let $\square ABCD$ be a rhombus (all four sides are equal), and let E be the point of intersection between $\overline{AC}$ and $\overline{DB}$.[1] We must show that $\triangle AEB \cong \triangle CEB \cong \triangle CED \cong \triangle AED$. Now $\angle BAC \cong \angle ACB$ and $\angle CAD \cong \angle ACD$ by Proposition 5. By addition we can see that $\angle BAD \cong \angle BCD$ and similarly, $\angle ADC \cong \angle ABC$. Now we know that $\triangle ABC \cong \triangle ADC$ by Proposition 4. Similarly, $\triangle DBA \cong \triangle DBC$. This implies that $\angle BAC \cong \angle DAC \cong \angle BCA \cong \angle DCA$ and $\angle BDA \cong \angle DBA \cong \angle BDC \cong \angle DBC$.

[1] In this solution and the next, the existence of the point E is taken for granted. Its existence is obvious from the diagram. Proving that E exists is one of the gaps that must be filled in these proofs. This point will be addressed in Chapter 6.

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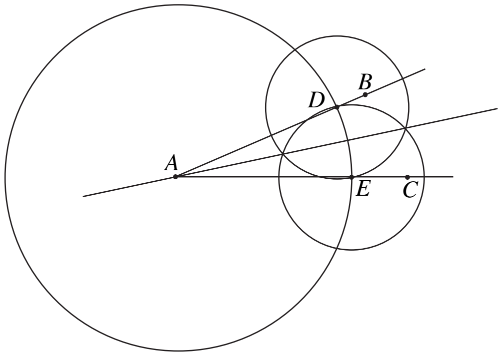

FIGURE S1.5: Exercise 1.6.6(c) Construction of an angle bisector

Thus $\triangle AEB \cong \triangle AED \cong \triangle CEB \cong \triangle CED$, again by Proposition 4.[2]

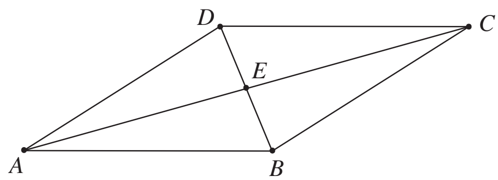

FIGURE S1.6: Exercise 1.6.9 Rhombus $\square ABCD$

1.6.10 Let $\square ABCD$ be a rectangle, and let E be the point of intersection of $\overline{AC}$ and $\overline{BD}$. We must prove that $\overline{AC} \cong \overline{BD}$ and that $\overline{AC}$ and $\overline{BD}$ bisect each other (i.e., $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$). By Proposition 28, $\overleftrightarrow{DA} \parallel \overleftrightarrow{CB}$ and $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$. Therefore, by Proposition 29, $\angle CAB \cong \angle ACD$ and $\angle DAC \cong \angle ACB$. Hence $\triangle ABC \cong \triangle CDA$ and $\triangle ADB \cong \triangle CBD$ by Proposition 26 (ASA). Since those triangles are congruent we know that opposite sides of the rectangle are congruent and $\triangle ABD \cong \triangle BAC$ (by Proposition 4), and therefore $\overline{BD} \cong \overline{AC}$.
Now we must prove that the segments bisect each other. By Proposition 29, $\angle CAB \cong \angle ACD$ and $\angle DBA \cong \angle BDC$. Hence $\triangle ABE \cong \triangle CDE$ (by Proposition 26) which implies that $\overline{AE} \cong \overline{CE}$ and $\overline{DE} \cong \overline{BE}$. Therefore the diagonals are equal and bisect each other.

1.6.11 The argument works for the first case. This is the case in which the triangle actually is isosceles. The second case never occurs (D is never inside the triangle). The flaw lies in the third case (D is outside the triangle). If the triangle is not isosceles then either E will be outside the triangle and F will be on the edge $\overline{AC}$, or E will be on the edge $\overline{AB}$ and F will be outside. They cannot both be outside as shown in the diagram. This can be checked by drawing a careful diagram by hand or by drawing the diagram using GeoGebra (or similar software).

[2]It should be noted that the fact about rhombi can be proved using just propositions that come early in Book I and do not depend on the Fifth Postulate, whereas the proof in the next exercise requires propositions about parallelism that Euclid proves much later in Book I using his Fifth Postulate.

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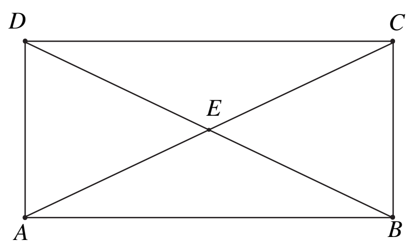

FIGURE S1.7: Exercise 1.6.10 Rectangle $\square ABCD$

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Solutions to Exercises in Chapter 2

2.4.1 This is not a model for Incidence Geometry since it does not satisfy Incidence Axiom 3. This example is isomorphic to the 3-point line.

2.4.2 One-point Geometry satisfies Axioms 1 and 2 but not Axiom 3. Every pair of distinct points defines a unique line (vacuously—there is no pair of distinct points). There do not exist three distinct points, so there cannot be three noncollinear points. One-point Geometry satisfies all three parallel postulates (vacuously—there is no line).

2.4.3 It helps to draw a schematic diagram of the relationships.

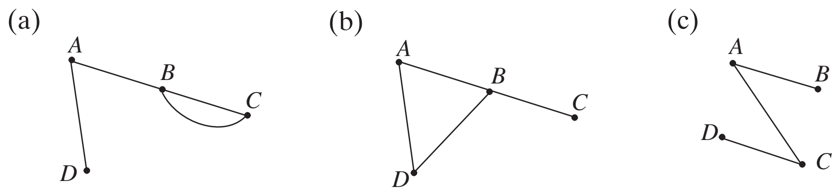

FIGURE S2.1: A schematic representation of the committee structures

- **(a)** Not a model. There is no line containing B and D. There are two lines containing B and C.
- **(b)** Not a model. There is no line containing C and D.
- **(c)** Not a model. There is no line containing A and D.

2.4.4
- **(a)** The Three-point plane is a model for Three-point geometry.
- **(b)** Every model for Three-point geometry has 3 lines. If there are 3 points, then there are also 3 pairs of points
- **(c)** Suppose there are two models for Three-point geometry, model A and model B. Choose any 1-1 correspondence of the points in model A to the points in model B. Any line in A is determined by two points. These two points correspond to two points in B. Those two points determine a line in B. The isomorphism should map the given line in A to this line in B. Then the function will preserve betweennes. Therefore models A and B are isomorphic to one another.

2.4.5 Axiom 1 does not hold, but Axioms 2 and 3 do. The Euclidean Parallel Postulate holds. The other parallel postulates are false in this interpretation.

2.4.6 See Figure S2.2.

2.4.7 Fano's Geometry satisfies the Elliptic Parallel Postulate because every line shares at least one point with every other line; there are no parallel lines. It does not satisfy either of the other parallel postulates.

2.4.8 The three-point line satisfies all three parallel postulates (vacuously).

2.4.9 If there are so few points and lines that there is no line with an external point, then all three parallel postulates are satisfied (vacuously). If there is a line with an external point, then there will either be a parallel line through the external point or there will not be. Hence at most one of the parallel postulates can be satisfied in

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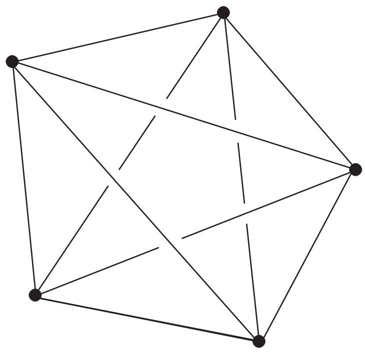

FIGURE S2.2: Five-point Geometry

that case. Since every incidence geometry contains three noncollinear points, there must be a line with an external point. Hence an incidence geometry can satisfy at most one of the parallel postulates.

2.10 Start with a line with three points on it. There must exist another point not on that line (Incidence Axiom 3). That point, together with the points on the original line, determines three more lines (Incidence Axiom 1). But each of those lines must have a third point on it. So there must be at least three more points, for a total of at least seven points. Since Fano's Geometry has exactly seven points, seven is the minimum.

2.4.11 See Figures S2.3 and S2.4.

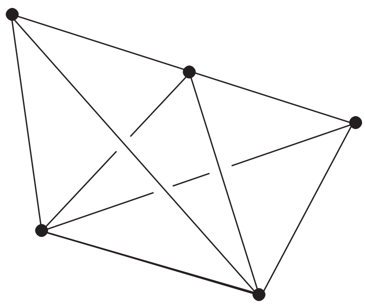

FIGURE S2.3: An unbalanced geometry

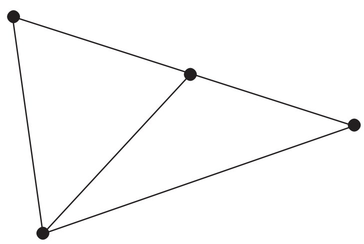

FIGURE S2.4: A simpler example

2.4.12 **(a)** The three-point line (Example 2.2.3).

(b) The square (Exercise 2.4.5) or the sphere (Example 2.2.9).

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(c) One-point geometry (Exercise 2.4.2).

2.4.13 **(a)** The dual of the Three-point plane is another Three-point plane. It is a model for incidence geometry.

(b) The dual of the Three-point line is a point which is incident with 3 lines. This is not a model for incidence geometry.

(c) The dual of Four-point Geometry has 6 points and 4 lines. Each point is incident with exactly 2 lines, and each line is incident with 3 points. It is not a model for incidence geometry because it does not satisfy Incidence Axiom 1.

(d) The dual of Fano's Geometry is isomorphic to Fano's Geometry, so it is a model for incidence geometry.

2.5.1 **(a)** H : it rains
C : I get wet

(b) H : the sun shines
C : we go hiking and biking

(c) H : $x > 0$
C : $\exists$ a y such that $y^2 = 0$

(d) H : $2x + 1 = 5$
C : $x = 2$ or $x = 3$

2.5.2 **(a)** Converse : If I get wet, then it rained.
Contrapositive : If I do not get wet, then it did not rain.

(b) Converse : If we go hiking and biking, then the sun shines.
Contrapositive : If we do not go hiking and biking, then the sun does not shine.

(c) Converse : If $\exists$ a y such that $y^2 = 0$, then $x > 0$.
Contrapositive : If $\forall$ y, $y^2 \neq 0$, then $x \leqslant 0$.

(d) Converse : If $x = 2$ or $x = 3$, then $2x + 1 = 5$.
Contrapositive : If $x \neq 2$ and $x \neq 3$, then $2x + 1 \neq 5$

2.5.3 **(a)** It rains and I do not get wet.

(b) The sun shines but we do not go hiking or biking.

(c) $x > 0$ and $\forall y$, $y^2 \neq 0$.

(d) $2x + 1 = 5$ but x is not equal to 2 or 3.

2.5.4 **(a)** If the grade is an A, then the score is at least 90%.

(b) If the score is at least 50%, then the grade is a passing grade.

(c) If you fail, then you scored less than 50%.

(d) If you try hard, then you succeed.

2.5.5 **(a)** Converse: If the score is at least 90%, then the grade is an A.
Contrapositive: If the score is less than 90%, then the grade is not an A.

(b) Converse: If the grade is a passing grade, then the score is at least 50%.
Contrapositive: If the grades is not a passing grade, then the score is less than 50%.

(c) Converse: If you score less than 50%, then you fail.
Contrapositive: If you score at least 50%, then you pass.

(d) Converse: If you succeed, then you tried hard.
Contrapositive: If you do not succeed, then you did not try hard.

2.5.6 **(a)** The grade is an A but the score is less than 90%.

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(b) The grade is at least 50% but the grade is not a passing grade.

(c) You fail and your score is at least 50%.

(d) You try hard but do not succeed.

2.5.8 **(a)** H : I pass geometry
C : I can take topology

(b) H : it rains
C : I get wet

(c) H : the number x is divisible by 4
C : x is even

2.5.9 **(a)** $\forall$ triangles T, the angle sum of T is $180°$.

(b) $\exists$ triangle T such that the angle sum of T is less than $180°$.

(c) $\exists$ triangle T such that the angle sum of T is not equal to $180°$.

(d) $\forall$ great circles α and β, $\alpha \bigcap \beta \neq \emptyset$.

(e) $\forall$ point P and $\forall$ line ℓ $\exists$ line m such that P lies on ℓ and $m \perp \ell$.

2.5.10 **(a)** $\forall$ model for incidence geometry, the Euclidean Parallel Postulate does not hold in that model.

(b) $\exists$ a model for incidence geometry in which there are not exactly 7 points (the number of points is either $\leqslant 6$ or $\geqslant 8$).

(c) $\exists$ a triangle whose angle sum is not $180°$.

(d) $\exists$ a triangle whose angle sum is at least $180°$.

(e) It is not hot or it is not humid outside.

(f) My favorite color is not red and it is not green.

(g) The sun shines and (but?) we do not go hiking. (See explanation in last full paragraph on page 36.)

(h) $\exists$ a geometry student who does not know how to write proofs.

2.5.11 **(a)** Negation of Euclidean Parallel Postulate. There exist a line ℓ and a point P not on ℓ such that either there is no line m such that P lies on m and m is parallel to ℓ or there are (at least) two lines m and n such that P lies on both m and n, $m \parallel \ell$, and $n \parallel \ell$.

(b) Negation of Elliptic Parallel Postulate. There exist a line ℓ and a point P that does not lie on ℓ such that there is at least one line m such that P lies on m and $m \parallel \ell$.

(c) Negation of Hyperbolic Parallel Postulate. There exist a line ℓ and a point P that does not lie on ℓ such that either there is no line m such that P lies on m and $m \parallel \ell$ or there is exactly one line m with these properties.

Note. You could emphasize the separate existence of ℓ and P by starting each of the statements above with, "There exist a line ℓ and there exists a point P not on ℓ such that ...."

2.5.12 *not* (S *and* T) ≡ (*not* S) *or* (*not* T).

S	T	S and T	*not* (S and T)	*not* S	*not* T	(*not* C) or (*not* H)
True	True	True	False	False	False	False
True	False	False	True	False	True	True
False	True	False	True	True	False	True
False	False	False	True	True	True	True

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not $(S$ *or* $T) \equiv ($*not* $S)$ *and* $($*not* $T)$.

S	T	S or T	*not* $(S$ or $T)$	*not* S	*not* T	$($*not* $C)$ and $($*not* $H)$
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

2.5.13 $H \Rightarrow C \equiv ($*not* $H)$ *or* C.

H	C	$H \Rightarrow C$	*not* H	$($*not* $H)$ or C
True	True	True	False	True
True	False	False	False	False
False	True	True	True	True
False	False	True	True	True

By De Morgan, *not* $(H$ *and* $($*not* $C))$ is logically equivalent to $($*not* $H)$ *or* C.

2.5.14 *not* $(H \Rightarrow C) \equiv (H$ *and* $($*not* $C))$.

H	C	$H \Rightarrow C$	*not*$(H \Rightarrow C)$	*not* C	H *and* $($*not* $C)$
True	True	True	False	False	False
True	False	False	True	True	True
False	True	True	False	False	False
False	False	True	False	True	False

2.5.15 If $\triangle ABC$ is right triangle with right angle at vertex C, and a, b, and c are the lengths of the sides opposite vertices A, B, and C respectively, then $a^2 + b^2 = c^2$.

2.5.16 **(a)** Euclidean. If ℓ is a line and P is a point that does not lie on ℓ, then there exists exactly one line m such that P lies on m and $m \parallel \ell$.

(b) Elliptic. If ℓ is a line and P is a point that does not lie on ℓ, then there does not exist a line m such that P lies on m and $m \parallel \ell$.

(c) Hyperbolic. If ℓ is a line and P is a point that does not lie on ℓ, then there exist at least two distinct lines m and n such that P lies on both m and n and ℓ is parallel to both m and n.

2.6.1 Converse to Theorem 2.6.2.

Proof. Let ℓ and m be two lines (notation). Assume there exists exactly one point that lies on both ℓ and m (hypothesis). We must show that $\ell \neq m$ and $\ell \nparallel m$.

Suppose $\ell = m$ (RAA hypothesis). There exists two distinct points Q and R that lie on ℓ (Incidence Axiom 2). Since $m = \ell$, Q and R lie on both ℓ and m. This contradicts the hypothesis that ℓ and m intersect in exactly one point, so we can reject the RAA hypothesis and conclude that $\ell \neq m$.

Since ℓ and m have a point in common, $\ell \nparallel m$ (definition of parallel). □

Note. In the next few proofs it is convenient to introduce the notation $\overleftrightarrow{AB}$ for the line determined by A and B (Incidence Axiom 1). This notation is not defined in the textbook until Chapter 3, but it fits with Incidence Axiom 1 and allows the proofs below to be written more succinctly.

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2.6.2 Theorem 2.6.3.

Proof. Let ℓ be a line (hypothesis). We must prove that there exists a point P such that P does not lie on ℓ. There exist three noncollinear points A, B, and C (Axiom 3). The three points A, B and C cannot all lie on ℓ because if they did then they would be collinear (definition of collinear). Hence at least one of them does not lie on ℓ and the proof is complete. □

Note. Many (or most) students will get the last proof wrong. The reason is that they want to assume some relationship between the line given in the hypothesis of the theorem and the points given by Axiom 3. In particular, many students will start with the first three sentences of the proof above, but will then either assert or assume that $\ell = \overleftrightarrow{AB}$.

This is an important opportunity to point out that the axioms mean exactly what they say and we must be careful not to impose our own assumptions on them. The fact that we are imposing an unstated assumption on the axioms is often covered up by the fact that the same letters are used for the three points given by Axiom 3 and the two points given by Axiom 2. So this exercise is also an opportunity to discuss notation and how we assign names to mathematical objects. (The fact that we happen to use the same letter to name two different points does not make them the same point.)

Similar comments apply to the next three proofs.

2.6.3 Theorem 2.6.4.

Proof. Let P be a point (hypothesis). We must show that there are two distinct lines ℓ and m such that P lies on both ℓ and m. There exist three noncollinear points A, B, and C (Axiom 3). There are two cases to consider: either P is equal to one of the three points A, B, and C or it is not.

Suppose, first, that $P = A$. Define $\ell = \overleftrightarrow{AB}$ and $m = \overleftrightarrow{AC}$ (Axiom 1). Obviously $P = A$ lies on both these lines. It cannot be that $\ell = m$ because in that case A, B, and C would be collinear. So the proof of this case is complete. The proofs of the cases in which $P = B$ and $P = C$ are similar.

Now suppose that P is distinct from all three of the points A, B, and C. In that case we can define three lines $\ell = \overleftrightarrow{PA}$, $m = \overleftrightarrow{PB}$, and $n = \overleftrightarrow{PC}$ (Axiom 1). These three lines cannot all be the same because if they were then A, B, and C would be collinear. Therefore at least two of them are distinct and the proof is complete. □

Note. Many students will assert that the three lines in the last paragraph are distinct. But that is not necessarily the case, as Fig. S2.5 shows.

Outline of an alternative proof: Find one line ℓ such that P lies on ℓ. Then use the previous theorem to find a point R that does not lie on ℓ. The line through P and R is the second line. This proof is simpler and better than the one given above, but most students do not think of it. You might want to lead them in that direction by suggesting that they prove the existence of one line first rather than proving the existence of both at the same time.

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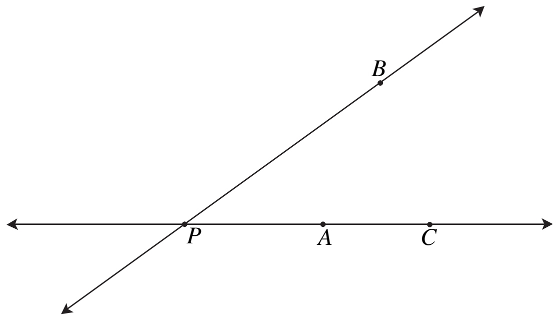

FIGURE S2.5: Two of the lines could be the same

2.6.4 Theorem 2.6.5.

Proof. Let ℓ be a line (hypothesis). We must show that there exist two lines m and n such that ℓ, m and n are distinct and both m and n intersect ℓ.

There exist two distinct points P and Q such that P and Q lie on ℓ (Axiom 2). There exists a point R such that R does not lie on ℓ (Theorem 2.6.3). Let m be the line determined by P and R and let n be the line determined by Q and R (Axiom 1). Since P lies on both ℓ amd m, m intersects ℓ. Since Q lies on both ℓ amd n, n intersects ℓ. Since R does not lie on ℓ, neither m nor n is equal to ℓ. To complete the proof we must show that $m \neq n$.

Suppose $m = n$ (RAA hypothesis). Then both P and Q lie on m as well as ℓ, so $m = \ell$ (the uniqueness part of Axiom 1). But this contradicts an earlier statement in the proof, so we must reject the RAA hypothesis and conclude that $m \neq n$. □

2.6.5 Theorem 2.6.6.

Proof. Let P be a point (hypothesis). We must prove that there is at least one line such that P does not lie on that line.

Let A, B and C be three non-collinear points (Axiom 3). Define $\ell = \overleftrightarrow{AB}$, $m = \overleftrightarrow{AC}$ and $n = \overleftrightarrow{BC}$ (Axiom 1). There are two cases to consider: either P is equal to one of the three points A, B or C or it is not.

Suppose, first, that $P = A$. Then $P = A$ does not lie on n because if it did, A, B and C would be collinear. So the proof for the first case is complete. There are similar proofs for $P = B$ and $P = C$.

Now assume P is distinct from A, B and C. We will show that P cannot lie on all three of the lines ℓ, m, and n. Suppose P lies on all three of ℓ, m, and n (RAA hypothesis). The fact that P and A lie on both m and ℓ implies that $m = \ell$ (Axiom 1). The fact that P and B lie on both n and ℓ implies that $n = \ell$ (Axiom 1). Thus $\ell = m = n$. But this cannot be because A, B and C are non-collinear. Therefore we must reject the RAA hypothesis and the proof is complete. □

2.6.6 Theorem 2.6.7.

Proof. Let A, B and C be three non-collinear points (Axiom 3). Define $\ell = \overleftrightarrow{AB}$, $m = \overleftrightarrow{AC}$ and $n = \overleftrightarrow{BC}$ (Axiom 1). In order to complete the proof we must show that ℓ, m, and n are distinct lines and that there does not exist a point P such that P lies on all three of the lines.

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No two of the lines can be equal because A, B, and C are noncollinear. The argument that there is no point P that lies on all three of the lines is the same as that in the last two paragraphs of the preceding proof. □

2.6.7 Theorem 2.6.8.

Proof. Let P be a point (hypothesis). We must prove that there exist points Q and R such that P, Q, and R are noncollinear.

There exists a point Q such that $Q \neq P$ (by Axiom 3 there are at least three points, so P is not the only point). There exists a point R such that R does not lie on $\overleftrightarrow{PQ}$ (Theorem 2.6.3). There is no line on which all three of P, Q, and R lie because the only line on which the first two lie is $\overleftrightarrow{PQ}$ and R does not lie on that line. □

2.6.8 Theorem 2.6.9.

Proof. Let P and Q be two points such that $P \neq Q$ (hypothesis). We must prove that there exists a point R such that P, Q, and R are noncollinear.

There exists a unique line ℓ such that both P and Q lie on ℓ (Incidence Axiom 1). There is a point R such that R does not lie on ℓ (Theorem 2.6.3). We will complete the proof by showing that P, Q, and R are noncollinear. Suppose there exists a line m such that all three of the points P, Q, and R lie on m (RAA hypothesis). Then $m = \ell$ by the uniqueness part of Incidence Axiom 1. But this is impossible since R does not lie on ℓ but does lie on m. Hence we must reject the RAA hypothesis and conclude that P, Q, and R are noncollinear. □

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