Thomas Calculus 14th edition Hass Solutions Manual

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CHAPTER 2 LIMITS AND CONTINUITY

2.1 RATES OF CHANGE AND TANGENTS TO CURVES

1. (a) $\frac{\Delta f}{\Delta x} = \frac{f(3)-f(2)}{3-2} = \frac{28-9}{1} = 19$ (b) $\frac{\Delta f}{\Delta x} = \frac{f(1)-f(-1)}{1-(-1)} = \frac{2-0}{2} = 1$

2. (a) $\frac{\Delta g}{\Delta x} = \frac{g(3)-g(1)}{3-1} = \frac{3-(-1)}{2} = 2$ (b) $\frac{\Delta g}{\Delta x} = \frac{g(4)-g(-2)}{4-(-2)} = \frac{8-8}{6} = 0$

3. (a) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{3\pi}{4}\right)-h\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4}-\frac{\pi}{4}} = \frac{-1-1}{\frac{\pi}{2}} = -\frac{4}{\pi}$ (b) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{\pi}{2}\right)-h\left(\frac{\pi}{6}\right)}{\frac{\pi}{2}-\frac{\pi}{6}} = \frac{0-\sqrt{3}}{\frac{\pi}{3}} = \frac{-3\sqrt{3}}{\pi}$

4. (a) $\frac{\Delta g}{\Delta t} = \frac{g(\pi)-g(0)}{\pi-0} = \frac{(2-1)-(2+1)}{\pi-0} = -\frac{2}{\pi}$ (b) $\frac{\Delta g}{\Delta t} = \frac{g(\pi)-g(-\pi)}{\pi-(-\pi)} = \frac{(2-1)-(2-1)}{2\pi} = 0$

5. $\frac{\Delta R}{\Delta \theta} = \frac{R(2)-R(0)}{2-0} = \frac{\sqrt{8+1}-\sqrt{1}}{2} = \frac{3-1}{2} = 1$

6. $\frac{\Delta P}{\Delta \theta} = \frac{P(2)-P(1)}{2-1} = \frac{(8-16+10)-(1-4+5)}{1} = 2-2 = 0$

7. (a) $\frac{\Delta y}{\Delta x} = \frac{((2+h)^2-5)-(2^2-5)}{h} = \frac{4+4h+h^2-5+1}{h} = \frac{4h+h^2}{h} = 4+h.$ As $h \to 0,\ 4+h \to 4 \Rightarrow$ at $P(2,-1)$ the slope is 4.
 (b) $y-(-1) = 4(x-2) \Rightarrow y+1 = 4x-8 \Rightarrow y = 4x-9$

8. (a) $\frac{\Delta y}{\Delta x} = \frac{(7-(2+h)^2)-(7-2^2)}{h} = \frac{7-4-4h-h^2-3}{h} = \frac{-4h-h^2}{h} = -4-h.$ As $h \to 0, -4-h \to -4 \Rightarrow$ at $P(2,3)$ the slope is -4.
 (b) $y-3 = (-4)(x-2) \Rightarrow y-3 = -4x+8 \Rightarrow y = -4x+11$

9. (a) $\frac{\Delta y}{\Delta x} = \frac{((2+h)^2-2(2+h)-3)-(2^2-2(2)-3)}{h} = \frac{4+4h+h^2-4-2h-3-(-3)}{h} = \frac{2h+h^2}{h} = 2+h.$ As $h \to 0,\ 2+h \to 2 \Rightarrow$ at $P(2,-3)$ the slope is 2.
 (b) $y-(-3) = 2(x-2) \Rightarrow y+3 = 2x-4 \Rightarrow y = 2x-7.$

10. (a) $\frac{\Delta y}{\Delta x} = \frac{((1+h)^2-4(1+h))-(1^2-4(1))}{h} = \frac{1+2h+h^2-4-4h-(-3)}{h} = \frac{h^2-2h}{h} = h-2.$ As $h \to 0,\ h-2 \to -2 \Rightarrow$ at $P(1,-3)$ the slope is -2.
 (b) $y-(-3) = (-2)(x-1) \Rightarrow y+3 = -2x+2 \Rightarrow y = -2x-1.$

11. (a) $\frac{\Delta y}{\Delta x} = \frac{(2+h)^3-2^3}{h} = \frac{8+12h+4h^2+h^3-8}{h} = \frac{12h+4h^2+h^3}{h} = 12+4h+h^2.$ As $h \to 0,\ 12+4h+h^2 \to 12, \Rightarrow$ at $P(2,8)$ the slope is 12.
 (b) $y-8 = 12(x-2) \Rightarrow y-8 = 12x-24 \Rightarrow y = 12x-16.$

12. (a) $\frac{\Delta y}{\Delta x} = \frac{2-(1+h)^3-(2-1^3)}{h} = \frac{2-1-3h-3h^2-h^3-1}{h} = \frac{-3h-3h^2-h^3}{h} = -3-3h-h^2.$ As $h \to 0, -3-3h-h^2 \to -3, \Rightarrow$ at $P(1,1)$ the slope is -3.
 (b) $y-1 = (-3)(x-1) \Rightarrow y-1 = -3x+3 \Rightarrow y = -3x+4.$

13. (a) $\frac{\Delta y}{\Delta x} = \frac{(1+h)^3 - 12(1+h) - (1^3 - 12(1))}{h} = \frac{1+3h+3h^2+h^3-12-12h-(-11)}{h} = \frac{-9h+3h^2+h^3}{h} = -9+3h+h^2$.
As $h \to 0$, $-9+3h+h^2 \to -9 \Rightarrow$ at $P(1,-11)$ the slope is -9.
(b) $y-(-11) = (-9)(x-1) \Rightarrow y+11 = -9x+9 \Rightarrow y = -9x-2$.

14. (a) $\frac{\Delta y}{\Delta x} = \frac{(2+h)^3 - 3(2+h)^2 + 4 - (2^3 - 3(2)^2 + 4)}{h} = \frac{8+12h+6h^2+h^3-12-12h-3h^2+4-0}{h} = \frac{3h^2+h^3}{h} = 3h+h^2$.
As $h \to 0$, $3h+h^2 \to 0 \Rightarrow$ at $P(2, 0)$ the slope is 0.
(b) $y-0 = 0(x-2) \Rightarrow y = 0$.

15. (a) $\frac{\Delta y}{\Delta x} = \frac{\frac{1}{-2+h} - \frac{1}{-2}}{h} = \frac{2+(-2+h)}{2(-2+h)} \cdot \frac{1}{h} = \frac{1}{2(-2+h)}$.
As $h \to 0$, $\frac{1}{2(-2+h)} \to \frac{-1}{4}$, $\Rightarrow$ at $P\left(-2, \frac{-1}{2}\right)$ the slope is $\frac{-1}{4}$.
(b) $y - \left(\frac{-1}{2}\right) = \frac{-1}{4}(x-(-2)) \Rightarrow y + \frac{1}{2} = \frac{-1}{4}x - \frac{1}{2} \Rightarrow y = \frac{-1}{4}x - 1$

16. (a) $\frac{\Delta y}{\Delta x} = \frac{\frac{(4+h)}{2-(4+h)} - \frac{4}{2-4}}{h} = \left(\frac{4+h}{-2-h} + \frac{2}{1}\right) \cdot \frac{1}{h} = \frac{4+h+2(-2-h)}{-2-h} \cdot \frac{1}{h} = \frac{-1}{-2-h} = \frac{1}{2+h}$.
As $h \to 0$, $\frac{1}{2+h} \to \frac{1}{2}$, $\Rightarrow$ at $P(4,-2)$ the slope is $\frac{1}{2}$.
(b) $y-(-2) = \frac{1}{2}(x-4) \Rightarrow y+2 = \frac{1}{2}x - 2 \Rightarrow y = \frac{1}{2}x - 4$

17. (a) $\frac{\Delta y}{\Delta x} = \frac{\sqrt{4+h} - \sqrt{4}}{h} = \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}$.
As $h \to 0$, $\frac{1}{\sqrt{4+h}+2} \to \frac{1}{\sqrt{4}+2} = \frac{1}{4}$, $\Rightarrow$ at $P(4,2)$ the slope is $\frac{1}{4}$.
(b) $y-2 = \frac{1}{4}(x-4) \Rightarrow y-2 = \frac{1}{4}x - 1 \Rightarrow y = \frac{1}{4}x + 1$

18. (a) $\frac{\Delta y}{\Delta x} = \frac{\sqrt{7-(-2+h)} - \sqrt{7-(-2)}}{h} = \frac{\sqrt{9-h}-3}{h} = \frac{\sqrt{9-h}-3}{h} \cdot \frac{\sqrt{9-h}+3}{\sqrt{9-h}+3} = \frac{(9-h)-9}{h(\sqrt{9-h}+3)} = \frac{-1}{\sqrt{9-h}+3}$.
As $h \to 0$, $\frac{-1}{\sqrt{9-h}+3} \to \frac{-1}{\sqrt{9}+3} = \frac{-1}{6}$, $\Rightarrow$ at $P(-2,3)$ the slope is $\frac{-1}{6}$.
(b) $y-3 = \frac{-1}{6}(x-(-2)) \Rightarrow y-3 = \frac{-1}{6}x - \frac{1}{3} \Rightarrow y = \frac{-1}{6}x + \frac{8}{3}$

19. (a)

Q	Slope of $PQ = \frac{\Delta p}{\Delta t}$
$Q_1(10, 225)$	$\frac{650-225}{20-10} = 42.5$ m/sec
$Q_2(14, 375)$	$\frac{650-375}{20-14} = 45.83$ m/sec
$Q_3(16.5, 475)$	$\frac{650-475}{20-16.5} = 50.00$ m/sec
$Q_4(18, 550)$	$\frac{650-550}{20-18} = 50.00$ m/sec

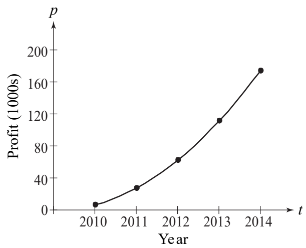
(b) At $t = 20$, the sportscar was traveling approximately 50 m/sec or 180 km/h.

20. (a)

Q	Slope of $PQ = \frac{\Delta p}{\Delta t}$
$Q_1(5, 20)$	$\frac{80-20}{10-5} = 12$ m/sec
$Q_2(7, 39)$	$\frac{80-39}{10-7} = 13.7$ m/sec
$Q_3(8.5, 58)$	$\frac{80-58}{10-8.5} = 14.7$ m/sec
$Q_4(9.5, 72)$	$\frac{80-72}{10-9.5} = 16$ m/sec

(b) Approximately 16 m/sec

21. (a)



(b) $\frac{\Delta p}{\Delta t} = \frac{174-62}{2014-2012} = \frac{112}{2} = 56$ thousand dollars per year

(c) The average rate of change from 2011 to 2012 is $\frac{\Delta p}{\Delta t} = \frac{62-27}{2012-2011} = 35$ thousand dollars per year.

The average rate of change from 2012 to 2013 is $\frac{\Delta p}{\Delta t} = \frac{111-62}{2013-2012} = 49$ thousand dollars per year.

So, the rate at which profits were changing in 2012 is approximately $\frac{1}{2}(35+49) = 42$ thousand dollars per year.

22. (a) $F(x) = (x+2)/(x-2)$

x	1.2	1.1	1.01	1.001	1.0001	1
$F(x)$	-4.0	$-3.\overline{4}$	$-3.\overline{04}$	$-3.\overline{004}$	$-3.\overline{0004}$	-3

$\frac{\Delta F}{\Delta x} = \frac{-4.0-(-3)}{1.2-1} = -5.0$; $\frac{\Delta F}{\Delta x} = \frac{-3.\overline{4}-(-3)}{1.1-1} = -4.\overline{4}$;

$\frac{\Delta F}{\Delta x} = \frac{-3.04-(-3)}{1.01-1} = -4.\overline{04}$; $\frac{\Delta F}{\Delta x} = \frac{-3.004-(-3)}{1.001-1} = -4.\overline{004}$;

$\frac{\Delta F}{\Delta x} = \frac{-3.0004-(-3)}{1.0001-1} = -4.\overline{0004}$;

(b) The rate of change of $F(x)$ at $x = 1$ is -4.

23. (a) $\frac{\Delta g}{\Delta x} = \frac{g(2)-g(1)}{2-1} = \frac{\sqrt{2}-1}{2-1} \approx 0.414213$ $\frac{\Delta g}{\Delta x} = \frac{g(1.5)-g(1)}{1.5-1} = \frac{\sqrt{1.5}-1}{0.5} \approx 0.449489$

$\frac{\Delta g}{\Delta x} = \frac{g(1+h)-g(1)}{(1+h)-1} = \frac{\sqrt{1+h}-1}{h}$

(b) $g(x) = \sqrt{x}$

$1+h$	1.1	1.01	1.001	1.0001	1.00001	1.000001
$\sqrt{1+h}$	1.04880	1.004987	1.0004998	1.0000499	1.000005	1.0000005
$\left(\sqrt{1+h}-1\right)/h$	0.4880	0.4987	0.4998	0.499	0.5	0.5

(c) The rate of change of $g(x)$ at $x = 1$ is 0.5.

(d) The calculator gives $\lim\limits_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \frac{1}{2}$.

24. (a) i) $\frac{f(3)-f(2)}{3-2} = \frac{\frac{1}{3}-\frac{1}{2}}{1} = \frac{\frac{-1}{6}}{1} = -\frac{1}{6}$

ii) $\frac{f(T)-f(2)}{T-2} = \frac{\frac{1}{T}-\frac{1}{2}}{T-2} = \frac{\frac{2}{2T}-\frac{T}{2T}}{T-2} = \frac{2-T}{2T(T-2)} = \frac{2-T}{-2T(2-T)} = -\frac{1}{2T}, T \neq 2$

(b)

T	2.1	2.01	2.001	2.0001	2.00001	2.000001
$f(T)$	0.476190	0.497512	0.499750	0.4999750	0.499997	0.499999
$(f(T)-f(2))/(T-2)$	-0.2381	-0.2488	-0.2500	-0.2500	-0.2500	-0.2500

(c) The table indicates the rate of change is -0.25 at $t = 2$.

(d) $\lim\limits_{T\to 2}\left(\frac{1}{-2T}\right) = -\frac{1}{4}$

NOTE: Answers will vary in Exercises 25 and 26.

25. (a) $[0, 1]$: $\frac{\Delta s}{\Delta t} = \frac{15-0}{1-0} = 15$ mph; $[1, 2.5]$: $\frac{\Delta s}{\Delta t} = \frac{20-15}{2.5-1} = \frac{10}{3}$ mph; $[2.5, 3.5]$: $\frac{\Delta s}{\Delta t} = \frac{30-20}{3.5-2.5} = 10$ mph

(b) At $P\left(\frac{1}{2}, 7.5\right)$: Since the portion of the graph from $t = 0$ to $t = 1$ is nearly linear, the instantaneous rate of change will be almost the same as the average rate of change, thus the instantaneous speed at $t = \frac{1}{2}$ is $\frac{15-7.5}{1-0.5} = 15$ mi/hr. At $P(2, 20)$: Since the portion of the graph from $t = 2$ to $t = 2.5$ is nearly linear, the instantaneous rate of change will be nearly the same as the average rate of change, thus $v = \frac{20-20}{2.5-2} = 0$ mi/hr. For values of t less than 2, we have

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(1, 15)$	$\frac{15-20}{1-2} = 5$ mi/hr
$Q_2(1.5, 19)$	$\frac{19-20}{1.5-2} = 2$ mi/hr
$Q_3(1.9, 19.9)$	$\frac{19.9-20}{1.9-2} = 1$ mi/hr

Thus, it appears that the instantaneous speed at $t = 2$ is 0 mi/hr.
At $P(3, 22)$:

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-22}{4-3} = 13$ mi/hr
$Q_2(3.5, 30)$	$\frac{30-22}{3.5-3} = 16$ mi/hr
$Q_3(3.1, 23)$	$\frac{23-22}{3.1-3} = 10$ mi/hr

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(2, 20)$	$\frac{20-22}{2-3} = 2$ mi/hr
$Q_2(2.5, 20)$	$\frac{20-22}{2.5-3} = 4$ mi/hr
$Q_3(2.9, 21.6)$	$\frac{21.6-22}{2.9-3} = 4$ mi/hr

Thus, it appears that the instantaneous speed at $t = 3$ is about 7 mi/hr.

(c) It appears that the curve is increasing the fastest at $t = 3.5$. Thus for $P(3.5, 30)$

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-30}{4-3.5} = 10$ mi/hr
$Q_2(3.75, 34)$	$\frac{34-30}{3.75-3.5} = 16$ mi/hr
$Q_3(3.6, 32)$	$\frac{32-30}{3.6-3.5} = 20$ mi/hr

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(3, 22)$	$\frac{22-30}{3-3.5} = 16$ mi/hr
$Q_2(3.25, 25)$	$\frac{25-30}{3.25-3.5} = 20$ mi/hr
$Q_3(3.4, 28)$	$\frac{28-30}{3.4-3.5} = 20$ mi/hr

Thus, it appears that the instantaneous speed at $t = 3.5$ is about 20 mi/hr.

26. (a) $[0, 3]$: $\frac{\Delta A}{\Delta t} = \frac{10-15}{3-0} \approx -1.67 \frac{\text{gal}}{\text{day}}$; $[0, 5]$: $\frac{\Delta A}{\Delta t} = \frac{3.9-15}{5-0} \approx -2.2 \frac{\text{gal}}{\text{day}}$; $[7, 10]$: $\frac{\Delta A}{\Delta t} = \frac{0-1.4}{10-7} \approx -0.5 \frac{\text{gal}}{\text{day}}$

(b) At $P(1, 14)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(2, 12.2)$	$\frac{12.2-14}{2-1} = -1.8$ gal/day
$Q_2(1.5, 13.2)$	$\frac{13.2-14}{1.5-1} = -1.6$ gal/day
$Q_3(1.1, 13.85)$	$\frac{13.85-14}{1.1-1} = -1.5$ gal/day

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(0, 15)$	$\frac{15-14}{0-1} = -1$ gal/day
$Q_2(0.5, 14.6)$	$\frac{14.6-14}{0.5-1} = -1.2$ gal/day
$Q_3(0.9, 14.86)$	$\frac{14.86-14}{0.9-1} = -1.4$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is about -1.45 gal/day.
At $P(4, 6)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(5, 3.9)$	$\frac{3.9-6}{5-4} = -2.1$ gal/day
$Q_2(4.5, 4.8)$	$\frac{4.8-6}{4.5-4} = -2.4$ gal/day
$Q_3(4.1, 5.7)$	$\frac{5.7-6}{4.1-4} = -3$ gal/day

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(3, 10)$	$\frac{10-6}{3-4} = -4$ gal/day
$Q_2(3.5, 7.8)$	$\frac{7.8-6}{3.5-4} = -3.6$ gal/day
$Q_3(3.9, 6.3)$	$\frac{6.3-6}{3.9-4} = -3$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is -3 gal/day.

At $P(8,1)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(9, 0.5)$	$\frac{0.5-1}{9-8} = -0.5$ gal/day
$Q_2(8.5, 0.7)$	$\frac{0.7-1}{8.5-8} = -0.6$ gal/day
$Q_3(8.1, 0.95)$	$\frac{0.95-1}{8.1-8} = -0.5$ gal/day

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(7, 1.4)$	$\frac{1.4-1}{7-8} = -0.6$ gal/day
$Q_2(7.5, 1.3)$	$\frac{1.3-1}{7.5-8} = -0.6$ gal/day
$Q_3(7.9, 1.04)$	$\frac{1.04-1}{7.9-8} = -0.6$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is -0.55 gal/day.

(c) It appears that the curve (the consumption) is decreasing the fastest at $t = 3.5$. Thus for $P(3.5, 7.8)$

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(4.5, 4.8)$	$\frac{4.8-7.8}{4.5-3.5} = -3$ gal/day
$Q_2(4, 6)$	$\frac{6-7.8}{4-3.5} = -3.6$ gal/day
$Q_3(3.6, 7.4)$	$\frac{7.4-7.8}{3.6-3.5} = -4$ gal/day

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(2.5, 11.2)$	$\frac{11.2-7.8}{2.5-3.5} = -3.4$ gal/day
$Q_2(3, 10)$	$\frac{10-7.8}{3-3.5} = -4.4$ gal/day
$Q_3(3.4, 8.2)$	$\frac{8.2-7.8}{3.4-3.5} = -4$ gal/day

Thus, it appears that the rate of consumption at $t = 3.5$ is about -4 gal/day.

2.2 LIMIT OF A FUNCTION AND LIMIT LAWS

1. (a) Does not exist. As x approaches 1 from the right, $g(x)$ approaches 0. As x approaches 1 from the left, $g(x)$ approaches 1. There is no single number L that all the values $g(x)$ get arbitrarily close to as $x \to 1$.
   (b) 1 (c) 0 (d) 0.5

2. (a) 0 (b) -1
   (c) Does not exist. As t approaches 0 from the left, $f(t)$ approaches -1. As t approaches 0 from the right, $f(t)$ approaches 1. There is no single number L that $f(t)$ gets arbitrarily close to as $t \to 0$.
   (d) -1

3. (a) True (b) True (c) False
   (d) False (e) False (f) True
   (g) True (h) False (i) True
   (j) True (k) False

4. (a) False (b) False (c) True
   (d) True (e) True (f) True
   (g) False (h) True (i) False

5. $\lim\limits_{x\to 0} \frac{x}{|x|}$ does not exist because $\frac{x}{|x|} = \frac{x}{x} = 1$ if $x > 0$ and $\frac{x}{|x|} = \frac{x}{-x} = -1$ if $x < 0$. As x approaches 0 from the left, $\frac{x}{|x|}$ approaches -1. As x approaches 0 from the right, $\frac{x}{|x|}$ approaches 1. There is no single number L that all the function values get arbitrarily close to as $x \to 0$.

6. As x approaches 1 from the left, the values of $\frac{1}{x-1}$ become increasingly large and negative. As x approaches 1 from the right, the values become increasingly large and positive. There is no number L that all the function values get arbitrarily close to as $x \to 1$, so $\lim\limits_{x\to 1} \frac{1}{x-1}$ does not exist.

7. Nothing can be said about $f(x)$ because the existence of a limit as $x \to x_0$ does not depend on how the function is defined at x_0. In order for a limit to exist, $f(x)$ must be arbitrarily close to a single real number L when x is close enough to x_0. That is, the existence of a limit depends on the values of $f(x)$ for x near x_0, not on the definition of $f(x)$ at x_0 itself.

8. Nothing can be said. In order for $\lim_{x\to 0} f(x)$ to exist, $f(x)$ must close to a single value for x near 0 regardless of the value $f(0)$ itself.

9. No, the definition does not require that f be defined at $x=1$ in order for a limiting value to exist there. If $f(1)$ is defined, it can be any real number, so we can conclude nothing about $f(1)$ from $\lim_{x\to 1} f(x)=5$.

10. No, because the existence of a limit depends on the values of $f(x)$ when x is near 1, not on $f(1)$ itself. If $\lim_{x\to 1} f(x)$ exists, its value may be some number other than $f(1)=5$. We can conclude nothing about $\lim_{x\to 1} f(x)$, whether it exists or what its value is if it does exist, from knowing the value of $f(1)$ alone.

11. $\lim_{x\to -3}(x^2-13)=(-3)^2-13=9-13=-4$

12. $\lim_{x\to 2}(-x^2+5x-2)=-(2)^2+5(2)-2=-4+10-2=4$

13. $\lim_{t\to 6} 8(t-5)(t-7)=8(6-5)(6-7)=-8$

14. $\lim_{x\to -2}(x^3-2x^2+4x+8)=(-2)^3-2(-2)^2+4(-2)+8=-8-8-8+8=-16$

15. $\lim_{x\to 2}\frac{2x+5}{11-x^3}=\frac{2(2)+5}{11-(2)^3}=\frac{9}{3}=3$

16. $\lim_{t\to 2/3}(8-3s)(2s-1)=\left(8-5\left(\frac{2}{3}\right)\right)\left(2\left(\frac{2}{3}\right)-1\right)=(8-2)\left(\left(\frac{4}{3}\right)-1\right)=(6)\left(\frac{1}{3}\right)=2$

17. $\lim_{x\to -1/2} 4x(3x+4)^2=4\left(-\frac{1}{2}\right)\left(3\left(-\frac{1}{2}\right)+4\right)^2=(-2)\left(-\frac{3}{2}+4\right)^2=(-2)\left(\frac{5}{2}\right)^2=-\frac{25}{2}$

18. $\lim_{y\to 2}\frac{y+2}{y^2+5y+6}=\frac{2+2}{(2)^2+5(2)+6}=\frac{4}{4+10+6}=\frac{4}{20}=\frac{1}{5}$

19. $\lim_{y\to -3}(5-y)^{4/3}=[5-(-3)]^{4/3}=(8)^{4/3}=\left((8)^{1/3}\right)^4=2^4=16$

20. $\lim_{z\to 4}\sqrt{z^2-10}=\sqrt{4^2-10}=\sqrt{16-10}=\sqrt{6}$

21. $\lim_{h\to 0}\frac{3}{\sqrt{3h+1}+1}=\frac{3}{\sqrt{3(0)+1}+1}=\frac{3}{\sqrt{1}+1}=\frac{3}{2}$

22. $\lim_{h\to 0}\frac{\sqrt{5h+4}-2}{h}=\lim_{h\to 0}\frac{\sqrt{5h+4}-2}{h}\cdot\frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2}=\lim_{h\to 0}\frac{(5h+4)-4}{h\left(\sqrt{5h+4}+2\right)}=\lim_{h\to 0}\frac{5h}{h\left(\sqrt{5h+4}+2\right)}=\lim_{h\to 0}\frac{5}{\sqrt{5h+4}+2}=\frac{5}{\sqrt{4}+2}=\frac{5}{4}$

23. $\lim_{x\to 5}\frac{x-5}{x^2-25}=\lim_{x\to 5}\frac{x-5}{(x+5)(x-5)}=\lim_{x\to 5}\frac{1}{x+5}=\frac{1}{5+5}=\frac{1}{10}$

24. $\lim_{x\to -3}\frac{x+3}{x^2+4x+3}=\lim_{x\to -3}\frac{x+3}{(x+3)(x+1)}=\lim_{x\to -3}\frac{1}{x+1}=\frac{1}{-3+1}=-\frac{1}{2}$

25. $\lim\limits_{x\to-5} \frac{x^2+3x-10}{x+5} = \lim\limits_{x\to-5} \frac{(x+5)(x-2)}{x+5} = \lim\limits_{x\to-5} (x-2) = -5-2 = -7$

26. $\lim\limits_{x\to2} \frac{x^2-7x-10}{x-2} = \lim\limits_{x\to2} \frac{(x-5)(x-2)}{x-2} = \lim\limits_{x\to2} (x-5) = 2-5 = -3$

27. $\lim\limits_{t\to1} \frac{t^2+t-2}{t^2-1} = \lim\limits_{t\to1} \frac{(t+2)(t-1)}{(t-1)(t+1)} = \lim\limits_{t\to1} \frac{t+2}{t+1} = \frac{1+2}{1+1} = \frac{3}{2}$

28. $\lim\limits_{t\to-1} \frac{t^2+3t+2}{t^2-t-2} = \lim\limits_{t\to-1} \frac{(t+2)(t+1)}{(t-2)(t+1)} = \lim\limits_{t\to-1} \frac{t+2}{t-2} = \frac{-1+2}{-1-2} = -\frac{1}{3}$

29. $\lim\limits_{x\to-2} \frac{-2x-4}{x^3+2x^2} = \lim\limits_{x\to-2} \frac{-2(x+2)}{x^2(x+2)} = \lim\limits_{x\to-2} \frac{-2}{x^2} = \frac{-2}{4} = -\frac{1}{2}$

30. $\lim\limits_{y\to0} \frac{5y^3+8y^2}{3y^4-16y^2} = \lim\limits_{y\to0} \frac{y^2(5y+8)}{y^2(3y^2-16)} = \lim\limits_{y\to0} \frac{5y+8}{3y^2-16} = \frac{8}{-16} = -\frac{1}{2}$

31. $\lim\limits_{x\to1} \frac{x^{-1}-1}{x-1} = \lim\limits_{x\to1} \frac{\frac{1-x}{x}}{x-1} = \lim\limits_{x\to1} \left(\frac{1-x}{x} \cdot \frac{1}{x-1}\right) = \lim\limits_{x\to1} -\frac{1}{x} = -1$

32. $\lim\limits_{x\to0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} = \lim\limits_{x\to0} \frac{\frac{(x+1)+(x-1)}{(x-1)(x+1)}}{x} = \lim\limits_{x\to0} \left(\frac{2x}{(x-1)(x+1)} \cdot \frac{1}{x}\right) = \lim\limits_{x\to0} \frac{2}{(x-1)(x+1)} = \frac{2}{-1} = -2$

33. $\lim\limits_{u\to1} \frac{u^4-1}{u^3-1} = \lim\limits_{u\to1} \frac{(u^2+1)(u+1)(u-1)}{(u^2+u+1)(u-1)} = \lim\limits_{u\to1} \frac{(u^2+1)(u+1)}{u^2+u+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$

34. $\lim\limits_{v\to2} \frac{v^3-8}{v^4-16} = \lim\limits_{v\to2} \frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} = \lim\limits_{v\to2} \frac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{4+4+4}{(4)(8)} = \frac{12}{32} = \frac{3}{8}$

35. $\lim\limits_{x\to9} \frac{\sqrt{x}-3}{x-9} = \lim\limits_{x\to9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim\limits_{x\to9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

36. $\lim\limits_{x\to4} \frac{4x-x^2}{2-\sqrt{x}} = \lim\limits_{x\to4} \frac{x(4-x)}{2-\sqrt{x}} = \lim\limits_{x\to4} \frac{x(2+\sqrt{x})(2-\sqrt{x})}{2-\sqrt{x}} = \lim\limits_{x\to4} x\left(2+\sqrt{x}\right) = 4(2+2) = 16$

37. $\lim\limits_{x\to1} \frac{x-1}{\sqrt{x+3}-2} = \lim\limits_{x\to1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}-2\right)\left(\sqrt{x+3}+2\right)} = \lim\limits_{x\to1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{(x+3)-4} = \lim\limits_{x\to1}\left(\sqrt{x+3}+2\right) = \sqrt{4}+2 = 4$

38. $\lim\limits_{x\to-1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim\limits_{x\to-1} \frac{\left(\sqrt{x^2+8}-3\right)\left(\sqrt{x^2+8}+3\right)}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim\limits_{x\to-1} \frac{(x^2+8)-9}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim\limits_{x\to-1} \frac{(x+1)(x-1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$

$= \lim\limits_{x\to-1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}$

39. $\lim\limits_{x\to2} \frac{\sqrt{x^2+12}-4}{x-2} = \lim\limits_{x\to2} \frac{\left(\sqrt{x^2+12}-4\right)\left(\sqrt{x^2+12}+4\right)}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim\limits_{x\to2} \frac{(x^2+12)-16}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim\limits_{x\to2} \frac{(x-2)(x+2)}{(x-2)\left(\sqrt{x^2+12}+4\right)}$

$= \lim\limits_{x\to2} \frac{x+2}{\sqrt{x^2+12}+4} = \frac{4}{\sqrt{16}+4} = \frac{1}{2}$

40. $\lim\limits_{x\to-2}\frac{x+2}{\sqrt{x^2+5}-3}=\lim\limits_{x\to-2}\frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{\left(\sqrt{x^2+5}-3\right)\left(\sqrt{x^2+5}+3\right)}=\lim\limits_{x\to-2}\frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{(x^2+5)-9}=\lim\limits_{x\to-2}\frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{(x+2)(x-2)}$
$=\lim\limits_{x\to-2}\frac{\sqrt{x^2+5}+3}{x-2}=\frac{\sqrt{9}+3}{-4}=-\frac{3}{2}$

41. $\lim\limits_{x\to-3}\frac{2-\sqrt{x^2-5}}{x+3}=\lim\limits_{x\to-3}\frac{\left(2-\sqrt{x^2-5}\right)\left(2+\sqrt{x^2-5}\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)}=\lim\limits_{x\to-3}\frac{4-(x^2-5)}{(x+3)\left(2+\sqrt{x^2-5}\right)}=\lim\limits_{x\to-3}\frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)}$
$=\lim\limits_{x\to-3}\frac{(3-x)(3+x)}{(x+3)\left(2+\sqrt{x^2-5}\right)}=\lim\limits_{x\to-3}\frac{3-x}{2+\sqrt{x^2-5}}=\frac{6}{2+\sqrt{4}}=\frac{3}{2}$

42. $\lim\limits_{x\to4}\frac{4-x}{5-\sqrt{x^2+9}}=\lim\limits_{x\to4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{\left(5-\sqrt{x^2+9}\right)\left(5+\sqrt{x^2+9}\right)}=\lim\limits_{x\to4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{25-(x^2+9)}=\lim\limits_{x\to4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{16-x^2}$
$=\lim\limits_{x\to4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{(4-x)(4+x)}=\lim\limits_{x\to4}\frac{5+\sqrt{x^2+9}}{4+x}=\frac{5+\sqrt{25}}{8}=\frac{5}{4}$

43. $\lim\limits_{x\to0}(2\sin x-1)=2\sin 0-1=0-1=-1$

44. $\lim\limits_{x\to\pi/4}\sin^2 x=\left(\lim\limits_{x\to\pi/4}\sin x\right)^2=(\sin 0)^2=0^2=0$

45. $\lim\limits_{x\to0}\sec x=\lim\limits_{x\to0}\frac{1}{\cos x}=\frac{1}{\cos 0}=\frac{1}{1}=1$

46. $\lim\limits_{x\to\pi/3}\tan x=\lim\limits_{x\to\pi/3}\frac{\sin x}{\cos x}=\frac{\sin 0}{\cos 0}=\frac{0}{1}=0$

47. $\lim\limits_{x\to0}\frac{1+x+\sin x}{3\cos x}=\frac{1+0+\sin 0}{3\cos 0}=\frac{1+0+0}{3}=\frac{1}{3}$

48. $\lim\limits_{x\to0}(x^2-1)(2-\cos x)=(0^2-1)(2-\cos 0)=(-1)(2-1)=(-1)(1)=-1$

49. $\lim\limits_{x\to-\pi}\sqrt{x+4}\cos(x+\pi)=\lim\limits_{x\to-\pi}\sqrt{x+4}\cdot\lim\limits_{x\to-\pi}\cos(x+\pi)=\sqrt{-\pi+4}\cdot\cos 0=\sqrt{4-\pi}\cdot 1=\sqrt{4-\pi}$

50. $\lim\limits_{x\to0}\sqrt{7+\sec^2 x}=\sqrt{\lim\limits_{x\to0}(7+\sec^2 x)}=\sqrt{7+\lim\limits_{x\to0}\sec^2 x}=\sqrt{7+\sec^2 0}=\sqrt{7+(1)^2}=2\sqrt{2}$

51. (a) quotient rule
(b) difference and power rules
(c) sum and constant multiple rules

52. (a) quotient rule
(b) power and product rules
(c) difference and constant multiple rules

53. (a) $\lim\limits_{x\to c}f(x)g(x)=\left[\lim\limits_{x\to c}f(x)\right]\left[\lim\limits_{x\to c}g(x)\right]=(5)(-2)=-10$
(b) $\lim\limits_{x\to c}2f(x)g(x)=2\left[\lim\limits_{x\to c}f(x)\right]\left[\lim\limits_{x\to c}g(x)\right]=2(5)(-2)=-20$
(c) $\lim\limits_{x\to c}[f(x)+3g(x)]=\lim\limits_{x\to c}f(x)+3\lim\limits_{x\to c}g(x)=5+3(-2)=-1$
(d) $\lim\limits_{x\to c}\frac{f(x)}{f(x)-g(x)}=\frac{\lim\limits_{x\to c}f(x)}{\lim\limits_{x\to c}f(x)-\lim\limits_{x\to c}g(x)}=\frac{5}{5-(-2)}=\frac{5}{7}$

54. (a) $\lim_{x\to 4}[g(x)+3] = \lim_{x\to 4} g(x) + \lim_{x\to 4} 3 = -3+3 = 0$

(b) $\lim_{x\to 4} xf(x) = \lim_{x\to 4} x \cdot \lim_{x\to 4} f(x) = (4)(0) = 0$

(c) $\lim_{x\to 4}[g(x)]^2 = \left[\lim_{x\to 4} g(x)\right]^2 = [-3]^2 = 9$

(d) $\lim_{x\to 4} \frac{g(x)}{f(x)-1} = \frac{\lim_{x\to 4} g(x)}{\lim_{x\to 4} f(x) - \lim_{x\to 4} 1} = \frac{-3}{0-1} = 3$

55. (a) $\lim_{x\to b}[f(x)+g(x)] = \lim_{x\to b} f(x) + \lim_{x\to b} g(x) = 7+(-3) = 4$

(b) $\lim_{x\to b} f(x)\cdot g(x) = \left[\lim_{x\to b} f(x)\right]\left[\lim_{x\to b} g(x)\right] = (7)(-3) = -21$

(c) $\lim_{x\to b} 4g(x) = \left[\lim_{x\to b} 4\right]\left[\lim_{x\to b} g(x)\right] = (4)(-3) = -12$

(d) $\lim_{x\to b} f(x)/g(x) = \lim_{x\to b} f(x)/\lim_{x\to b} g(x) = \frac{7}{-3} = -\frac{7}{3}$

56. (a) $\lim_{x\to -2}[p(x)+r(x)+s(x)] = \lim_{x\to -2} p(x) + \lim_{x\to -2} r(x) + \lim_{x\to -2} s(x) = 4+0+(-3) = 1$

(b) $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x) = \left[\lim_{x\to -2} p(x)\right]\left[\lim_{x\to -2} r(x)\right]\left[\lim_{x\to -2} s(x)\right] = (4)(0)(-3) = 0$

(c) $\lim_{x\to -2}[-4p(x)+5r(x)]/s(x) = \left[-4\lim_{x\to -2} p(x) + 5\lim_{x\to -2} r(x)\right]\Big/\lim_{x\to -2} s(x) = [-4(4)+5(0)]/-3 = \frac{16}{3}$

57. $\lim_{h\to 0} \frac{(1+h)^2-1^2}{h} = \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} \frac{h(2+h)}{h} = \lim_{h\to 0}(2+h) = 2$

58. $\lim_{h\to 0} \frac{(-2+h)^2-(-2)^2}{h} = \lim_{h\to 0} \frac{4-4h+h^2-4}{h} = \lim_{h\to 0} \frac{h(h-4)}{h} = \lim_{h\to 0}(h-4) = -4$

59. $\lim_{h\to 0} \frac{[3(2+h)-4]-[3(2)-4]}{h} = \lim_{h\to 0} \frac{3h}{h} = 3$

60. $\lim_{h\to 0} \frac{\left(\frac{1}{-2+h}\right)-\left(\frac{1}{-2}\right)}{h} = \lim_{h\to 0} \frac{\frac{-2}{-2+h}-1}{-2h} = \lim_{h\to 0} \frac{-2-(-2+h)}{-2h(-2+h)} = \lim_{h\to 0} \frac{-h}{h(4-2h)} = -\frac{1}{4}$

61. $\lim_{h\to 0} \frac{\sqrt{7+h}-\sqrt{7}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{7+h}-\sqrt{7}\right)\left(\sqrt{7+h}+\sqrt{7}\right)}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim_{h\to 0} \frac{(7+h)-7}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{7+h}+\sqrt{7}} = \frac{1}{2\sqrt{7}}$

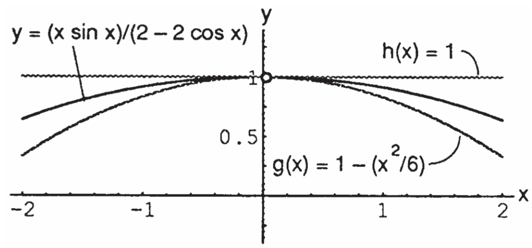
62. $\lim_{h\to 0} \frac{\sqrt{3(0+h)+1}-\sqrt{3(0)+1}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{3h+1}-1\right)\left(\sqrt{3h+1}+1\right)}{h\left(\sqrt{3h+1}+1\right)} = \lim_{h\to 0} \frac{(3h+1)-1}{h\left(\sqrt{3h+1}+1\right)} = \lim_{h\to 0} \frac{3h}{h\left(\sqrt{3h+1}+1\right)} = \lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1} = \frac{3}{2}$

63. $\lim_{x\to 0}\sqrt{5-2x^2} = \sqrt{5-2(0)^2} = \sqrt{5}$ and $\lim_{x\to 0}\sqrt{5-x^2} = \sqrt{5-(0)^2} = \sqrt{5}$; by the sandwich theorem, $\lim_{x\to 0} f(x) = \sqrt{5}$

64. $\lim_{x\to 0}(2-x^2) = 2-0 = 2$ and $\lim_{x\to 0} 2\cos x = 2(1) = 2$; by the sandwich theorem, $\lim_{x\to 0} g(x) = 2$

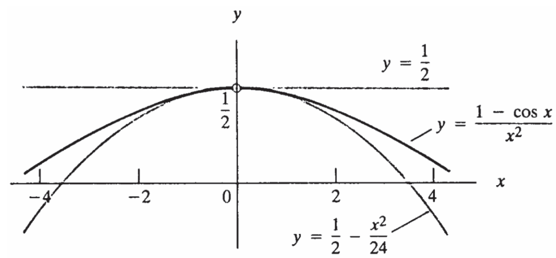
65. (a) $\lim_{x\to 0}\left(1-\frac{x^2}{6}\right) = 1-\frac{0}{6} = 1$ and $\lim_{x\to 0} 1 = 1$; by the sandwich theorem, $\lim_{x\to 0} \frac{x\sin x}{2-2\cos x} = 1$

(b) For $x \neq 0$, $y = (x \sin x)/(2 - 2\cos x)$ lies between the other two graphs in the figure, and the graphs converge as $x \to 0$.



66. (a) $\lim\limits_{x\to 0}\left(\frac{1}{2} - \frac{x^2}{24}\right) = \lim\limits_{x\to 0}\frac{1}{2} - \lim\limits_{x\to 0}\frac{x^2}{24} = \frac{1}{2} - 0 = \frac{1}{2}$ and $\lim\limits_{x\to 0}\frac{1}{2} = \frac{1}{2}$; by the sandwich theorem, $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$.

(b) For all $x \neq 0$, the graph of $f(x) = (1-\cos x)/x^2$ lies between the line $y = \frac{1}{2}$ and the parabola $y = \frac{1}{2} - x^2/24$, and the graphs converge as $x \to 0$.



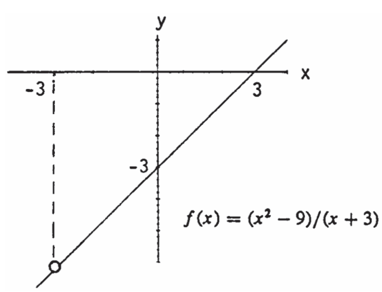
67. (a) $f(x) = (x^2 - 9)/(x+3)$

x	-3.1	-3.01	-3.001	-3.0001	-3.00001	-3.000001
$f(x)$	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001

x	-2.9	-2.99	-2.999	-2.9999	-2.99999	-2.999999
$f(x)$	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999

The estimate is $\lim\limits_{x\to -3} f(x) = -6$.

(b)

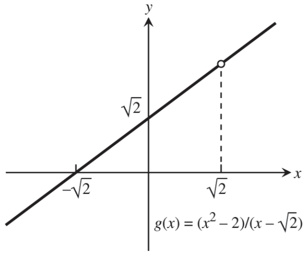


(c) $f(x) = \frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3} = x - 3$ if $x \neq -3$, and $\lim\limits_{x\to -3}(x-3) = -3 - 3 = -6$.

68. (a) $g(x) = (x^2 - 2)/\left(x - \sqrt{2}\right)$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
$g(x)$	2.81421	2.82421	2.82821	2.828413	2.828423	2.828426

(b)



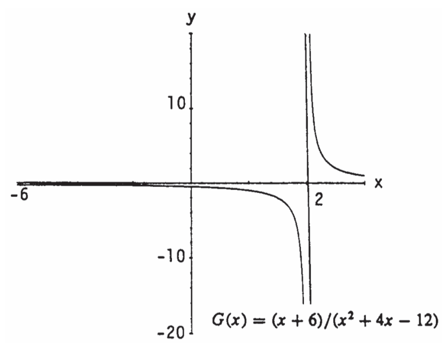
(c) $g(x)=\frac{x^2-2}{x-\sqrt{2}}=\frac{(x+\sqrt{2})(x-\sqrt{2})}{(x-\sqrt{2})}=x+\sqrt{2}$ if $x\neq\sqrt{2}$, and $\lim_{x\to\sqrt{2}}(x+\sqrt{2})=\sqrt{2}+\sqrt{2}=2\sqrt{2}$.

69. (a) $G(x)=(x+6)/(x^2+4x-12)$

x	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999
$G(x)$	$-.126582$	$-.1251564$	$-.1250156$	$-.1250015$	$-.1250001$	$-.1250000$

x	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001
$G(x)$	$-.123456$	$-.124843$	$-.124984$	$-.124998$	$-.124999$	$-.124999$

(b)



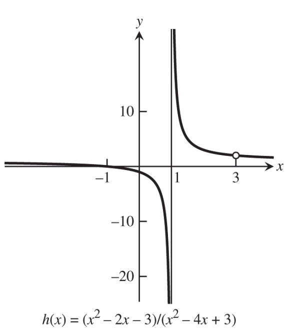
(c) $G(x)=\frac{x+6}{(x^2+4x-12)}=\frac{x+6}{(x+6)(x-2)}=\frac{1}{x-2}$ if $x\neq-6$, and $\lim_{x\to-6}\frac{1}{x-2}=\frac{1}{-6-2}=-\frac{1}{8}=-0.125$.

70. (a) $h(x)=(x^2-2x-3)/(x^2-4x+3)$

x	2.9	2.99	2.999	2.9999	2.99999	2.999999
$h(x)$	2.052631	2.005025	2.000500	2.000050	2.000005	2.0000005

x	3.1	3.01	3.001	3.0001	3.00001	3.000001
$h(x)$	1.952380	1.995024	1.999500	1.999950	1.999995	1.999999

(b)



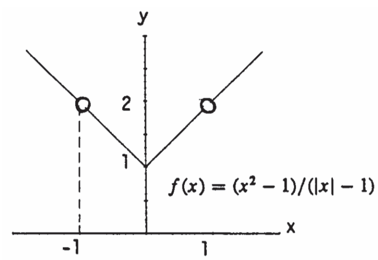
(c) $h(x)=\frac{x^2-2x-3}{x^2-4x+3}=\frac{(x-3)(x+1)}{(x-3)(x-1)}=\frac{x+1}{x-1}$ if $x\neq3$, and $\lim_{x\to3}\frac{x+1}{x-1}=\frac{3+1}{3-1}=\frac{4}{2}=2$.

71. (a) $f(x) = (x^2 - 1)/(|x| - 1)$

x	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001
$f(x)$	2.1	2.01	2.001	2.0001	2.00001	2.000001

x	$-.9$	$-.99$	$-.999$	$-.9999$	$-.99999$	$-.999999$
$f(x)$	1.9	1.99	1.999	1.9999	1.99999	1.999999

(b)



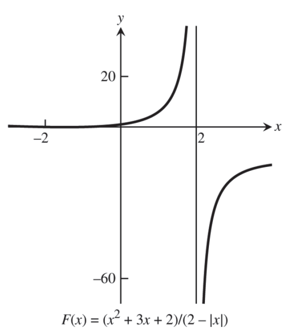
(c) $f(x) = \frac{x^2-1}{|x|-1} = \begin{cases} \frac{(x+1)(x-1)}{x-1} = x+1, \ x \ge 0 \text{ and } x \ne 1 \\ \frac{(x+1)(x-1)}{-(x+1)} = 1-x, \ x < 0 \text{ and } x \ne -1 \end{cases}$, and $\lim\limits_{x \to -1} (1-x) = 1-(-1) = 2.$

72. (a) $F(x) = (x^2 + 3x + 2)/(2 - |x|)$

x	-2.1	-2.01	-2.001	-2.0001	-2.00001	-2.000001
$F(x)$	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001

x	-1.9	-1.99	-1.999	-1.9999	-1.99999	-1.999999
$F(x)$	$-.9$	$-.99$	$-.999$	$-.9999$	$-.99999$	$-.999999$

(b)



(c) $F(x) = \frac{x^2+3x+2}{2-|x|} = \begin{cases} \frac{(x+2)(x+1)}{2-x}, & x \ge 0 \\ \frac{(x+2)(x+1)}{2+x} = x+1, & x < 0 \text{ and } x \ne -2 \end{cases}$, and $\lim\limits_{x \to -2} (x+1) = -2+1 = -1.$

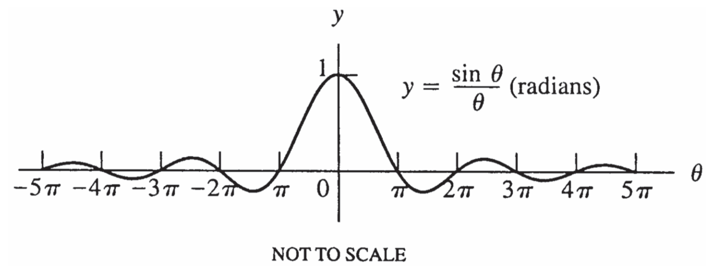
73. (a) $g(\theta) = (\sin\theta)/\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

θ	$-.1$	$-.01$	$-.001$	$-.0001$	$-.00001$	$-.000001$
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim\limits_{\theta \to 0} g(\theta) = 1$

(b)



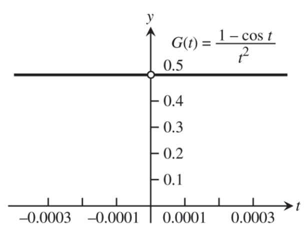
74. (a) $G(t) = (1 - \cos t)/t^2$

t	.1	.01	.001	.0001	.00001	.000001
$G(t)$	.499583	.499995	.499999	.5	.5	.5

t	$-.1$	$-.01$	$-.001$	$-.0001$	$-.00001$	$-.000001$
$G(t)$	.499583	.499995	.499999	.5	.5	.5

$\lim_{t\to 0} G(t) = 0.5$

(b)



75. $\lim_{x\to c} f(x)$ exists at those points c where $\lim_{x\to c} x^4 = \lim_{x\to c} x^2$. Thus, $c^4 = c^2 \Rightarrow c^2(1-c^2) = 0 \Rightarrow c = 0, 1,$ or -1. Moreover, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$ and $\lim_{x\to -1} f(x) = \lim_{x\to 1} f(x) = 1$.

76. Nothing can be concluded about the values of f, g, and h at $x = 2$. Yes, $f(2)$ could be 0. Since the conditions of the sandwich theorem are satisfied, $\lim_{x\to 2} f(x) = -5 \neq 0$.

77. $1 = \lim_{x\to 4} \frac{f(x)-5}{x-2} = \frac{\lim_{x\to 4} f(x) - \lim_{x\to 4} 5}{\lim_{x\to 4} x - \lim_{x\to 4} 2} = \frac{\lim_{x\to 4} f(x)-5}{4-2} \Rightarrow \lim_{x\to 4} f(x) - 5 = 2(1) \Rightarrow \lim_{x\to 4} f(x) = 2+5 = 7.$

78. (a) $1 = \lim_{x\to -2} \frac{f(x)}{x^2} = \frac{\lim_{x\to -2} f(x)}{\lim_{x\to -2} x^2} = \frac{\lim_{x\to -2} f(x)}{4} \Rightarrow \lim_{x\to -2} f(x) = 4.$

(b) $1 = \lim_{x\to -2} \frac{f(x)}{x^2} = \left[\lim_{x\to -2} \frac{f(x)}{x}\right]\left[\lim_{x\to -2} \frac{1}{x}\right] = \left[\lim_{x\to -2} \frac{f(x)}{x}\right]\left(\frac{1}{-2}\right) \Rightarrow \lim_{x\to -2} \frac{f(x)}{x} = -2.$

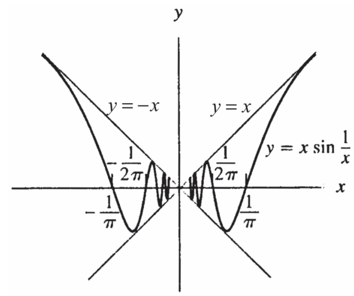
79. (a) $0 = 3\cdot 0 = \left[\lim_{x\to 2} \frac{f(x)-5}{x-2}\right]\left[\lim_{x\to 2} (x-2)\right] = \lim_{x\to 2}\left[\left(\frac{f(x)-5}{x-2}\right)(x-2)\right] = \lim_{x\to 2} [f(x)-5]$
$= \lim_{x\to 2} f(x) - 5 \Rightarrow \lim_{x\to 2} f(x) = 5.$

(b) $0 = 4\cdot 0 = \left[\lim_{x\to 2} \frac{f(x)-5}{x-2}\right]\left[\lim_{x\to 2} (x-2)\right] \Rightarrow \lim_{x\to 2} f(x) = 5$ as in part (a).

80. (a) $0 = 1\cdot 0 = \left[\lim_{x\to 0}\frac{f(x)}{x^2}\right]\left[\lim_{x\to 0} x\right]^2 = \left[\lim_{x\to 0}\frac{f(x)}{x^2}\right]\left[\lim_{x\to 0} x^2\right] = \lim_{x\to 0}\left[\frac{f(x)}{x^2}\cdot x^2\right] = \lim_{x\to 0} f(x).$
That is, $\lim_{x\to 0} f(x) = 0.$

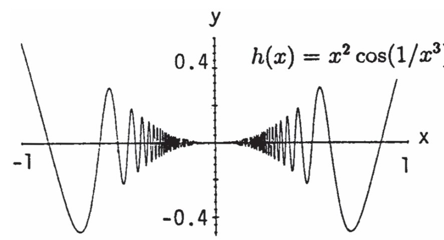
(b) $0 = 1\cdot 0 = \left[\lim_{x\to 0}\frac{f(x)}{x^2}\right]\left[\lim_{x\to 0} x\right] = \lim_{x\to 0}\left[\frac{f(x)}{x^2}\cdot x\right] = \lim_{x\to 0}\frac{f(x)}{x}$. That is, $\lim_{x\to 0}\frac{f(x)}{x} = 0.$

81. (a) $\lim_{x\to 0} x\sin\frac{1}{x} = 0$



(b) $-1 \le \sin\frac{1}{x} \le 1$ for $x \ne 0$:
$x > 0 \Rightarrow -x \le x\sin\frac{1}{x} \le x \Rightarrow \lim_{x\to 0} x\sin\frac{1}{x} = 0$ by the sandwich theorem;
$x < 0 \Rightarrow -x \ge x\sin\frac{1}{x} \ge x \Rightarrow \lim_{x\to 0} x\sin\frac{1}{x} = 0$ by the sandwich theorem.

82. (a) $\lim_{x\to 0} x^2\cos\left(\frac{1}{x^3}\right) = 0$



(b) $-1 \le \cos\left(\frac{1}{x^3}\right) \le 1$ for $x \ne 0 \Rightarrow -x^2 \le x^2\cos\left(\frac{1}{x^3}\right) \le x^2 \Rightarrow \lim_{x\to 0} x^2\cos\left(\frac{1}{x^3}\right) = 0$ by the sandwich theorem since $\lim_{x\to 0} x^2 = 0.$

83–88. Example CAS commands:
<u>Maple</u>:

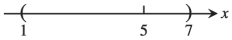
```
f := x -> (x^4 − 16)/(x − 2);
x0 := 2;
plot( f(x), x = x0-1..x0+1, color = black,
      title = "Section 2.2, #83(a)" );
limit( f(x), x = x0 );
```

In Exercise 85, note that the standard cube root, x^(1/3), is not defined for x<0 in many CASs. This can be overcome in Maple by entering the function as f := x -> (surd(x+1, 3) − 1)/x.

<u>Mathematica</u>: (assigned function and values for x0 and h may vary)

```
Clear[f, x]
f[x_]:=(x^3 − x^2 − 5x − 3)/(x + 1)^2
x0= −1; h = 0.1;
Plot[f[x],{x, x0 − h, x0 + h}]
Limit[f[x], x → x0]
```

2.3 THE PRECISE DEFINITION OF A LIMIT

1. 

Step 1: $|x-5|<\delta \Rightarrow -\delta<x-5<\delta \Rightarrow -\delta+5<x<\delta+5$

Step 2: $\delta+5=7 \Rightarrow \delta=2$, or $-\delta+5=1 \Rightarrow \delta=4$.
The value of δ which assures $|x-5|<\delta \Rightarrow 1<x<7$ is the smaller value, $\delta=2$.

2. 

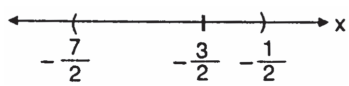
Step 1: $|x-2|<\delta \Rightarrow -\delta<x-2<\delta \Rightarrow -\delta+2<x<\delta+2$

Step 2: $-\delta+2=1 \Rightarrow \delta=1$, or $\delta+2=7 \Rightarrow \delta=5$.
The value of δ which assures $|x-2|<\delta \Rightarrow 1<x<7$ is the smaller value, $\delta=1$.

3. 

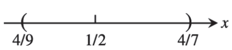
Step 1: $|x-(-3)|<\delta \Rightarrow -\delta<x+3<\delta \Rightarrow -\delta-3<x<\delta-3$

Step 2: $-\delta-3=-\frac{7}{2} \Rightarrow \delta=\frac{1}{2}$, or $\delta-3=-\frac{1}{2} \Rightarrow \delta=\frac{5}{2}$.
The value of δ which assures $|x-(-3)|<\delta \Rightarrow -\frac{7}{2}<x<-\frac{1}{2}$ is the smaller value, $\delta=\frac{1}{2}$.

4. 

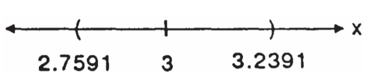
Step 1: $\left|x-\left(-\frac{3}{2}\right)\right|<\delta \Rightarrow -\delta<x+\frac{3}{2}<\delta \Rightarrow -\delta-\frac{3}{2}<x<\delta-\frac{3}{2}$

Step 2: $-\delta-\frac{3}{2}=-\frac{7}{2} \Rightarrow \delta=2$, or $\delta-\frac{3}{2}=-\frac{1}{2} \Rightarrow \delta=1$.
The value of δ which assures $\left|x-\left(-\frac{3}{2}\right)\right|<\delta \Rightarrow -\frac{7}{2}<x<-\frac{1}{2}$ is the smaller value, $\delta=1$.

5. 

Step 1: $\left|x-\frac{1}{2}\right|<\delta \Rightarrow -\delta<x-\frac{1}{2}<\delta \Rightarrow -\delta+\frac{1}{2}<x<\delta+\frac{1}{2}$

Step 2: $-\delta+\frac{1}{2}=\frac{4}{9} \Rightarrow \delta=\frac{1}{18}$, or $\delta+\frac{1}{2}=\frac{4}{7} \Rightarrow \delta=\frac{1}{14}$.
The value of δ which assures $\left|x-\frac{1}{2}\right|<\delta \Rightarrow \frac{4}{9}<x<\frac{4}{7}$ is the smaller value, $\delta=\frac{1}{18}$.

6. 

Step 1: $|x-3|<\delta \Rightarrow -\delta<x-3<\delta \Rightarrow -\delta+3<x<\delta+3$

Step 2: $-\delta+3=2.7591 \Rightarrow \delta=0.2409$, or $\delta+3=3.2391 \Rightarrow \delta=0.2391$.
The value of δ which assures $|x-3|<\delta \Rightarrow 2.7591<x<3.2391$ is the smaller value, $\delta=0.2391$.

7. Step 1: $|x-5|<\delta \Rightarrow -\delta<x-5<\delta \Rightarrow -\delta+5<x<\delta+5$

Step 2: From the graph, $-\delta+5=4.9 \Rightarrow \delta=0.1$, or $\delta+5=5.1 \Rightarrow \delta=0.1$; thus $\delta=0.1$ in either case.

8. Step 1: $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta - 3 < x < \delta - 3$
Step 2: From the graph, $-\delta - 3 = -3.1 \Rightarrow \delta = 0.1$, or $\delta - 3 = -2.9 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$.

9. Step 1: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow -\delta + 1 < x < \delta + 1$
Step 2: From the graph, $-\delta + 1 = \frac{9}{16} \Rightarrow \delta = \frac{7}{16}$, or $\delta + 1 = \frac{25}{16} \Rightarrow \delta = \frac{9}{16}$; thus $\delta = \frac{7}{16}$.

10. Step 1: $|x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3$
Step 2: From the graph, $-\delta + 3 = 2.61 \Rightarrow \delta = 0.39$, or $\delta + 3 = 3.41 \Rightarrow \delta = 0.41$; thus $\delta = 0.39$.

11. Step 1: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2$
Step 2: From the graph,$-\delta + 2 = \sqrt{3} \Rightarrow \delta = 2 - \sqrt{3} \approx 0.2679$, or $\delta + 2 = \sqrt{5} \Rightarrow \delta = \sqrt{5} - 2 \approx 0.2361$; thus $\delta = \sqrt{5} - 2$.

12. Step 1: $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$
Step 2: From the graph, $-\delta - 1 = -\frac{\sqrt{5}}{2} \Rightarrow \delta = \frac{\sqrt{5}-2}{2} \approx 0.118$ or $\delta - 1 = -\frac{\sqrt{3}}{2} \Rightarrow \delta = \frac{2-\sqrt{3}}{2} \approx 0.1340$; thus $\delta = \frac{\sqrt{5}-2}{2}$.

13. Step 1: $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$
Step 2: From the graph, $-\delta - 1 = -\frac{16}{9} \Rightarrow \delta = \frac{7}{9} \approx 0.77$, or $\delta - 1 = -\frac{16}{25} \Rightarrow \frac{9}{25} = 0.36$; thus $\delta = \frac{9}{25} = 0.36$.

14. Step 1: $\left|x - \frac{1}{2}\right| < \delta \Rightarrow -\delta < x - \frac{1}{2} < \delta \Rightarrow -\delta + \frac{1}{2} < x < \delta + \frac{1}{2}$
Step 2: From the graph, $-\delta + \frac{1}{2} = \frac{1}{2.01} \Rightarrow \delta = \frac{1}{2} - \frac{1}{2.01} \approx 0.00248$, or $\delta + \frac{1}{2} = \frac{1}{1.99} \Rightarrow \delta = \frac{1}{1.99} - \frac{1}{2} \approx 0.00251$; thus $\delta = 0.00248$.

15. Step 1: $|(x+1)-5| < 0.01 \Rightarrow |x-4| < 0.01 \Rightarrow -0.01 < x-4 < 0.01 \Rightarrow 3.99 < x < 4.01$
Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta + 4 < x < \delta + 4 \Rightarrow \delta = 0.01$.

16. Step 1: $|(2x-2)-(-6)| < 0.02 \Rightarrow |2x+4| < 0.02 \Rightarrow -0.02 < 2x+4 < 0.02$
$\Rightarrow -4.02 < 2x < -3.98 \Rightarrow -2.01 < x < -1.99$
Step 2: $|x-(-2)| < \delta \Rightarrow -\delta < x+2 < \delta \Rightarrow -\delta - 2 < x < \delta - 2 \Rightarrow \delta = 0.01$.

17. Step 1: $\left|\sqrt{x+1} - 1\right| < 0.1 \Rightarrow -0.1 < \sqrt{x+1} - 1 < 0.1 \Rightarrow 0.9 < \sqrt{x+1} < 1.1 \Rightarrow 0.81 < x+1 < 1.21$
$\Rightarrow -0.19 < x < 0.21$
Step 2: $|x-0| < \delta \Rightarrow -\delta < x < \delta$. Then, $-\delta = -0.19 \Rightarrow \delta = 0.19$ or $\delta = 0.21$; thus, $\delta = 0.19$.

18. Step 1: $\left|\sqrt{x} - \frac{1}{2}\right| < 0.1 \Rightarrow -0.1 < \sqrt{x} - \frac{1}{2} < 0.1 \Rightarrow 0.4 < \sqrt{x} < 0.6 \Rightarrow 0.16 < x < 0.36$
Step 2: $\left|x - \frac{1}{4}\right| < \delta \Rightarrow -\delta < x - \frac{1}{4} < \delta \Rightarrow -\delta + \frac{1}{4} < x < \delta + \frac{1}{4}$.
Then $-\delta + \frac{1}{4} = 0.16 \Rightarrow \delta = 0.09$ or $\delta + \frac{1}{4} = 0.36 \Rightarrow \delta = 0.11$; thus $\delta = 0.09$.

19. Step 1: $\left|\sqrt{19-x} - 3\right| < 1 \Rightarrow -1 < \sqrt{19-x} - 3 < 1 \Rightarrow 2 < \sqrt{19-x} < 4 \Rightarrow 4 < 19-x < 16$
$\Rightarrow -4 > x-19 > -16 \Rightarrow 15 > x > 3$ or $3 < x < 15$
Step 2: $|x-10| < \delta \Rightarrow -\delta < x-10 < \delta \Rightarrow -\delta + 10 < x < \delta + 10$.
Then $-\delta + 10 = 3 \Rightarrow \delta = 7$, or $\delta + 10 = 15 \Rightarrow \delta = 5$; thus $\delta = 5$.

20. Step 1: $\left|\sqrt{x-7}-4\right|<1 \Rightarrow -1<\sqrt{x-7}-4<1 \Rightarrow 3<\sqrt{x-7}<5 \Rightarrow 9<x-7<25 \Rightarrow 16<\mathrm{x}<32$
Step 2: $|x-23|<\delta \Rightarrow -\delta<x-23<\delta \Rightarrow -\delta+23<\mathrm{x}<\delta+23$.
Then $-\delta+23=16 \Rightarrow \delta=7$, or $\delta+23=32 \Rightarrow \delta=9$; thus $\delta=7$.

21. Step 1: $\left|\frac{1}{x}-\frac{1}{4}\right|<0.05 \Rightarrow -0.05<\frac{1}{x}-\frac{1}{4}<0.05 \Rightarrow 0.2<\frac{1}{x}<0.3 \Rightarrow \frac{10}{2}>x>\frac{10}{3}$ or $\frac{10}{3}<x<5$.
Step 2: $|x-4|<\delta \Rightarrow -\delta<x-4<\delta \Rightarrow -\delta+4<x<\delta+4$.
Then $-\delta+4=\frac{10}{3}$ or $\delta=\frac{2}{3}$, or $\delta+4=5$ or $\delta=1$; thus $\delta=\frac{2}{3}$.

22. Step 1: $\left|x^2-3\right|<0.1 \Rightarrow -0.1<x^2-3<0.1 \Rightarrow 2.9<x^2<3.1 \Rightarrow \sqrt{2.9}<x<\sqrt{3.1}$
Step 2: $\left|x-\sqrt{3}\right|<\delta \Rightarrow -\delta<x-\sqrt{3}<\delta \Rightarrow -\delta+\sqrt{3}<x<\delta+\sqrt{3}$.
Then $-\delta+\sqrt{3}=\sqrt{2.9} \Rightarrow \delta=\sqrt{3}-\sqrt{2.9}\approx 0.0291$, or $\delta+\sqrt{3}=\sqrt{3.1} \Rightarrow \delta=\sqrt{3.1}-\sqrt{3}\approx 0.0286$;
thus $\delta=0.0286$

23. Step 1: $\left|x^2-4\right|<0.5 \Rightarrow -0.5<x^2-4<0.5 \Rightarrow 3.5<x^2<4.5 \Rightarrow \sqrt{3.5}<|x|<\sqrt{4.5} \Rightarrow -\sqrt{4.5}<x<-\sqrt{3.5}$,
for x near -2.
Step 2: $|x-(-2)|<\delta \Rightarrow -\delta<x+2<\delta \Rightarrow -\delta-2<x<\delta-2$.
Then $-\delta-2=-\sqrt{4.5} \Rightarrow \delta=\sqrt{4.5}-2\approx 0.1213$, or $\delta-2=-\sqrt{3.5} \Rightarrow \delta=2-\sqrt{3.5}\approx 0.1292$;
thus $\delta=\sqrt{4.5}-2\approx 0.12$.

24. Step 1: $\left|\frac{1}{x}-(-1)\right|<0.1 \Rightarrow -0.1<\frac{1}{x}+1<0.1 \Rightarrow -\frac{11}{10}<\frac{1}{x}<-\frac{9}{10} \Rightarrow -\frac{10}{11}>x>-\frac{10}{9}$ or $-\frac{10}{9}<x<-\frac{10}{11}$.
Step 2: $|x-(-1)|<\delta \Rightarrow -\delta<x+1<\delta \Rightarrow -\delta-1<x<\delta-1$.
Then $-\delta-1=-\frac{10}{9} \Rightarrow \delta=\frac{1}{9}$, or $\delta-1=-\frac{10}{11} \Rightarrow \delta=\frac{1}{11}$; thus $\delta=\frac{1}{11}$.

25. Step 1: $\left|(x^2-5)-11\right|<1 \Rightarrow \left|x^2-16\right|<1 \Rightarrow -1<x^2-16<1 \Rightarrow 15<x^2<17 \Rightarrow \sqrt{15}<x<\sqrt{17}$.
Step 2: $|x-4|<\delta \Rightarrow -\delta<x-4<\delta \Rightarrow -\delta+4<x<\delta+4$.
Then $-\delta+4=\sqrt{15} \Rightarrow \delta=4-\sqrt{15}\approx 0.1270$, or $\delta+4=\sqrt{17} \Rightarrow \delta=\sqrt{17}-4\approx 0.1231$; thus
$\delta=\sqrt{17}-4\approx 0.12$.

26. Step 1: $\left|\frac{120}{x}-5\right|<1 \Rightarrow -1<\frac{120}{x}-5<1 \Rightarrow 4<\frac{120}{x}<6 \Rightarrow \frac{1}{4}>\frac{x}{120}>\frac{1}{6} \Rightarrow 30>x>20$ or $20<x<30$.
Step 2: $|x-24|<\delta \Rightarrow -\delta<x-24<\delta \Rightarrow -\delta+24<x<\delta+24$.
Then $-\delta+24=20 \Rightarrow \delta=4$, or $\delta+24=30 \Rightarrow \delta=6$; thus $\Rightarrow \delta=4$.

27. Step 1: $|mx-2m|<0.03 \Rightarrow -0.03<mx-2m<0.03 \Rightarrow -0.03+2m<mx<0.03+2m \Rightarrow 2-\frac{0.03}{m}<x<2+\frac{0.03}{m}$.
Step 2: $|x-2|<\delta \Rightarrow -\delta<x-2<\delta \Rightarrow -\delta+2<x<\delta+2$.
Then $-\delta+2=2-\frac{0.03}{m} \Rightarrow \delta=\frac{0.03}{m}$, or $\delta+2=2+\frac{0.03}{m} \Rightarrow \delta=\frac{0.03}{m}$. In either case, $\delta=\frac{0.03}{m}$.

28. Step 1: $|mx-3m|<c \Rightarrow -c<mx-3m<c \Rightarrow -c+3m<mx<c+3m \Rightarrow 3-\frac{c}{m}<x<3+\frac{c}{m}$
Step 2: $|x-3|<\delta \Rightarrow -\delta<x-3<\delta \Rightarrow -\delta+3<x<\delta+3$.
Then $-\delta+3=3-\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$, or $\delta+3=3+\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$. In either case, $\delta=\frac{c}{m}$.

29. Step 1: $\left|(mx+b)-\left(\frac{m}{2}+b\right)\right|<c \Rightarrow -c<mx-\frac{m}{2}<c \Rightarrow -c+\frac{m}{2}<mx<c+\frac{m}{2} \Rightarrow \frac{1}{2}-\frac{c}{m}<x<\frac{1}{2}+\frac{c}{m}$.
Step 2: $\left|x-\frac{1}{2}\right|<\delta \Rightarrow -\delta<x-\frac{1}{2}<\delta \Rightarrow -\delta+\frac{1}{2}<x<\delta+\frac{1}{2}$.
Then $-\delta+\frac{1}{2}=\frac{1}{2}-\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$, or $\delta+\frac{1}{2}=\frac{1}{2}+\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$. In either case, $\delta=\frac{c}{m}$.

30. Step 1: $|(mx+b)-(m+b)|<0.05 \Rightarrow -0.05<mx-m<0.05 \Rightarrow -0.05+m<mx<0.05+m$
$\Rightarrow 1-\frac{0.05}{m}<x<1+\frac{0.05}{m}$.
Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow -\delta+1<x<\delta+1$.
Then $-\delta+1=1-\frac{0.05}{m} \Rightarrow \delta=\frac{0.05}{m}$, or $\delta+1=1+\frac{0.05}{m} \Rightarrow \delta=\frac{0.05}{m}$. In either case, $\delta=\frac{0.05}{m}$.

31. $\lim\limits_{x\to 3}(3-2x)=3-2(3)=-3$
Step 1: $|(3-2x)-(-3)|<0.02 \Rightarrow -0.02<6-2x<0.02 \Rightarrow -6.02<-2x<-5.98 \Rightarrow 3.01>x>2.99$ or $2.99<x<3.01$.
Step 2: $0<|x-3|<\delta \Rightarrow -\delta<x-3<\delta \Rightarrow -\delta+3<x<\delta+3$.
Then $-\delta+3=2.99 \Rightarrow \delta=0.01$, or $\delta+3=3.01 \Rightarrow \delta=0.01$; thus $\delta=0.01$.

32. $\lim\limits_{x\to -1}(-3x-2)=(-3)(-1)-2=1$
Step 1: $|(-3x-2)-1|<0.03 \Rightarrow -0.03<-3x-3<0.03 \Rightarrow 0.01>x+1>-0.01 \Rightarrow -1.01<x<-0.99$.
Step 2: $|x-(-1)|<\delta \Rightarrow -\delta<x+1<\delta \Rightarrow -\delta-1<x<\delta-1$.
Then $-\delta-1=-1.01 \Rightarrow \delta=0.01$, or $\delta-1=-0.99 \Rightarrow \delta=0.01$; thus $\delta=0.01$.

33. $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}=\lim\limits_{x\to 2}\frac{(x+2)(x-2)}{(x-2)}=\lim\limits_{x\to 2}(x+2)=2+2=4,\ x\neq 2$
Step 1: $\left|\left(\frac{x^2-4}{x-2}\right)-4\right|<0.05 \Rightarrow -0.05<\frac{(x+2)(x-2)}{(x-2)}-4<0.05 \Rightarrow 3.95<x+2<4.05,\ x\neq 2$
$\Rightarrow 1.95<x<2.05,\ x\neq 2$.
Step 2: $|x-2|<\delta \Rightarrow -\delta<x-2<\delta \Rightarrow -\delta+2<x<\delta+2$.
Then $-\delta+2=1.95 \Rightarrow \delta=0.05$, or $\delta+2=2.05 \Rightarrow \delta=0.05$; thus $\delta=0.05$.

34. $\lim\limits_{x\to -5}\frac{x^2+6x+5}{x+5}=\lim\limits_{x\to -5}\frac{(x+5)(x+1)}{(x+5)}=\lim\limits_{x\to -5}(x+1)=-4,\ x\neq -5$.
Step 1: $\left|\left(\frac{x^2+6x+5}{x+5}\right)-(-4)\right|<0.05 \Rightarrow -0.05<\frac{(x+5)(x+1)}{(x+5)}+4<0.05 \Rightarrow -4.05<x+1<-3.95,\ \ x\neq -5$
$\Rightarrow -5.05<x<-4.95,\ x\neq -5$.
Step 2: $|x-(-5)|<\delta \Rightarrow -\delta<x+5<\delta \Rightarrow -\delta-5<x<\delta-5$.
Then $-\delta-5=-5.05 \Rightarrow \delta=0.05$, or $\delta-5=-4.95 \Rightarrow \delta=0.05$; thus $\delta=0.05$.

35. $\lim\limits_{x\to -3}\sqrt{1-5x}=\sqrt{1-5(-3)}=\sqrt{16}=4$
Step 1: $\left|\sqrt{1-5x}-4\right|<0.5 \Rightarrow -0.5<\sqrt{1-5x}-4<0.5 \Rightarrow 3.5<\sqrt{1-5x}<4.5 \Rightarrow 12.25<1-5x<20.25$
$\Rightarrow 11.25<-5x<19.25 \Rightarrow -3.85<x<2.25$.
Step 2: $|x-(-3)|<\delta \Rightarrow -\delta<x+3<\delta \Rightarrow -\delta-3<x<\delta-3$.
Then $-\delta-3=-3.85 \Rightarrow \delta=0.85$, or $\delta-3=-2.25 \Rightarrow 0.75$; thus $\delta=0.75$.

36. $\lim\limits_{x\to 2}\frac{4}{x}=\frac{4}{2}=2$
Step 1: $\left|\frac{4}{x}-2\right|<0.4 \Rightarrow -0.4<\frac{4}{x}-2<0.4 \Rightarrow 1.6<\frac{4}{x}<2.4 \Rightarrow \frac{10}{16}>\frac{x}{4}>\frac{10}{24} \Rightarrow \frac{10}{4}>x>\frac{10}{6}$ or $\frac{5}{3}<x<\frac{5}{2}$.
Step 2: $|x-2|<\delta \Rightarrow -\delta<x-2<\delta \Rightarrow -\delta+2<x<\delta+2$.
Then $-\delta+2=\frac{5}{3} \Rightarrow \delta=\frac{1}{3}$, or $\delta+2=\frac{5}{2} \Rightarrow \delta=\frac{1}{2}$; thus $\delta=\frac{1}{3}$.

37. Step 1: $|(9-x)-5|<\epsilon \Rightarrow -\epsilon<4-x<\epsilon \Rightarrow -\epsilon-4<-x<\epsilon-4 \Rightarrow \epsilon+4>x>4-\epsilon \Rightarrow 4-\epsilon<x<4+\epsilon$.
Step 2: $|x-4|<\delta \Rightarrow -\delta<x-4<\delta \Rightarrow -\delta+4<x<\delta+4$.
Then $-\delta+4=-\epsilon+4 \Rightarrow \delta=\epsilon$, or $\delta+4=\epsilon+4 \Rightarrow \delta=\epsilon$. Thus choose $\delta=\epsilon$.

38. Step 1: $|(3x-7)-2|<\epsilon \Rightarrow -\epsilon<3x-9<\epsilon \Rightarrow 9-\epsilon<3x<9+\epsilon \Rightarrow 3-\frac{\epsilon}{3}<x<3+\frac{\epsilon}{3}$.
Step 2: $|x-3|<\delta \Rightarrow -\delta<x-3<\delta \Rightarrow -\delta+3<x<\delta+3$.
Then $-\delta+3=3-\frac{\epsilon}{3} \Rightarrow \delta=\frac{\epsilon}{3}$, or $\delta+3=3+\frac{\epsilon}{3} \Rightarrow \delta=\frac{\epsilon}{3}$. Thus choose $\delta=\frac{\epsilon}{3}$.

39. Step 1: $\left|\sqrt{x-5}-2\right|<\epsilon \Rightarrow -\epsilon<\sqrt{x-5}-2<\epsilon \Rightarrow 2-\epsilon<\sqrt{x-5}<2+\epsilon \Rightarrow (2-\epsilon)^2<x-5<(2+\epsilon)^2$
$\Rightarrow (2-\epsilon)^2+5<x<(2+\epsilon)^2+5$.
Step 2: $|x-9|<\delta \Rightarrow -\delta<x-9<\delta \Rightarrow -\delta+9<x<\delta+9$.
Then $-\delta+9=\epsilon^2-4\epsilon+9 \Rightarrow \delta=4\epsilon-\epsilon^2$, or $\delta+9=\epsilon^2+4\epsilon+9 \Rightarrow \delta=4\epsilon+\epsilon^2$. Thus choose the smaller distance, $\delta=4\epsilon-\epsilon^2$.

40. Step 1: $\left|\sqrt{4-x}-2\right|<\epsilon \Rightarrow -\epsilon<\sqrt{4-x}-2<\epsilon \Rightarrow 2-\epsilon<\sqrt{4-x}<2+\epsilon \Rightarrow (2-\epsilon)^2<4-x<(2+\epsilon)^2$
$\Rightarrow -(2+\epsilon)^2<x-4<-(2-\epsilon)^2 \Rightarrow -(2+\epsilon)^2+4<x<-(2-\epsilon)^2+4$.
Step 2: $|x-0|<\delta \Rightarrow -\delta<x<\delta$.
Then $-\delta=-(2+\epsilon)^2+4=-\epsilon^2-4\epsilon \Rightarrow \delta=4\epsilon+\epsilon^2$, or $\delta=-(2-\epsilon)^2+4=4\epsilon-\epsilon^2$. Thus choose the smaller distance, $\delta=4\epsilon-\epsilon^2$.

41. Step 1: For $x\neq 1$, $\left|x^2-1\right|<\epsilon \Rightarrow -\epsilon<x^2-1<\epsilon \Rightarrow 1-\epsilon<x^2<1+\epsilon \Rightarrow \sqrt{1-\epsilon}<|x|<\sqrt{1+\epsilon}$
$\Rightarrow \sqrt{1-\epsilon}<x<\sqrt{1+\epsilon}$ near $x=1$.
Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow -\delta+1<x<\delta+1$.
Then $-\delta+1=\sqrt{1-\epsilon} \Rightarrow \delta=1-\sqrt{1-\epsilon}$, or $\delta+1=\sqrt{1+\epsilon} \Rightarrow \delta=\sqrt{1+\epsilon}-1$. Choose $\delta=\min\left\{1-\sqrt{1-\epsilon}, \sqrt{1+\epsilon}-1\right\}$, that is, the smaller of the two distances.

42. Step 1: For $x\neq -2$, $\left|x^2-4\right|<\epsilon \Rightarrow -\epsilon<x^2-4<\epsilon \Rightarrow 4-\epsilon<x^2<4+\epsilon \Rightarrow \sqrt{4-\epsilon}<|x|<\sqrt{4+\epsilon} \Rightarrow -\sqrt{4+\epsilon}<x$
$<-\sqrt{4-\epsilon}$ near $x=-2$.
Step 2: $|x-(-2)|<\delta \Rightarrow -\delta<x+2<\delta \Rightarrow -\delta-2<x<\delta-2$.
Then $-\delta-2=-\sqrt{4+\epsilon} \Rightarrow \delta=\sqrt{4+\epsilon}-2$, or $\delta-2=-\sqrt{4-\epsilon} \Rightarrow \delta=2-\sqrt{4-\epsilon}$. Choose $\delta=\min\left\{\sqrt{4+\epsilon}-2, 2-\sqrt{4-\epsilon}\right\}$.

43. Step 1: $\left|\frac{1}{x}-1\right|<\epsilon \Rightarrow -\epsilon<\frac{1}{x}-1<\epsilon \Rightarrow 1-\epsilon<\frac{1}{x}<1+\epsilon \Rightarrow \frac{1}{1+\epsilon}<x<\frac{1}{1-\epsilon}$.
Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow 1-\delta<x<1+\delta$.
Then $1-\delta=\frac{1}{1+\epsilon} \Rightarrow \delta=1-\frac{1}{1+\epsilon}=\frac{\epsilon}{1+\epsilon}$, or $1+\delta=\frac{1}{1-\epsilon} \Rightarrow \delta=\frac{1}{1-\epsilon}-1=\frac{\epsilon}{1-\epsilon}$.
Choose $\delta=\frac{\epsilon}{1+\epsilon}$, the smaller of the two distances.

44. Step 1: $\left|\frac{1}{x^2}-\frac{1}{3}\right|<\epsilon \Rightarrow -\epsilon<\frac{1}{x^2}-\frac{1}{3}<\epsilon \Rightarrow \frac{1}{3}-\epsilon<\frac{1}{x^2}<\frac{1}{3}+\epsilon \Rightarrow \frac{1-3\epsilon}{3}<\frac{1}{x^2}<\frac{1+3\epsilon}{3}$
$\Rightarrow \frac{3}{1-3\epsilon}>x^2>\frac{3}{1+3\epsilon} \Rightarrow \sqrt{\frac{3}{1+3\epsilon}}<|x|<\sqrt{\frac{3}{1-3\epsilon}}$, or $\sqrt{\frac{3}{1+3\epsilon}}<x<\sqrt{\frac{3}{1-3\epsilon}}$ for x near $\sqrt{3}$.
Step 2: $\left|x-\sqrt{3}\right|<\delta \Rightarrow -\delta<x-\sqrt{3}<\delta \Rightarrow \sqrt{3}-\delta<x<\sqrt{3}+\delta$.
Then $\sqrt{3}-\delta=\sqrt{\frac{3}{1+3\epsilon}} \Rightarrow \delta=\sqrt{3}-\sqrt{\frac{3}{1+3\epsilon}}$, or $\sqrt{3}+\delta=\sqrt{\frac{3}{1-3\epsilon}} \Rightarrow \delta=\sqrt{\frac{3}{1-3\epsilon}}-\sqrt{3}$.
Choose $\delta=\min\left\{\sqrt{3}-\sqrt{\frac{3}{1+3\epsilon}}, \sqrt{\frac{3}{1-3\epsilon}}-\sqrt{3}\right\}$.

45. Step 1: $\left|\left(\frac{x^2-9}{x+3}\right)-(-6)\right|<\epsilon \Rightarrow -\epsilon<(x-3)+6<\epsilon,\ x\neq -3 \Rightarrow -\epsilon<x+3<\epsilon \Rightarrow -\epsilon-3<x<\epsilon-3$.
Step 2: $|x-(-3)|<\delta \Rightarrow -\delta<x+3<\delta \Rightarrow -\delta-3<x<\delta-3$.
Then $-\delta-3=-\epsilon-3 \Rightarrow \delta=\epsilon$, or $\delta-3=\epsilon-3 \Rightarrow \delta=\epsilon$. Choose $\delta=\epsilon$.

46. Step 1: $\left|\left(\frac{x^2-1}{x-1}\right)-2\right|<\epsilon \Rightarrow -\epsilon<(x+1)-2<\epsilon,\ x\neq 1 \Rightarrow 1-\epsilon<x<1+\epsilon.$

Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow 1-\delta<x<1+\delta.$
Then $1-\delta=1-\epsilon \Rightarrow \delta=\epsilon$, or $1+\delta=1+\epsilon \Rightarrow \delta=\epsilon$. Choose $\delta=\epsilon$.

47. Step 1: $x<1$: $|(4-2x)-2|<\epsilon \Rightarrow 0<2-2x<\epsilon$ since $x<1$. Thus, $1-\frac{\epsilon}{2}<x<0$;
$x\geq 1$: $|(6x-4)-2|<\epsilon \Rightarrow 0\leq 6x-6<\epsilon$ since $x\geq 1$. Thus, $1\leq x<1+\frac{\epsilon}{6}$.

Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow 1-\delta<x<1+\delta.$
Then $1-\delta=1-\frac{\epsilon}{2} \Rightarrow \delta=\frac{\epsilon}{2}$, or $1+\delta=1+\frac{\epsilon}{6} \Rightarrow \delta=\frac{\epsilon}{6}$. Choose $\delta=\frac{\epsilon}{6}$.

48. Step 1: $x<0$: $|2x-0|<\epsilon \Rightarrow -\epsilon<2x<0 \Rightarrow -\frac{\epsilon}{2}<x<0$;
$x\geq 0$: $\left|\frac{x}{2}-0\right|<\epsilon \Rightarrow 0\leq x<2\epsilon$.

Step 2: $|x-0|<\delta \Rightarrow -\delta<x<\delta$.
Then $-\delta=-\frac{\epsilon}{2} \Rightarrow \delta=\frac{\epsilon}{2}$, or $\delta=2\epsilon \Rightarrow \delta=2\epsilon$. Choose $\delta=\frac{\epsilon}{2}$.

49. By the figure, $-x\leq x\sin\frac{1}{x}\leq x$ for all $x>0$ and $-x\geq x\sin\frac{1}{x}\geq x$ for $x<0$. Since $\lim\limits_{x\to 0}(-x)=\lim\limits_{x\to 0}x=0$, then by the sandwich theorem, in either case, $\lim\limits_{x\to 0}x\sin\frac{1}{x}=0$.

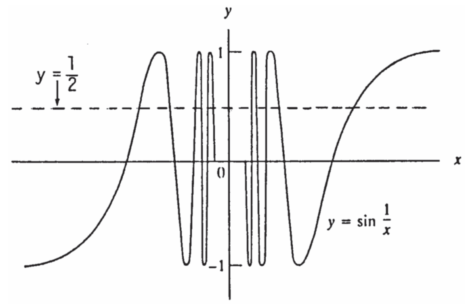
50. By the figure, $-x^2\leq x^2\sin\frac{1}{x}\leq x^2$ for all x except possibly at $x=0$. Since $\lim\limits_{x\to 0}(-x^2)=\lim\limits_{x\to 0}x^2=0$, then by the sandwich theorem, $\lim\limits_{x\to 0}x^2\sin\frac{1}{x}=0$.

51. As x approaches the value 0, the values of $g(x)$ approach k. Thus for every number $\epsilon>0$, there exists a $\delta>0$ such that $0<|x-0|<\delta \Rightarrow |g(x)-k|<\epsilon$.

52. Write $x=h+c$. Then $0<|x-c|<\delta \Leftrightarrow -\delta<x-c<\delta,\ x\neq c \Leftrightarrow -\delta<(h+c)-c<\delta,\ h+c\neq c \Leftrightarrow -\delta<h<\delta,$ $h\neq 0 \Leftrightarrow 0<|h-0|<\delta$.
Thus, $\lim\limits_{x\to c}f(x)=L \Leftrightarrow$ for any $\epsilon>0$, there exists $\delta>0$ such that $|f(x)-L|<\epsilon$ whenever $0<|x-c|<\delta \Leftrightarrow$ $|f(h+c)-L|<\epsilon$ whenever $0<|h-0|<\delta \Leftrightarrow \lim\limits_{h\to 0}f(h+c)=L$.

53. Let $f(x)=x^2$. The function values do get closer to -1 as x approaches 0, but $\lim\limits_{x\to 0}f(x)=0$, not -1. The function $f(x)=x^2$ never gets arbitrarily close to -1 for x near 0.

54. Let $f(x)=\sin x$, $L=\frac{1}{2}$, and $x_0=0$. There exists a value of x (namely $x=\frac{\pi}{6}$) for which $\left|\sin x-\frac{1}{2}\right|<\epsilon$ for any given $\epsilon>0$. However, $\lim\limits_{x\to 0}\sin x=0$, not $\frac{1}{2}$. The wrong statement does not require x to be arbitrarily close to x_0. As another example, let $g(x)=\sin\frac{1}{x}$, $L=\frac{1}{2}$, and $x_0=0$. We can choose infinitely many values of x near 0 such that $\sin\frac{1}{x}=\frac{1}{2}$ as you can see from the accompanying figure. However, $\lim\limits_{x\to 0}\sin\frac{1}{x}$ fails to exist. The wrong statement does not require all values of x arbitrarily close to $x_0=0$ to lie within $\epsilon>0$ of $L=\frac{1}{2}$. Again you can see from the figure that there are also infinitely many values of x near 0 such that $\sin\frac{1}{x}=0$. If we choose $\epsilon<\frac{1}{4}$ we cannot satisfy the inequality $\left|\sin\frac{1}{x}-\frac{1}{2}\right|<\epsilon$ for all values of x sufficiently near $x_0=0$.



55. $|A-9| \le 0.01 \Rightarrow -0.01 \le \pi\left(\frac{x}{2}\right)^2 - 9 \le 0.01 \Rightarrow 8.99 \le \frac{\pi x^2}{4} \le 9.01 \Rightarrow \frac{4}{\pi}(8.99) \le x^2 \le \frac{4}{\pi}(9.01)$ $\Rightarrow 2\sqrt{\frac{8.99}{\pi}} \le x \le 2\sqrt{\frac{9.01}{\pi}}$ or $3.384 \le x \le 3.387$. To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

56. $V = RI \Rightarrow \frac{V}{R} = I \Rightarrow \left|\frac{V}{R} - 5\right| \le 0.1 \Rightarrow -0.1 \le \frac{120}{R} - 5 \le 0.1 \Rightarrow 4.9 \le \frac{120}{R} \le 5.1 \Rightarrow \frac{10}{49} \ge \frac{R}{120} \ge \frac{10}{51}$ $\Rightarrow \frac{(120)(10)}{51} \le R \le \frac{(120)(10)}{49} \Rightarrow 23.53 \le R \le 24.48$.

To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

57. (a) $-\delta < x-1 < 0 \Rightarrow 1-\delta < x < 1 \Rightarrow f(x) = x$. Then $|f(x)-2| = |x-2| = 2-x > 2-1 = 1$. That is, $|f(x)-2| \ge 1 \ge \frac{1}{2}$ no matter how small δ is taken when $1-\delta < x < 1 \Rightarrow \lim_{x\to1} f(x) \ne 2$.

(b) $0 < x-1 < \delta \Rightarrow 1 < x < 1+\delta \Rightarrow f(x) = x+1$. Then $|f(x)-1| = |(x+1)-1| = |x| = x > 1$. That is, $|f(x)-1| \ge 1$ no matter how small δ is taken when $1 < x < 1+\delta \Rightarrow \lim_{x\to1} f(x) \ne 1$.

(c) $-\delta < x-1 < 0 \Rightarrow 1-\delta < x < 1 \Rightarrow f(x) = x$. Then $|f(x)-1.5| = |x-1.5| = 1.5-x > 1.5-1 = 0.5$. Also, $0 < x-1 < \delta \Rightarrow 1 < x < 1+\delta \Rightarrow f(x) = x+1$. Then $|f(x)-1.5| = |(x+1)-1.5| = |x-0.5|$ $= x-0.5 > 1-0.5 = 0.5$. Thus, no matter how small δ is taken, there exists a value of x such that $-\delta < x-1 < \delta$ but $|f(x)-1.5| \ge \frac{1}{2} \Rightarrow \lim_{x\to1} f(x) \ne 1.5$.

58. (a) For $2 < x < 2+\delta \Rightarrow h(x) = 2 \Rightarrow |h(x)-4| = 2$. Thus for $\epsilon < 2$, $|h(x)-4| \ge \epsilon$ whenever $2 < x < 2+\delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to2} h(x) \ne 4$.

(b) For $2 < x < 2+\delta \Rightarrow h(x) = 2 \Rightarrow |h(x)-3| = 1$. Thus for $\epsilon < 1$, $|h(x)-3| \ge \epsilon$ whenever $2 < x < 2+\delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to2} h(x) \ne 3$.

(c) For $2-\delta < x < 2 \Rightarrow h(x) = x^2$ so $|h(x)-2| = \left|x^2-2\right|$. No matter how small $\delta > 0$ is chosen, x^2 is close to 4 when x is near 2 and to the left on the real line $\Rightarrow \left|x^2-2\right|$ will be close to 2. Thus if $\epsilon < 1$, $|h(x)-2| \ge \epsilon$ whenever $2-\delta < x < 2$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to2} h(x) \ne 2$.

59. (a) For $3-\delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x)-4| \ge 0.8$. Thus for $\epsilon < 0.8$, $|f(x)-4| \ge \epsilon$ whenever $3-\delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to3} f(x) \ne 4$.

(b) For $3 < x < 3+\delta \Rightarrow f(x) < 3 \Rightarrow |f(x)-4.8| \ge 1.8$. Thus for $\epsilon < 1.8$, $|f(x)-4.8| \ge \epsilon$ whenever $3 < x < 3+\delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to3} f(x) \ne 4.8$.

(c) For $3-\delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x)-3| \ge 1.8$. Again, for $\epsilon < 1.8$, $|f(x)-3| \ge \epsilon$ whenever $3-\delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to3} f(x) \ne 3$.

60. (a) No matter how small we choose $\delta > 0$, for x near -1 satisfying $-1-\delta < x < -1+\delta$, the values of $g(x)$ are near $1 \Rightarrow |g(x)-2|$ is near 1. Then, for $\epsilon = \frac{1}{2}$ we have $|g(x)-2| \geq \frac{1}{2}$ for some x satisfying $-1-\delta < x < -1+\delta$, or $0 < |x+1| < \delta \Rightarrow \lim_{x\to -1} g(x) \neq 2$.

(b) Yes, $\lim_{x\to -1} g(x) = 1$ because from the graph we can find a $\delta > 0$ such that $|g(x)-1| < \epsilon$ if $0 < |x-(-1)| < \delta$.

61–66. Example CAS commands (values of del may vary for a specified eps):

Maple:

```
f := x -> (x^4-81)/(x-3); x0 := 3;
plot( f(x), x=x0-1..x0+1, color=black,                    # (a)
title="Section 2.3, #61(a)" );
L := limit( f(x), x=x0 );                                 # (b)
epsilon := 0.2;                                           # (c)
plot( [f(x), L-epsilon,L+epsilon], x=x0-0.01..x0+0.01,
color=black, linestyle=[1,3,3], title="Section 2.3, #61(c)" );
q := fsolve( abs( f(x)-L ) = epsilon, x=x0-1..x0+1 );     # (d)
delta := abs(x0-q);
plot( [f(x), L-epsilon, L+epsilon], x=x0-delta..x0+delta, color=black, title="Section 2.3, #61(d)" );
for eps in [0.1, 0.005, 0.001 ] do                        # (e)
q := fsolve( abs( f(x)-L ) = eps, x=x0-1..x0+1 );
delta := abs(x0-q);
head := sprintf("Section 2.3, #61(e)\n epsilon = %5f, delta = %5f \n", eps, delta );
print(plot( [f(x), L-eps, L+eps], x=x0-delta..x0+delta,
color=black, linestyle=[1,3,3], title=head ));
end do:
```

Mathematica (assigned function and values for x0, eps and del may vary):

```
Clear[f, x]
y1: = L − eps; y2: = L + eps; x0 = 1;
f[x_]: = (3x^2 − (7x + 1)Sqrt[x] + 5)/(x − 1)
Plot[f[x], {x, x0 − 0.2, x0 + 0.2}]
L: = Limit[f[x], x → x0]
eps = 0.1; del = 0.2;
Plot[{f[x], y1, y2}, {x, x0 − del, x0 + del}, PlotRange → {L − 2eps, L + 2eps}]
```

2.4 ONE-SIDED LIMITS

1. (a) True (b) True (c) False (d) True
(e) True (f) True (g) False (h) False
(i) False (j) False (k) True (l) False

2. (a) True (b) False (c) False (d) True
(e) True (f) True (g) True (h) True
(i) True (j) False (k) True

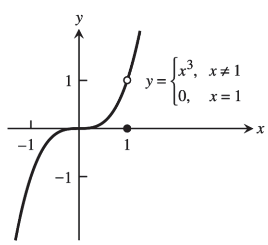
3. (a) $\lim\limits_{x\to 2^+} f(x) = \frac{2}{2}+1 = 2,\ \lim\limits_{x\to 2^-} f(x) = 3-2 = 1$
   (b) No, $\lim\limits_{x\to 2} f(x)$ does not exist because $\lim\limits_{x\to 2^+} f(x) \neq \lim\limits_{x\to 2^-} f(x)$
   (c) $\lim\limits_{x\to 4^-} f(x) = \frac{4}{2}+1 = 3,\ \lim\limits_{x\to 4^+} f(x) = \frac{4}{2}+1 = 3$
   (d) Yes, $\lim\limits_{x\to 4} f(x) = 3$ because $3 = \lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^+} f(x)$

4. (a) $\lim\limits_{x\to 2^+} f(x) = \frac{2}{2} = 1,\ \lim\limits_{x\to 2^-} f(x) = 3-2 = 1,\ f(2) = 2$
   (b) Yes, $\lim\limits_{x\to 2} f(x) = 1$ because $1 = \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x)$
   (c) $\lim\limits_{x\to -1^-} f(x) = 3-(-1) = 4,\ \lim\limits_{x\to -1^+} f(x) = 3-(-1) = 4$
   (d) Yes, $\lim\limits_{x\to -1} f(x) = 4$ because $4 = \lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^+} f(x)$

5. (a) No, $\lim\limits_{x\to 0^+} f(x)$ does not exist since $\sin\left(\frac{1}{x}\right)$ does not approach any single value as x approaches 0
   (b) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 0 = 0$
   (c) $\lim\limits_{x\to 0} f(x)$ does not exist because $\lim\limits_{x\to 0^+} f(x)$ does not exist

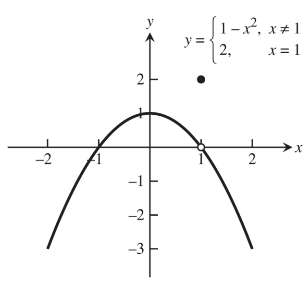
6. (a) Yes, $\lim\limits_{x\to 0^+} g(x) = 0$ by the sandwich theorem since $-\sqrt{x} \le g(x) \le \sqrt{x}$ when $x > 0$
   (b) No, $\lim\limits_{x\to 0^-} g(x)$ does not exist since $\sqrt{x}$ is not defined for $x < 0$
   (c) No, $\lim\limits_{x\to 0} g(x)$ does not exist since $\lim\limits_{x\to 0^-} g(x)$ does not exist

7. (a)



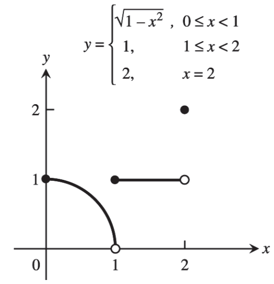
   (b) $\lim\limits_{x\to 1^-} f(x) = 1 = \lim\limits_{x\to 1^+} f(x)$
   (c) Yes, $\lim\limits_{x\to 1} f(x) = 1$ since the right-hand and left-hand limits exist and equal 1

8. (a)

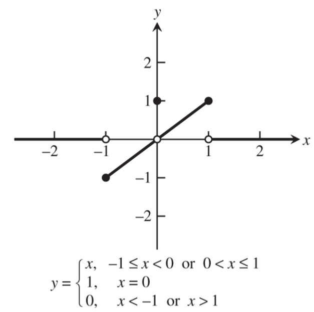


   (b) $\lim\limits_{x\to 1^+} f(x) = 0 = \lim\limits_{x\to 1^-} f(x)$
   (c) Yes, $\lim\limits_{x\to 1} f(x) = 0$ since the right-hand and left-hand limits exist and equal 0

9. (a) domain: $0 \le x \le 2$
range: $0 < y \le 1$ and $y = 2$
(b) $\lim\limits_{x\to c} f(x)$ exists for c belonging to $(0, 1) \cup (1, 2)$
(c) $x = 2$
(d) $x = 0$



10. (a) domain: $-\infty < x < \infty$
range: $-1 \le y \le 1$
(b) $\lim\limits_{x\to c} f(x)$ exists for c belonging to
$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
(c) none
(d) none



11. $\lim\limits_{x\to -0.5^-} \sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{-0.5+2}{-0.5+1}} = \sqrt{\frac{3/2}{1/2}} = \sqrt{3}$

12. $\lim\limits_{x\to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}} = \sqrt{0} = 0$

13. $\lim\limits_{x\to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right) = (2)\left(\frac{1}{2}\right) = 1$

14. $\lim\limits_{x\to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right) = \left(\frac{1}{1+1}\right)\left(\frac{1+6}{1}\right)\left(\frac{3-1}{7}\right) = \left(\frac{1}{2}\right)\left(\frac{7}{1}\right)\left(\frac{2}{7}\right) = 1$

15. $$\lim_{h\to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h} = \lim_{h\to 0^+} \left(\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}\right)\left(\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}}\right) = \lim_{h\to 0^+} \frac{(h^2+4h+5)-5}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)}$$
$$= \lim_{h\to 0^+} \frac{h(h+4)}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} = \frac{0+4}{\sqrt{5}+\sqrt{5}} = \frac{2}{\sqrt{5}}$$

16. $$\lim_{h\to 0^-} \frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h} = \lim_{h\to 0^-} \left(\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}\right)\left(\frac{\sqrt{6}+\sqrt{5h^2+11h+6}}{\sqrt{6}+\sqrt{5h^2+11h+6}}\right)$$
$$= \lim_{h\to 0^-} \frac{6-(5h^2+11h+6)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \lim_{h\to 0^-} \frac{-h(5h+11)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \frac{-(0+11)}{\sqrt{6}+\sqrt{6}} = -\frac{11}{2\sqrt{6}}$$

17. (a) $$\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^+} (x+3)\frac{(x+2)}{(x+2)} \qquad (|x+2| = (x+2) \text{ for } x > -2)$$
$$= \lim_{x\to -2^+} (x+3) = ((-2)+3) = 1$$

(b) $$\lim_{x\to -2^-} (x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^-} (x+3)\left[\frac{-(x+2)}{(x+2)}\right] \qquad (|x+2| = -(x+2) \text{ for } x < -2)$$
$$= \lim_{x\to -2^-} (x+3)(-1) = -(-2+3) = -1$$

18. (a) $\lim\limits_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim\limits_{x\to -1^+} \frac{\sqrt{2x}(x-1)}{(x-1)}$ $\quad (|x-1| = x-1 \text{ for } x>1)$

$= \lim\limits_{x\to -1^+} \sqrt{2x} = \sqrt{2}$

(b) $\lim\limits_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim\limits_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{-(x-1)}$ $\quad (|x-1| = -(x-1) \text{ for } x<1)$

$= \lim\limits_{x\to 1^-} -\sqrt{2x} = -\sqrt{2}$

19. (a) If $0 < x < \frac{\pi}{2}$, then $\sin x > 0$, so that $\lim\limits_{x\to 0^+} \frac{|\sin x|}{\sin x} = \lim\limits_{x\to 0^+} \frac{\sin x}{\sin x} = \lim\limits_{x\to 0^+} 1 = 1$

(b) If $\frac{-\pi}{2} < x < 0$, then $\sin x < 0$, so that $\lim\limits_{x\to 0^-} \frac{|\sin x|}{\sin x} = \lim\limits_{x\to 0^-} \frac{-\sin x}{\sin x} = \lim\limits_{x\to 0^-} -1 = -1$

20. (a) If $0 < x < \frac{\pi}{2}$, then $\cos x < 1$, so that $\lim\limits_{x\to 0^+} \frac{1-\cos x}{|\cos x - 1|} = \lim\limits_{x\to 0^+} \frac{1-\cos x}{-(\cos x - 1)} = \lim\limits_{x\to 0^+} \frac{1-\cos x}{1-\cos x} = \lim\limits_{x\to 0^+} 1 = 1$

(b) If $\frac{-\pi}{2} < x < 0$, then $\cos x < 1$, so that $\lim\limits_{x\to 0^-} \frac{\cos x - 1}{|\cos x - 1|} = \lim\limits_{x\to 0^-} \frac{\cos x - 1}{-(\cos x - 1)} = \lim\limits_{x\to 0^-} -1 = -1$

21. (a) $\lim\limits_{\theta\to 3^+} \frac{\lfloor\theta\rfloor}{\theta} = \frac{3}{3} = 1$ (b) $\lim\limits_{\theta\to 3^-} \frac{\lfloor\theta\rfloor}{\theta} = \frac{2}{3}$

22. (a) $\lim\limits_{t\to 4^+} (t - \lfloor t \rfloor) = 4 - 4 = 0$ (b) $\lim\limits_{t\to 4^-} (t - \lfloor t \rfloor) = 4 - 3 = 1$

23. $\lim\limits_{\theta\to 0} \frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta} = \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ (where $x = \sqrt{2}\theta$)

24. $\lim\limits_{t\to 0} \frac{\sin kt}{t} = \lim\limits_{t\to 0} \frac{k\sin kt}{kt} = \lim\limits_{\theta\to 0} \frac{k\sin\theta}{\theta} = k \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = k\cdot 1 = k$ (where $\theta = kt$)

25. $\lim\limits_{y\to 0} \frac{\sin 3y}{4y} = \frac{1}{4}\lim\limits_{y\to 0} \frac{3\sin 3y}{3y} = \frac{3}{4}\lim\limits_{y\to 0} \frac{\sin 3y}{3y} = \frac{3}{4}\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = \frac{3}{4}$ (where $\theta = 3y$)

26. $\lim\limits_{h\to 0^-} \frac{h}{\sin 3h} = \lim\limits_{h\to 0^-} \left(\frac{1}{3}\cdot\frac{3h}{\sin 3h}\right) = \frac{1}{3}\lim\limits_{h\to 0^-} \frac{1}{\left(\frac{\sin 3h}{3h}\right)} = \frac{1}{3}\left(\frac{1}{\lim\limits_{\theta\to 0^-} \frac{\sin\theta}{\theta}}\right) = \frac{1}{3}\cdot 1 = \frac{1}{3}$ (where $\theta = 3h$)

27. $\lim\limits_{x\to 0} \frac{\tan 2x}{x} = \lim\limits_{x\to 0} \frac{\left(\frac{\sin 2x}{\cos 2x}\right)}{x} = \lim\limits_{x\to 0} \frac{\sin 2x}{x\cos 2x} = \left(\lim\limits_{x\to 0} \frac{1}{\cos 2x}\right)\left(\lim\limits_{x\to 0} \frac{2\sin 2x}{2x}\right) = 1\cdot 2 = 2$

28. $\lim\limits_{t\to 0} \frac{2t}{\tan t} = 2\lim\limits_{t\to 0} \frac{t}{\left(\frac{\sin t}{\cos t}\right)} = 2\lim\limits_{t\to 0} \frac{t\cos t}{\sin t} = 2\left(\lim\limits_{t\to 0} \cos t\right)\left(\frac{1}{\lim\limits_{t\to 0} \frac{\sin t}{t}}\right) = 2\cdot 1\cdot 1 = 2$

29. $\lim\limits_{x\to 0} \frac{x\csc 2x}{\cos 5x} = \lim\limits_{x\to 0} \left(\frac{x}{\sin 2x}\cdot\frac{1}{\cos 5x}\right) = \left(\frac{1}{2}\lim\limits_{x\to 0} \frac{2x}{\sin 2x}\right)\left(\lim\limits_{x\to 0} \frac{1}{\cos 5x}\right) = \left(\frac{1}{2}\cdot 1\right)(1) = \frac{1}{2}$

30. $\lim\limits_{x\to 0} 6x^2(\cot x)(\csc 2x) = \lim\limits_{x\to 0} \frac{6x^2\cos x}{\sin x\sin 2x} = \lim\limits_{x\to 0} \left(3\cos x\cdot\frac{x}{\sin x}\cdot\frac{2x}{\sin 2x}\right) = 3\cdot 1\cdot 1 = 3$

31. $\lim\limits_{x\to 0} \frac{x+x\cos x}{\sin x\cos x} = \lim\limits_{x\to 0}\left(\frac{x}{\sin x\cos x} + \frac{x\cos x}{\sin x\cos x}\right) = \lim\limits_{x\to 0}\left(\frac{x}{\sin x}\cdot\frac{1}{\cos x}\right) + \lim\limits_{x\to 0}\frac{x}{\sin x}$

$= \lim\limits_{x\to 0}\left(\frac{1}{\frac{\sin x}{x}}\right)\cdot\lim\limits_{x\to 0}\left(\frac{1}{\cos x}\right) + \lim\limits_{x\to 0}\left(\frac{1}{\frac{\sin x}{x}}\right) = (1)(1)+1 = 2$

32. $\lim\limits_{x\to 0}\frac{x^2-x+\sin x}{2x} = \lim\limits_{x\to 0}\left(\frac{x}{2}-\frac{1}{2}+\frac{1}{2}\left(\frac{\sin x}{x}\right)\right) = 0-\frac{1}{2}+\frac{1}{2}(1) = 0$

33. $\lim\limits_{\theta\to 0}\frac{1-\cos\theta}{\sin 2\theta} = \lim\limits_{\theta\to 0}\frac{(1-\cos\theta)(1+\cos\theta)}{(2\sin\theta\cos\theta)(1+\cos\theta)} = \lim\limits_{\theta\to 0}\frac{1-\cos^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)} = \lim\limits_{\theta\to 0}\frac{\sin^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)}$

$= \lim\limits_{\theta\to 0}\frac{\sin\theta}{(2\cos\theta)(1+\cos\theta)} = \frac{0}{(2)(2)} = 0$

34. $\lim\limits_{x\to 0}\frac{x-x\cos x}{\sin^2 3x} = \lim\limits_{x\to 0}\frac{x(1-\cos x)}{\sin^2 3x} = \lim\limits_{x\to 0}\frac{\frac{x(1-\cos x)}{9x^2}}{\frac{\sin^2 3x}{9x^2}} = \lim\limits_{x\to 0}\frac{\frac{1-\cos x}{9x}}{\left(\frac{\sin 3x}{3x}\right)^2} = \frac{\frac{1}{9}\lim\limits_{x\to 0}\left(\frac{1-\cos x}{x}\right)}{\lim\limits_{x\to 0}\left(\frac{\sin 3x}{3x}\right)^2} = \frac{\frac{1}{9}(0)}{1^2} = 0$

34. $\lim\limits_{t\to 0}\frac{\sin(1-\cos t)}{1-\cos t} = \lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ since $\theta = 1-\cos t \to 0$ as $t\to 0$

36. $\lim\limits_{h\to 0}\frac{\sin(\sin h)}{\sin h} = \lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ since $\theta = \sin h \to 0$ as $h\to 0$

37. $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\sin 2\theta} = \lim\limits_{\theta\to 0}\left(\frac{\sin\theta}{\sin 2\theta}\cdot\frac{2\theta}{2\theta}\right) = \frac{1}{2}\lim\limits_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{2\theta}{\sin 2\theta}\right) = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$

38. $\lim\limits_{x\to 0}\frac{\sin 5x}{\sin 4x} = \lim\limits_{x\to 0}\left(\frac{\sin 5x}{\sin 4x}\cdot\frac{4x}{5x}\cdot\frac{5}{4}\right) = \frac{5}{4}\lim\limits_{x\to 0}\left(\frac{\sin 5x}{5x}\cdot\frac{4x}{\sin 4x}\right) = \frac{5}{4}\cdot 1\cdot 1 = \frac{5}{4}$

39. $\lim\limits_{\theta\to 0}\theta\cos\theta = 0\cdot 1 = 0$

40. $\lim\limits_{\theta\to 0}\sin\theta\cot 2\theta = \lim\limits_{\theta\to 0}\sin\theta\frac{\cos 2\theta}{\sin 2\theta} = \lim\limits_{\theta\to 0}\sin\theta\frac{\cos 2\theta}{2\sin\theta\cos\theta} = \lim\limits_{\theta\to 0}\frac{\cos 2\theta}{2\cos\theta} = \frac{1}{2}$

41. $\lim\limits_{x\to 0}\frac{\tan 3x}{\sin 8x} = \lim\limits_{x\to 0}\left(\frac{\sin 3x}{\cos 3x}\cdot\frac{1}{\sin 8x}\right) = \lim\limits_{x\to 0}\left(\frac{\sin 3x}{\cos 3x}\cdot\frac{1}{\sin 8x}\cdot\frac{8x}{3x}\cdot\frac{3}{8}\right)$

$= \frac{3}{8}\lim\limits_{x\to 0}\left(\frac{1}{\cos 3x}\right)\left(\frac{\sin 3x}{3x}\right)\left(\frac{8x}{\sin 8x}\right) = \frac{3}{8}\cdot 1\cdot 1\cdot 1 = \frac{3}{8}$

42. $\lim\limits_{y\to 0}\frac{\sin 3y\cot 5y}{y\cot 4y} = \lim\limits_{y\to 0}\frac{\sin 3y\sin 4y\cos 5y}{y\cos 4y\sin 5y} = \lim\limits_{y\to 0}\left(\frac{\sin 3y}{y}\right)\left(\frac{\sin 4y}{\cos 4y}\right)\left(\frac{\cos 5y}{\sin 5y}\right)\left(\frac{3\cdot 4\cdot 5y}{3\cdot 4\cdot 5y}\right)$

$= \lim\limits_{y\to 0}\left(\frac{\sin 3y}{3y}\right)\left(\frac{\sin 4y}{4y}\right)\left(\frac{5y}{\sin 5y}\right)\left(\frac{\cos 5y}{\cos 4y}\right)\left(\frac{3\cdot 4}{5}\right) = 1\cdot 1\cdot 1\cdot 1\cdot\frac{12}{5} = \frac{12}{5}$

43. $\lim\limits_{\theta\to 0}\frac{\tan\theta}{\theta^2\cot 3\theta} = \lim\limits_{\theta\to 0}\frac{\frac{\sin\theta}{\cos\theta}}{\theta^2\frac{\cos 3\theta}{\sin 3\theta}} = \lim\limits_{\theta\to 0}\frac{\sin\theta\sin 3\theta}{\theta^2\cos\theta\cos 3\theta} = \lim\limits_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin 3\theta}{3\theta}\right)\left(\frac{3}{\cos\theta\cos 3\theta}\right) = (1)(1)\left(\frac{3}{1\cdot 1}\right) = 3$

44. $\lim\limits_{\theta\to 0}\frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta} = \lim\limits_{\theta\to 0}\frac{\theta\frac{\cos 4\theta}{\sin 4\theta}}{\sin^2\theta\frac{\cos^2 2\theta}{\sin^2 2\theta}} = \lim\limits_{\theta\to 0}\frac{\theta\cos 4\theta\sin^2 2\theta}{\sin^2\theta\cos^2 2\theta\sin 4\theta} = \lim\limits_{\theta\to 0}\frac{\theta\cos 4\theta(2\sin\theta\cos\theta)^2}{\sin^2\theta\cos^2 2\theta\sin 4\theta} = \lim\limits_{\theta\to 0}\frac{\theta\cos 4\theta(4\sin^2\theta\cos^2\theta)}{\sin^2\theta\cos^2 2\theta\sin 4\theta}$

$= \lim\limits_{\theta\to 0}\frac{4\theta\cos 4\theta\cos^2\theta}{\cos^2 2\theta\sin 4\theta} = \lim\limits_{\theta\to 0}\left(\frac{4\theta}{\sin 4\theta}\right)\left(\frac{\cos 4\theta\cos^2\theta}{\cos^2 2\theta}\right) = \lim\limits_{\theta\to 0}\left(\frac{1}{\frac{\sin 4\theta}{4\theta}}\right)\left(\frac{\cos 4\theta\cos^2\theta}{\cos^2 2\theta}\right) = \left(\frac{1}{1}\right)\left(\frac{1\cdot 1^2}{1^2}\right) = 1$

45. $\lim\limits_{x\to 0}\frac{1-\cos 3x}{2x}=\lim\limits_{x\to 0}\frac{1-\cos 3x}{2x}\cdot\frac{1+\cos 3x}{1+\cos 3x}=\lim\limits_{x\to 0}\frac{1-\cos^2 3x}{2x(1+\cos 3x)}=\lim\limits_{x\to 0}\frac{\sin^2 3x}{2x(1+\cos 3x)}=\lim\limits_{x\to 0}\frac{3}{2}\cdot\frac{\sin 3x}{3x}\cdot\frac{\sin 3x}{1+\cos 3x}$

$=\lim\limits_{\theta\to 0}\frac{3}{2}\cdot\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}=\frac{3}{2}(1)\left(\frac{0}{1+1}\right)=0$ (where $\theta=3x$)

46. $\lim\limits_{x\to 0}\frac{\cos^2 x-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{\cos x(\cos x-1)}{x^2}=\lim\limits_{x\to 0}\frac{\cos x(\cos x-1)}{x^2}\cdot\frac{\cos x+1}{\cos x+1}=\lim\limits_{x\to 0}\frac{\cos x(\cos^2 x-1)}{x^2(\cos x+1)}=\lim\limits_{x\to 0}\frac{\cos x\cdot(-\sin^2 x)}{x^2(\cos x+1)}$

$=\lim\limits_{x\to 0}\left\{-\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{\cos x}{\cos x+1}\right\}=-(1)(1)\cdot\frac{1}{1+1}=\frac{-1}{2}$

47. Yes. If $\lim\limits_{x\to a^+} f(x)=L=\lim\limits_{x\to a^-} f(x)$, then $\lim\limits_{x\to a} f(x)=L$. If $\lim\limits_{x\to a^+} f(x)\neq\lim\limits_{x\to a^-} f(x)$, then $\lim\limits_{x\to a} f(x)$ does not exist.

48. Since $\lim\limits_{x\to c} f(x)=L$ if and only if $\lim\limits_{x\to c^+} f(x)=L$ and $\lim\limits_{x\to c^-} f(x)=L$, then $\lim\limits_{x\to c} f(x)$ can be found by calculating $\lim\limits_{x\to c^+} f(x)$.

49. If f is an odd function of x, then $f(-x)=-f(x)$. Given $\lim\limits_{x\to 0^+} f(x)=3$, then $\lim\limits_{x\to 0^-} f(x)=-3$.

50. If f is an even function of x, then $f(-x)=f(x)$. Given $\lim\limits_{x\to 2^-} f(x)=7$ then $\lim\limits_{x\to -2^+} f(x)=7$. However, nothing can be said about $\lim\limits_{x\to -2^-} f(x)$ because we don't know $\lim\limits_{x\to 2^+} f(x)$.

51. $I=(5,5+\delta)\Rightarrow 5<x<5+\delta$. Also, $\sqrt{x-5}<\epsilon\Rightarrow x-5<\epsilon^2\Rightarrow x<5+\epsilon^2$. Choose $\delta=\epsilon^2\Rightarrow\lim\limits_{x\to 5^+}\sqrt{x-5}=0$.

52. $I=(4-\delta,4)\Rightarrow 4-\delta<x<4$. Also, $\sqrt{4-x}<\epsilon\Rightarrow 4-x<\epsilon^2\Rightarrow x>4-\epsilon^2$. Choose $\delta=\epsilon^2\Rightarrow\lim\limits_{x\to 4^-}\sqrt{4-x}=0$.

53. As $x\to 0^-$ the number x is always negative. Thus, $\left|\frac{x}{|x|}-(-1)\right|<\epsilon\Rightarrow\left|\frac{x}{-x}+1\right|<\epsilon\Rightarrow 0<\epsilon$ which is always true independent of the value of x. Hence we can choose any $\delta>0$ with $-\delta<x<0\Rightarrow\lim\limits_{x\to 0^-}\frac{x}{|x|}=-1$.

54. Since $x\to 2^+$ we have $x>2$ and $|x-2|=x-2$. Then, $\left|\frac{x-2}{|x-2|}-1\right|=\left|\frac{x-2}{x-2}-1\right|<\epsilon\Rightarrow 0<\epsilon$ which is always true so long as $x>2$. Hence we can choose any $\delta>0$, and thus $2<x<2+\delta\Rightarrow\left|\frac{x-2}{|x-2|}-1\right|<\epsilon$. Thus, $\lim\limits_{x\to -2^+}\frac{x-2}{|x-2|}=1$.

55. (a) $\lim\limits_{x\to 400^+}\lfloor x\rfloor=400$. Just observe that if $400<x<401$, then $\lfloor x\rfloor=400$. Thus if we choose $\delta=1$, we have for any number $\epsilon>0$ that $400<x<400+\delta\Rightarrow|\lfloor x\rfloor-400|=|400-400|=0<\epsilon$.

(b) $\lim\limits_{x\to 400^-}\lfloor x\rfloor=399$. Just observe that if $399<x<400$ then $\lfloor x\rfloor=399$. Thus if we choose $\delta=1$, we have for any number $\epsilon>0$ that $400-\delta<x<400\Rightarrow|\lfloor x\rfloor-399|=|399-399|=0<\epsilon$.

(c) Since $\lim\limits_{x\to 400^+}\lfloor x\rfloor\neq\lim\limits_{x\to 400^-}\lfloor x\rfloor$ we conclude that $\lim\limits_{x\to 400}\lfloor x\rfloor$ does not exist.

56. (a) $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\sqrt{x}=\sqrt{0}=0$; $\left|\sqrt{x}-0\right|<\epsilon\Rightarrow -\epsilon<\sqrt{x}<\epsilon\Rightarrow 0<x<\epsilon^2$ for x positive. Choose $\delta=\epsilon^2$ $\Rightarrow\lim\limits_{x\to 0^+} f(x)=0$.

(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 \sin\left(\frac{1}{x}\right) = 0$ by the sandwich theorem since $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$ for all $x \ne 0$.
Since $\left|x^2 - 0\right| = \left|-x^2 - 0\right| = x^2 < \epsilon$ whenever $|x| < \sqrt{\epsilon}$, we choose $\delta = \sqrt{\epsilon}$ and obtain $\left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| < \epsilon$ if $-\delta < x < 0$.

(c) The function f has limit 0 at $x_0 = 0$ since both the right-hand and left-hand limits exist and equal 0.

2.5 CONTINUITY

1. No, discontinuous at $x = 2$, not defined at $x = 2$

2. No, discontinuous at $x = 3$, $1 = \lim_{x\to 3^-} g(x) \ne g(3) = 1.5$

3. Continuous on $[-1, 3]$

4. No, discontinuous at $x = 1$, $1.5 = \lim_{x\to 1^-} k(x) \ne \lim_{x\to 1^+} k(x) = 0$

5. (a) Yes (b) Yes, $\lim_{x\to -1^+} f(x) = 0$
(c) Yes (d) Yes

6. (a) Yes, $f(1) = 1$ (b) Yes, $\lim_{x\to 1} f(x) = 2$
(c) No (d) No

7. (a) No (b) No

8. $[-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3)$

9. $f(2) = 0$, since $\lim_{x\to 2^-} f(x) = -2(2) + 4 = 0 = \lim_{x\to 2^+} f(x)$

10. $f(1)$ should be changed to $2 = \lim_{x\to 1} f(x)$

11. Nonremovable discontinuity at $x = 1$ because $\lim_{x\to 1} f(x)$ fails to exist ($\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = 0$). Removable discontinuity at $x = 0$ by assigning the number $\lim_{x\to 0} f(x) = 0$ to be the value of $f(0)$ rather than $f(0) = 1$.

12. Nonremovable discontinuity at $x = 1$ because $\lim_{x\to 1} f(x)$ fails to exist ($\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 1$). Removable discontinuity at $x = 2$ by assigning the number $\lim_{x\to 2} f(x) = 1$ to be the value of $f(2)$ rather than $f(2) = 2$.

13. Discontinuous only when $x - 2 = 0 \Rightarrow x = 2$

14. Discontinuous only when $(x+2)^2 = 0 \Rightarrow x = -2$

15. Discontinuous only when $x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 3$ or $x = 1$

16. Discontinuous only when $x^2 - 3x - 10 = 0 \Rightarrow (x-5)(x+2) = 0 \Rightarrow x = 5$ or $x = -2$

17. Continuous everywhere. ($|x-1| + \sin x$ defined for all x; limits exist and are equal to function values.)

18. Continuous everywhere. ($|x|+1 \neq 0$ for all x; limits exist and are equal to function values.)

19. Discontinuous only at $x = 0$

20. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n-1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

21. Discontinuous when $2x$ is an integer multiple of π, i.e., $2x = n\pi$, n an integer $\Rightarrow x = \frac{n\pi}{2}$, n an integer, but continuous at all other x.

22. Discontinuous when $\frac{\pi x}{2}$ is an odd integer multiple of $\frac{\pi}{2}$, i.e., $\frac{\pi x}{2} = (2n-1)\frac{\pi}{2}$, n an integer $\Rightarrow x = 2n-1$, n an integer (i.e., x is an odd integer). Continuous everywhere else.

23. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n-1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

24. Continuous everywhere since $x^4+1 \geq 1$ and $-1 \leq \sin x \leq 1 \Rightarrow 0 \leq \sin^2 x \leq 1 \Rightarrow 1+\sin^2 x \geq 1$; limits exist and are equal to the function values.

25. Discontinuous when $2x+3 < 0$ or $x < -\frac{3}{2} \Rightarrow$ continuous on the interval $\left[-\frac{3}{2}, \infty\right)$.

26. Discontinuous when $3x-1 < 0$ or $x < \frac{1}{3} \Rightarrow$ continuous on the interval $\left[\frac{1}{3}, \infty\right)$.

27. Continuous everywhere: $(2x-1)^{1/3}$ is defined for all x; limits exist and are equal to function values.

28. Continuous everywhere: $(2-x)^{1/5}$ is defined for all x; limits exist and are equal to function values.

29. Continuous everywhere since $\lim\limits_{x\to 3} \frac{x^2-x-6}{x-3} = \lim\limits_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim\limits_{x\to 3} (x+2) = 5 = g(3)$

30. Discontinuous at $x = -2$ since $\lim\limits_{x\to -2} f(x)$ does not exist while $f(-2) = 4$.

31. $\lim\limits_{x\to\pi} \sin(x - \sin x) = \sin(\pi - \sin\pi) = \sin(\pi - 0) = \sin\pi = 0$, and function continuous at $x = \pi$.

32. $\lim\limits_{t\to 0} \sin(\frac{\pi}{2}\cos(\tan t)) = \sin(\frac{\pi}{2}\cos(\tan(0))) = \sin\left(\frac{\pi}{2}\cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$, and function continuous at $t = 0$.

33. $\lim\limits_{y\to 1} \sec(y\sec^2 y - \tan^2 y - 1) = \lim\limits_{y\to 1} \sec(y\sec^2 y - \sec^2 y) = \lim\limits_{y\to 1} \sec((y-1)\sec^2 y) = \sec((1-1)\sec^2 1) =$ $\sec 0 = 1$, and function continuous at $y = 1$.

34. $\lim\limits_{x\to 0} \tan\left[\frac{\pi}{4}\cos(\sin x^{1/3})\right] = \tan\left[\frac{\pi}{4}\cos(\sin(0))\right] = \tan\left(\frac{\pi}{4}\cos(0)\right) = \tan\left(\frac{\pi}{4}\right) = 1$, and function continuous at $x = 0$.

35. $\lim\limits_{t\to 0} \cos\left[\frac{\pi}{\sqrt{19-3\sec 2t}}\right] = \cos\left[\frac{\pi}{\sqrt{19-3\sec 0}}\right] = \cos\frac{\pi}{\sqrt{16}} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, and function continuous at $t = 0$.

36. $\lim\limits_{x\to\frac{\pi}{6}} \sqrt{\csc^2 x + 5\sqrt{3}\tan x} = \sqrt{\csc^2\left(\frac{\pi}{6}\right) + 5\sqrt{3}\tan\left(\frac{\pi}{6}\right)} = \sqrt{4 + 5\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)} = \sqrt{9} = 3$, and function continuous at $x = \frac{\pi}{6}$.

37. $g(x) = \frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{(x-3)} = x+3,\ x \neq 3 \Rightarrow g(3) = \lim\limits_{x\to 3}(x+3) = 6$

38. $h(t) = \frac{t^2+3t-10}{t-2} = \frac{(t+5)(t-2)}{t-2} = t+5,\ t \neq 2 \Rightarrow h(2) = \lim\limits_{t\to 2}(t+5) = 7$

39. $f(s) = \frac{s^3-1}{s^2-1} = \frac{(s^2+s+1)(s-1)}{(s+1)(s-1)} = \frac{s^2+s+1}{s+1},\ s \neq 1 \Rightarrow f(1) = \lim\limits_{s\to 1}\left(\frac{s^2+s+1}{s+1}\right) = \frac{3}{2}$

40. $g(x) = \frac{x^2-16}{x^2-3x-4} = \frac{(x+4)(x-4)}{(x-4)(x+1)} = \frac{x+4}{x+1},\ x \neq 4 \Rightarrow g(4) = \lim\limits_{x\to 4}\left(\frac{x+4}{x+1}\right) = \frac{8}{5}$

41. As defined, $\lim\limits_{x\to 3^-} f(x) = (3)^2 - 1 = 8$ and $\lim\limits_{x\to 3^+}(2a)(3) = 6a$. For $f(x)$ to be continuous we must have $6a = 8 \Rightarrow a = \frac{4}{3}$.

42. As defined, $\lim\limits_{x\to -2^-} g(x) = -2$ and $\lim\limits_{x\to -2^+} g(x) = b(-2)^2 = 4b$. For $g(x)$ to be continuous we must have $4b = -2 \Rightarrow b = -\frac{1}{2}$.

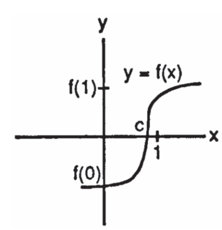
43. As defined, $\lim\limits_{x\to 2^-} f(x) = 12$ and $\lim\limits_{x\to 2^+} f(x) = a^2(2) - 2a = 2a^2 - 2a$. For $f(x)$ to be continuous we must have $12 = 2a^2 - 2a \Rightarrow a = 3$ or $a = -2$.

44. As defined, $\lim\limits_{x\to 0^-} g(x) = \frac{0-b}{b+1} = \frac{-b}{b+1}$ and $\lim\limits_{x\to 0^+} g(x) = (0)^2 + b = b$. For $g(x)$ to be continuous we must have $\frac{-b}{b+1} = b \Rightarrow b = 0$ or $b = -2$.

45. As defined, $\lim\limits_{x\to -1^-} f(x) = -2$ and $\lim\limits_{x\to -1^+} f(x) = a(-1) + b = -a + b$, and $\lim\limits_{x\to 1^-} f(x) = a(1) + b = a + b$ and $\lim\limits_{x\to 1^+} f(x) = 3$. For $f(x)$ to be continuous we must have $-2 = -a + b$ and $a + b = 3 \Rightarrow a = \frac{5}{2}$ and $b = \frac{1}{2}$.

46. As defined, $\lim\limits_{x\to 0^-} g(x) = a(0) + 2b = 2b$ and $\lim\limits_{x\to 0^+} g(x) = (0)^2 + 3a - b = 3a - b$, and $\lim\limits_{x\to 2^-} g(x) = (2)^2 + 3a - b = 4 + 3a - b$ and $\lim\limits_{x\to 0^+} g(x) = 3(2) - 5 = 1$. For $g(x)$ to be continuous we must have $2b = 3a - b$ and $4 + 3a - b = 1 \Rightarrow a = -\frac{3}{2}$ and $b = -\frac{3}{2}$.

47. $f(x)$ is continuous on $[0, 1]$ and $f(0) < 0$, $f(1) > 0 \Rightarrow$ by the Intermediate Value Theorem $f(x)$ takes on every value between $f(0)$ and $f(1) \Rightarrow$ the equation $f(x) = 0$ has at least one solution between $x = 0$ and $x = 1$.



48. $\cos x = x \Rightarrow (\cos x) - x = 0$. If $x = -\frac{\pi}{2}$, $\cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0$. If $x = \frac{\pi}{2}$, $\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} < 0$. Thus $\cos x - x = 0$ for some x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ according to the Intermediate Value Theorem, since the function $\cos x - x$ is continuous.

49. Let $f(x) = x^3 - 15x + 1$, which is continuous on $[-4, 4]$. Then $f(-4) = -3$, $f(-1) = 15$, $f(1) = -13$, and $f(4) = 5$. By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-4 < x < -1$, $-1 < x < 1$, and

$1 < x < 4$. That is, $x^3 - 15x + 1 = 0$ has three solutions in $[-4, 4]$. Since a polynomial of degree 3 can have at most 3 solutions, these are the only solutions.

50. Without loss of generality, assume that $a < b$. Then $F(x) = (x-a)^2(x-b)^2 + x$ is continuous for all values of x, so it is continuous on the interval $[a, b]$. Moreover $F(a) = a$ and $F(b) = b$. By the Intermediate Value Theorem, since $a < \frac{a+b}{2} < b$, there is a number c between a and b such that $F(x) = \frac{a+b}{2}$.

51. Answers may vary. Note that f is continuous for every value of x.
    (a) $f(0) = 10$, $f(1) = 1^3 - 8(1) + 10 = 3$. Since $3 < \pi < 10$, by the Intermediate Value Theorem, there exists a c so that $0 < c < 1$ and $f(c) = \pi$.
    (b) $f(0) = 10$, $f(-4) = (-4)^3 - 8(-4) + 10 = -22$. Since $-22 < -\sqrt{3} < 10$, by the Intermediate Value Theorem, there exists a c so that $-4 < c < 0$ and $f(c) = -\sqrt{3}$.
    (c) $f(0) = 10$, $f(1000) = (1000)^3 - 8(1000) + 10 = 999{,}992{,}010$. Since $10 < 5{,}000{,}000 < 999{,}992{,}010$, by the Intermediate Value Theorem, there exists a c so that $0 < c < 1000$ and $f(c) = 5{,}000{,}000$.

52. All five statements ask for the same information because of the intermediate value property of continuous functions.
    (a) A root of $f(x) = x^3 - 3x - 1$ is a point c where $f(c) = 0$.
    (b) The point where $y = x^3$ crosses $y = 3x + 1$ have the same y-coordinate, or $y = x^3 = 3x + 1 \Rightarrow f(x) = x^3 - 3x - 1 = 0$.
    (c) $x^3 - 3x = 1 \Rightarrow x^3 - 3x - 1 = 0$. The solutions to the equation are the roots of $f(x) = x^3 - 3x - 1$.
    (d) The points where $y = x^3 - 3x$ crosses $y = 1$ have common y-coordinates, or $y = x^3 - 3x = 1 \Rightarrow f(x) = x^3 - 3x - 1 = 0$.
    (e) The solutions of $x^3 - 3x - 1 = 0$ are those points where $f(x) = x^3 - 3x - 1$ has value 0.

53. Answers may vary. For example, $f(x) = \frac{\sin(x-2)}{x-2}$ is discontinuous at $x = 2$ because it is not defined there. However, the discontinuity can be removed because f has a limit (namely 1) as $x \to 2$.

54. Answers may vary. For example, $g(x) = \frac{1}{x+1}$ has a discontinuity at $x = -1$ because $\lim_{x \to -1} g(x)$ does not exist. $\left(\lim_{x \to -1^-} g(x) = -\infty \text{ and } \lim_{x \to -1^+} g(x) = +\infty.\right)$

55. (a) Suppose x_0 is rational $\Rightarrow f(x_0) = 1$. Choose $\epsilon = \frac{1}{2}$. For any $\delta > 0$ there is an irrational number x (actually infinitely many) in the interval $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 0$. Then $0 < |x - x_0| < \delta$ but $|f(x) - f(x_0)| = 1 > \frac{1}{2} = \epsilon$, so $\lim_{x \to x_0} f(x)$ fails to exist $\Rightarrow f$ is discontinuous at x_0 rational.
    On the other hand, x_0 irrational $\Rightarrow f(x_0) = 0$ and there is a rational number x in $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 1$. Again $\lim_{x \to x_0} f(x)$ fails to exist $\Rightarrow f$ is discontinuous at x_0 irrational. That is, f is discontinuous at every point.
    (b) f is neither right-continuous nor left-continuous at any point x_0 because in every interval $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$ there exist both rational and irrational real numbers. Thus neither limits $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist by the same arguments used in part (a).

56. Yes. Both $f(x) = x$ and $g(x) = x - \frac{1}{2}$ are continuous on $[0, 1]$. However $\frac{f(x)}{g(x)}$ is undefined at $x = \frac{1}{2}$ since $g\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{f(x)}{g(x)}$ is discontinuous at $x = \frac{1}{2}$.

57. No. For instance, if $f(x) = 0$, $g(x) = \lceil x \rceil$, then $h(x) = 0\left(\lceil x \rceil\right) = 0$ is continuous at $x = 0$ and $g(x)$ is not.

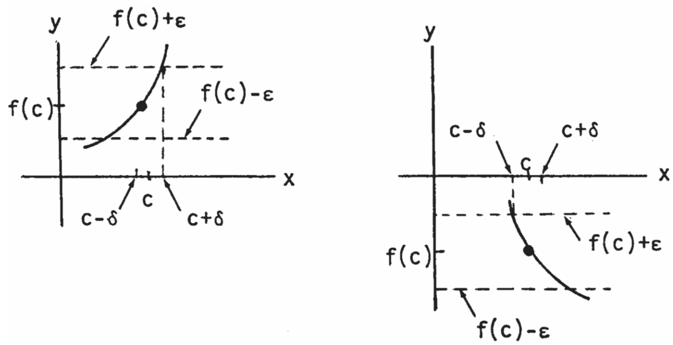
58. Let $f(x)=\frac{1}{x-1}$ and $g(x)=x+1$. Both functions are continuous at $x=0$. The composition $f\circ g=f(g(x))=\frac{1}{(x+1)-1}=\frac{1}{x}$ is discontinuous at $x=0$, since it is not defined there. Theorem 10 requires that $f(x)$ be continuous at $g(0)$, which is not the case here since $g(0)=1$ and f is undefined at 1.

59. Yes, because of the Intermediate Value Theorem. If $f(a)$ and $f(b)$ did have different signs then f would have to equal zero at some point between a and b since f is continuous on $[a, b]$.

60. Let $f(x)$ be the new position of point x and let $d(x)=f(x)-x$. The displacement function d is negative if x is the left-hand point of the rubber band and positive if x is the right-hand point of the rubber band. By the Intermediate Value Theorem, $d(x)=0$ for some point in between. That is, $f(x)=x$ for some point x, which is then in its original position.

61. If $f(0)=0$ or $f(1)=1$, we are done (i.e., $c=0$ or $c=1$ in those cases). Then let $f(0)=a>0$ and $f(1)=b<1$ because $0\le f(x)\le 1$. Define $g(x)=f(x)-x \Rightarrow g$ is continuous on $[0, 1]$. Moreover, $g(0)=f(0)-0=a>0$ and $g(1)=f(1)-1=b-1<0 \Rightarrow$ by the Intermediate Value Theorem there is a number c in $(0, 1)$ such that $g(c)=0 \Rightarrow f(c)-c=0$ or $f(c)=c$.

62. Let $\epsilon=\frac{|f(c)|}{2}>0$. Since f is continuous at $x=c$ there is a $\delta>0$ such that $|x-c|<\delta \Rightarrow |f(x)-f(c)|<\epsilon$ $\Rightarrow f(c)-\epsilon<f(x)<f(c)+\epsilon$.
If $f(c)>0$, then $\epsilon=\frac{1}{2}f(c)\Rightarrow \frac{1}{2}f(c)<f(x)<\frac{3}{2}f(c)\Rightarrow f(x)>0$ on the interval $(c-\delta, c+\delta)$.
If $f(c)<0$, then $\epsilon=-\frac{1}{2}f(c)\Rightarrow \frac{3}{2}f(c)<f(x)<\frac{1}{2}f(c)\Rightarrow f(x)<0$ on the interval $(c-\delta, c+\delta)$.



63. By Exercise 52 in Section 2.3, we have $\lim_{x\to c} f(x)=L \Leftrightarrow \lim_{h\to 0} f(c+h)=L$.
Thus, $f(x)$ is continuous at $x=c \Leftrightarrow \lim_{x\to c} f(x)=f(c) \Leftrightarrow \lim_{h\to 0} f(c+h)=f(c)$.

64. By Exercise 63, it suffices to show that $\lim_{h\to 0}\sin(c+h)=\sin c$ and $\lim_{h\to 0}\cos(c+h)=\cos c$.

Now $\lim_{h\to 0}\sin(c+h)=\lim_{h\to 0}\left[(\sin c)(\cos h)+(\cos c)(\sin h)\right]=(\sin c)\left(\lim_{h\to 0}\cos h\right)+(\cos c)\left(\lim_{h\to 0}\sin h\right)$.
By Example 11 Section 2.2, $\lim_{h\to 0}\cos h=1$ and $\lim_{h\to 0}\sin h=0$. So $\lim_{h\to 0}\sin(c+h)=\sin c$ and thus $f(x)=\sin x$ is continuous at $x=c$. Similarly,

$\lim_{h\to 0}\cos(c+h)=\lim_{h\to 0}\left[(\cos c)(\cos h)-(\sin c)(\sin h)\right]=(\cos c)\left(\lim_{h\to 0}\cos h\right)-(\sin c)\left(\lim_{h\to 0}\sin h\right)=\cos c$. Thus, $g(x)=\cos x$ is continuous at $x=c$.

65. $x\approx 1.8794, -1.5321, -0.3473$

66. $x\approx 1.4516, -0.8547, 0.4030$

67. $x \approx 1.7549$

68. $x \approx 3.5156$

69. $x \approx 0.7391$

70. $x \approx -1.8955, 0, 1.8955$

2.6 LIMITS INVOLVING INFINITY; ASYMPTOTES OF GRAPHS

1. (a) $\lim_{x\to 2} f(x) = 0$
   (b) $\lim_{x\to -3^+} f(x) = -2$
   (c) $\lim_{x\to -3^-} f(x) = 2$
   (d) $\lim_{x\to -3} f(x) =$ does not exist
   (e) $\lim_{x\to 0^+} f(x) = -1$
   (f) $\lim_{x\to 0^-} f(x) = +\infty$
   (g) $\lim_{x\to 0} f(x) =$ does not exist
   (h) $\lim_{x\to \infty} f(x) = 1$
   (i) $\lim_{x\to -\infty} f(x) = 0$

2. (a) $\lim_{x\to 4} f(x) = 2$
   (b) $\lim_{x\to 2^+} f(x) = -3$
   (c) $\lim_{x\to 2^-} f(x) = 1$
   (d) $\lim_{x\to 2} f(x) =$ does not exist
   (e) $\lim_{x\to -3^+} f(x) = +\infty$
   (f) $\lim_{x\to -3^-} f(x) = +\infty$
   (g) $\lim_{x\to -3} f(x) = +\infty$
   (h) $\lim_{x\to 0^+} f(x) = +\infty$
   (i) $\lim_{x\to 0^-} f(x) = -\infty$
   (j) $\lim_{x\to 0} f(x) =$ does not exist
   (k) $\lim_{x\to \infty} f(x) = 0$
   (l) $\lim_{x\to -\infty} f(x) = -1$

Note: In these exercises we use the result $\lim_{x\to\pm\infty} \frac{1}{x^{m/n}} = 0$ whenever $\frac{m}{n} > 0$. This result follows immediately from Theorem 8 and the power rule in Theorem 1: $\lim_{x\to\pm\infty}\left(\frac{1}{x^{m/n}}\right) = \lim_{x\to\pm\infty}\left(\frac{1}{x}\right)^{m/n} = \left(\lim_{x\to\pm\infty}\frac{1}{x}\right)^{m/n} = 0^{m/n} = 0.$

3. (a) -3
   (b) -3

4. (a) π
   (b) π

5. (a) $\frac{1}{2}$
   (b) $\frac{1}{2}$

6. (a) $\frac{1}{8}$
   (b) $\frac{1}{8}$

7. (a) $-\frac{5}{3}$
   (b) $-\frac{5}{3}$

8. (a) $\frac{3}{4}$
   (b) $\frac{3}{4}$

9. $-\frac{1}{x} \le \frac{\sin 2x}{x} \le \frac{1}{x} \Rightarrow \lim_{x\to\infty}\frac{\sin 2x}{x} = 0$ by the Sandwich Theorem

10. $-\frac{1}{3\theta} \le \frac{\cos\theta}{3\theta} \le \frac{1}{3\theta} \Rightarrow \lim_{\theta\to-\infty}\frac{\cos\theta}{3\theta} = 0$ by the Sandwich Theorem

11. $\lim_{t\to-\infty}\frac{2-t+\sin t}{t+\cos t} = \lim_{t\to\infty}\frac{\frac{2}{t}-1+\left(\frac{\sin t}{t}\right)}{1+\left(\frac{\cos t}{t}\right)} = \frac{0-1+0}{1+0} = -1$

12. $\lim\limits_{r\to\infty} \frac{r+\sin r}{2r+7-5\sin r} = \lim\limits_{r\to\infty} \frac{1+\left(\frac{\sin r}{r}\right)}{2+\frac{7}{r}-5\left(\frac{\sin r}{r}\right)} = \lim\limits_{r\to\infty} \frac{1+0}{2+0-0} = \frac{1}{2}$

13. (a) $\lim\limits_{x\to\infty} \frac{2x+3}{5x+7} = \lim\limits_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2}{5}$ (b) $\frac{2}{5}$ (same process as part (a))

14. (a) $\lim\limits_{x\to\infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim\limits_{x\to\infty} \frac{2+\left(\frac{7}{x^3}\right)}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}} = 2$
(b) 2 (same process as part (a))

15. (a) $\lim\limits_{x\to\infty} \frac{x+1}{x^2+3} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} = 0$ (b) 0 (same process as part (a))

16. (a) $\lim\limits_{x\to\infty} \frac{3x+7}{x^2-2} = \lim\limits_{x\to\infty} \frac{\frac{3}{x}+\frac{7}{x^2}}{1+\frac{2}{x^2}} = 0$ (b) 0 (same process as part (a))

17. (a) $\lim\limits_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim\limits_{x\to\infty} \frac{7}{1-\frac{3}{x}+\frac{9}{x^2}} = 7$ (b) 7 (same process as part (a))

18. (a) $\lim\limits_{x\to\infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim\limits_{x\to\infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}} = \frac{9}{2}$ (b) $\frac{9}{2}$ (same process as part (a))

19. (a) $\lim\limits_{x\to\infty} \frac{10x^5+x^4+31}{x^6} = \lim\limits_{x\to\infty} \frac{\frac{10}{x}+\frac{1}{x^2}+\frac{31}{x^6}}{1} = 0$ (b) 0 (same process as part (a))

20. (a) $\lim\limits_{x\to\infty} \frac{x^3+7x^2-2}{x^2-x-1} = \lim\limits_{x\to\infty} \frac{x+7-2x^{-1}}{1-x^{-1}-x^{-2}} = \infty$, since $x^{-n} \to 0$ and $x+7 \to \infty$. (b)
$\lim\limits_{x\to-\infty} \frac{x^3+7x^2-2}{x^2-x-1} = \lim\limits_{x\to-\infty} \frac{x+7-2x^{-1}}{1-x^{-1}-x^{-2}} = -\infty$, since $x^{-n} \to 0$ and $x+7 \to -\infty$.

21. (a) $\lim\limits_{x\to\infty} \frac{3x^7+5x^2-1}{6x^3-7x+3} = \lim\limits_{x\to\infty} \frac{3x^4+5x^{-1}-x^{-3}}{6-7x^{-2}+3x^{-3}} = \infty$, since $x^{-n} \to 0$ and $3x^4 \to \infty$.
(b) $\lim\limits_{x\to-\infty} \frac{3x^7+5x^2-1}{6x^3-7x+3} = \lim\limits_{x\to-\infty} \frac{3x^4+5x^{-1}-x^{-3}}{6-7x^{-2}+3x^{-3}} = \infty$, since $x^{-n} \to 0$ and $3x^4 \to \infty$.

22. (a) $\lim\limits_{x\to\infty} \frac{5x^8-2x^3+9}{3+x-4x^5} = \lim\limits_{x\to\infty} \frac{5x^3-2x^{-2}+9x^{-5}}{3x^{-5}+x^{-4}-4} = -\infty$, since $x^{-n} \to 0$, $5x^3 \to \infty$, and the denominator $\to -4$.
(b) $\lim\limits_{x\to-\infty} \frac{5x^8-2x^3+9}{3+x-4x^5} = \lim\limits_{x\to-\infty} \frac{5x^3-2x^{-2}+9x^{-5}}{3x^{-5}+x^{-4}-4} = \infty$, since $x^{-n} \to 0$, $5x^3 \to -\infty$, and the denominator $\to -4$.

23. $\lim\limits_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}} = \lim\limits_{x\to\infty} \sqrt{\frac{8-\frac{3}{x^2}}{2+\frac{1}{x}}} = \sqrt{\lim\limits_{x\to\infty} \frac{8-\frac{3}{x^2}}{2+\frac{1}{x}}} = \sqrt{\frac{8-0}{2+0}} = \sqrt{4} = 2$

24. $\lim\limits_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \lim\limits_{x\to-\infty} \left(\frac{1+\frac{1}{x}-\frac{1}{x^2}}{8-\frac{3}{x^2}}\right)^{1/3} = \left(\lim\limits_{x\to-\infty} \frac{1+\frac{1}{x}-\frac{1}{x^2}}{8-\frac{3}{x^2}}\right)^{1/3} = \left(\frac{1+0-0}{8-0}\right)^{1/3} = \left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$

25. $\lim\limits_{x\to-\infty}\left(\frac{1-x^3}{x^2-7x}\right)^5 = \lim\limits_{x\to-\infty}\left(\frac{\frac{1}{x^2}-x}{1-\frac{7}{x}}\right)^5 = \left(\lim\limits_{x\to-\infty}\frac{\frac{1}{x^2}-x}{1-\frac{7}{x}}\right)^5 = \left(\frac{0+\infty}{1-0}\right)^5 = \infty$

26. $\lim\limits_{x\to\infty}\sqrt{\frac{x^2-5x}{x^3+x-2}} = \lim\limits_{x\to\infty}\sqrt{\frac{\frac{1}{x}-\frac{5}{x^2}}{1+\frac{1}{x^2}-\frac{2}{x^3}}} = \sqrt{\lim\limits_{x\to\infty}\frac{\frac{1}{x}-\frac{5}{x^2}}{1+\frac{1}{x^2}-\frac{2}{x^3}}} = \sqrt{\frac{0-0}{1+0-0}} = \sqrt{0} = 0$

27. $\lim\limits_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7} = \lim\limits_{x\to\infty}\frac{\left(\frac{2}{x^{1/2}}\right)+\left(\frac{1}{x^2}\right)}{3-\frac{7}{x}} = 0$

28. $\lim\limits_{x\to\infty}\frac{2+\sqrt{x}}{2-\sqrt{x}} = \lim\limits_{x\to\infty}\frac{\left(\frac{2}{x^{1/2}}\right)+1}{\left(\frac{2}{x^{1/2}}\right)-1} = -1$

29. $\lim\limits_{x\to-\infty}\frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}} = \lim\limits_{x\to-\infty}\frac{1-x^{(1/5)-(1/3)}}{1+x^{(1/5)-(1/3)}} = \lim\limits_{x\to-\infty}\frac{1-\left(\frac{1}{x^{2/15}}\right)}{1+\left(\frac{1}{x^{2/15}}\right)} = 1$

30. $\lim\limits_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}} = \lim\limits_{x\to\infty}\frac{x+\frac{1}{x^2}}{1-\frac{1}{x}} = \infty$

31. $\lim\limits_{x\to\infty}\frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}} = \lim\limits_{x\to\infty}\frac{2x^{1/15}-\frac{1}{x^{19/15}}+\frac{7}{x^{8/5}}}{1+\frac{3}{x^{3/5}}+\frac{1}{x^{11/10}}} = \infty$

32. $\lim\limits_{x\to-\infty}\frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4} = \lim\limits_{x\to-\infty}\frac{\frac{1}{x^{2/3}}-5+\frac{3}{x}}{2+\frac{1}{x^{1/3}}-\frac{4}{x}} = -\frac{5}{2}$

33. $\lim\limits_{x\to\infty}\frac{\sqrt{x^2+1}}{x+1} = \lim\limits_{x\to\infty}\frac{\sqrt{x^2+1}/\sqrt{x^2}}{(x+1)/\sqrt{x^2}} = \lim\limits_{x\to\infty}\frac{\sqrt{(x^2+1)/x^2}}{(x+1)/x} = \lim\limits_{x\to\infty}\frac{\sqrt{1+1/x^2}}{(1+1/x)} = \frac{\sqrt{1+0}}{(1+0)} = 1$

34. $\lim\limits_{x\to-\infty}\frac{\sqrt{x^2+1}}{x+1} = \lim\limits_{x\to-\infty}\frac{\sqrt{x^2+1}/\sqrt{x^2}}{(x+1)/\sqrt{x^2}} = \lim\limits_{x\to-\infty}\frac{\sqrt{(x^2+1)/x^2}}{(x+1)/(-x)} = \lim\limits_{x\to\infty}\frac{\sqrt{1+1/x^2}}{(-1-1/x)} = \frac{\sqrt{1+0}}{(-1-0)} = -1$

35. $\lim\limits_{x\to\infty}\frac{x-3}{\sqrt{4x^2+25}} = \lim\limits_{x\to\infty}\frac{(x-3)/\sqrt{x^2}}{\sqrt{4x^2+25}/\sqrt{x^2}} = \lim\limits_{x\to\infty}\frac{(x-3)/x}{\sqrt{(4x^2+25)/x^2}} = \lim\limits_{x\to\infty}\frac{(1-3/x)}{\sqrt{4+25/x^2}} = \frac{(1-0)}{\sqrt{4+0}} = \frac{1}{2}$

36. $\lim\limits_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}} = \lim\limits_{x\to-\infty}\frac{(4-3x^3)/\sqrt{x^6}}{\sqrt{x^6+9}/\sqrt{x^6}} = \lim\limits_{x\to-\infty}\frac{(4-3x^3)/(-x^3)}{\sqrt{(x^6+9)/x^6}} = \lim\limits_{x\to\infty}\frac{(-4/x^3+3)}{\sqrt{1+9/x^6}} = \frac{(0+3)}{\sqrt{1+0}} = 3$

37. $\lim\limits_{x\to0^+}\frac{1}{3x} = \infty \qquad \left(\frac{\text{positive}}{\text{positive}}\right)$

38. $\lim\limits_{x\to0^-}\frac{5}{2x} = -\infty \qquad \left(\frac{\text{positive}}{\text{negative}}\right)$

39. $\lim\limits_{x\to2^-}\frac{3}{x-2} = -\infty \qquad \left(\frac{\text{positive}}{\text{negative}}\right)$

40. $\lim\limits_{x\to3^+}\frac{1}{x-3} = \infty \qquad \left(\frac{\text{positive}}{\text{positive}}\right)$

41. $\lim\limits_{x\to-8^+}\frac{2x}{x+8} = -\infty \qquad \left(\frac{\text{negative}}{\text{positive}}\right)$

42. $\lim\limits_{x\to-5^-}\frac{3x}{2x+10} = \infty \qquad \left(\frac{\text{negative}}{\text{negative}}\right)$

43. $\lim\limits_{x\to7}\frac{4}{(x-7)^2} = \infty \qquad \left(\frac{\text{positive}}{\text{positive}}\right)$

44. $\lim\limits_{x\to0}\frac{-1}{x^2(x+1)} = -\infty \qquad \left(\frac{\text{negative}}{\text{positive}\cdot\text{positive}}\right)$

45. (a) $\lim\limits_{x\to 0^+} \frac{2}{3x^{1/3}} = \infty$ (b) $\lim\limits_{x\to 0^-} \frac{2}{3x^{1/3}} = -\infty$

46. (a) $\lim\limits_{x\to 0^+} \frac{2}{x^{1/5}} = \infty$ (b) $\lim\limits_{x\to 0^-} \frac{2}{x^{1/5}} = -\infty$

47. $\lim\limits_{x\to 0} \frac{4}{x^{2/5}} = \lim\limits_{x\to 0} \frac{4}{(x^{1/5})^2} = \infty$

48. $\lim\limits_{x\to 0} \frac{1}{x^{2/3}} = \lim\limits_{x\to 0} \frac{1}{(x^{1/3})^2} = \infty$

49. $\lim\limits_{x\to \left(\frac{\pi}{2}\right)^-} \tan x = \infty$

50. $\lim\limits_{x\to \left(\frac{-\pi}{2}\right)^+} \sec x = \infty$

51. $\lim\limits_{\theta\to 0^-} (1+\csc\theta) = -\infty$

52. $\lim\limits_{\theta\to 0^+} (2-\cot\theta) = -\infty$ and $\lim\limits_{\theta\to 0^-} (2-\cot\theta) = \infty$, so the limit does not exist

53. (a) $\lim\limits_{x\to 2^+} \frac{1}{x^2-4} = \lim\limits_{x\to 2^+} \frac{1}{(x+2)(x-2)} = \infty$ $\left(\frac{1}{\text{positive}\cdot\text{positive}}\right)$

(b) $\lim\limits_{x\to 2^-} \frac{1}{x^2-4} = \lim\limits_{x\to 2^-} \frac{1}{(x+2)(x-2)} = -\infty$ $\left(\frac{1}{\text{positive}\cdot\text{negative}}\right)$

(c) $\lim\limits_{x\to -2^+} \frac{1}{x^2-4} = \lim\limits_{x\to -2^+} \frac{1}{(x+2)(x-2)} = -\infty$ $\left(\frac{1}{\text{positive}\cdot\text{negative}}\right)$

(d) $\lim\limits_{x\to -2^-} \frac{1}{x^2-4} = \lim\limits_{x\to -2^-} \frac{1}{(x+2)(x-2)} = \infty$ $\left(\frac{1}{\text{negative}\cdot\text{negative}}\right)$

54. (a) $\lim\limits_{x\to 1^+} \frac{x}{x^2-1} = \lim\limits_{x\to 1^+} \frac{x}{(x+1)(x-1)} = \infty$ $\left(\frac{\text{positive}}{\text{positive}\cdot\text{positive}}\right)$

(b) $\lim\limits_{x\to 1^-} \frac{x}{x^2-1} = \lim\limits_{x\to 1^-} \frac{x}{(x+1)(x-1)} = -\infty$ $\left(\frac{\text{positive}}{\text{positive}\cdot\text{negative}}\right)$

(c) $\lim\limits_{x\to -1^+} \frac{x}{x^2-1} = \lim\limits_{x\to -1^+} \frac{x}{(x+1)(x-1)} = \infty$ $\left(\frac{\text{negative}}{\text{positive}\cdot\text{negative}}\right)$

(d) $\lim\limits_{x\to -1^-} \frac{x}{x^2-1} = \lim\limits_{x\to -1^-} \frac{x}{(x+1)(x-1)} = -\infty$ $\left(\frac{\text{negative}}{\text{negative}\cdot\text{negative}}\right)$

55. (a) $\lim\limits_{x\to 0^+} \left(\frac{x^2}{2} - \frac{1}{x}\right) = 0 + \lim\limits_{x\to 0^+} \frac{1}{-x} = -\infty$ $\left(\frac{1}{\text{negative}}\right)$

(b) $\lim\limits_{x\to 0^-} \left(\frac{x^2}{2} - \frac{1}{x}\right) = 0 + \lim\limits_{x\to 0^-} \frac{1}{-x} = \infty$ $\left(\frac{1}{\text{positive}}\right)$

(c) $\lim\limits_{x\to \sqrt[3]{2}} \left(\frac{x^2}{2} - \frac{1}{x}\right) = \frac{2^{2/3}}{2} - \frac{1}{2^{1/3}} = 2^{-1/3} - 2^{-1/3} = 0$

(d) $\lim\limits_{x\to -1} \left(\frac{x^2}{2} - \frac{1}{x}\right) = \frac{1}{2} - \left(\frac{1}{-1}\right) = \frac{3}{2}$

56. (a) $\lim\limits_{x\to -2^+} \frac{x^2-1}{2x+4} = \infty$ $\left(\frac{\text{positive}}{\text{positive}}\right)$

(b) $\lim\limits_{x\to -2^-} \frac{x^2-1}{2x+4} = -\infty$ $\left(\frac{\text{positive}}{\text{negative}}\right)$

(c) $\lim\limits_{x\to 1^+} \frac{x^2-1}{2x+4} = \lim\limits_{x\to 1^+} \frac{(x+1)(x-1)}{2x+4} = \frac{2\cdot 0}{2+4} = 0$

(d) $\lim\limits_{x\to 0^-} \frac{x^2-1}{2x+4} = \frac{-1}{4}$

57. (a) $\lim\limits_{x\to 0^+} \frac{x^2-3x+2}{x^3-2x^2} = \lim\limits_{x\to 0^+} \frac{(x-2)(x-1)}{x^2(x-2)} = -\infty \qquad \left(\frac{\text{negative}\cdot\text{negative}}{\text{positive}\cdot\text{negative}}\right)$

(b) $\lim\limits_{x\to 2^+} \frac{x^2-3x+2}{x^3-2x^2} = \lim\limits_{x\to 2^+} \frac{(x-2)(x-1)}{x^2(x-2)} = \lim\limits_{x\to 2^+} \frac{x-1}{x^2} = \frac{1}{4}, x \neq 2$

(c) $\lim\limits_{x\to 2^-} \frac{x^2-3x+2}{x^3-2x^2} = \lim\limits_{x\to 2^-} \frac{(x-2)(x-1)}{x^2(x-2)} = \lim\limits_{x\to 2^-} \frac{x-1}{x^2} = \frac{1}{4}, x \neq 2$

(d) $\lim\limits_{x\to 2} \frac{x^2-3x+2}{x^3-2x^2} = \lim\limits_{x\to 2} \frac{(x-2)(x-1)}{x^2(x-2)} = \lim\limits_{x\to 2} \frac{x-1}{x^2} = \frac{1}{4}, x \neq 2$

(e) $\lim\limits_{x\to 0} \frac{x^2-3x+2}{x^3-2x^2} = \lim\limits_{x\to 0} \frac{(x-2)(x-1)}{x^2(x-2)} = -\infty \qquad \left(\frac{\text{negative}\cdot\text{negative}}{\text{positive}\cdot\text{negative}}\right)$

58. (a) $\lim\limits_{x\to 2^+} \frac{x^2-3x+2}{x^3-4x} = \lim\limits_{x\to 2^+} \frac{(x-2)(x-1)}{x(x-2)(x+2)} = \lim\limits_{x\to 2^+} \frac{(x-1)}{x(x+2)} = \frac{1}{2(4)} = \frac{1}{8}$

(b) $\lim\limits_{x\to -2^+} \frac{x^2-3x+2}{x^3-4x} = \lim\limits_{x\to -2^+} \frac{(x-2)(x-1)}{x(x-2)(x+2)} = \lim\limits_{x\to -2^+} \frac{(x-1)}{x(x+2)} = \infty \qquad \left(\frac{\text{negative}}{\text{negative}\cdot\text{positive}}\right)$

(c) $\lim\limits_{x\to 0^-} \frac{x^2-3x+2}{x^3-4x} = \lim\limits_{x\to 0^-} \frac{(x-2)(x-1)}{x(x-2)(x+2)} = \lim\limits_{x\to 0^-} \frac{(x-1)}{x(x+2)} = \infty \qquad \left(\frac{\text{negative}}{\text{negative}\cdot\text{positive}}\right)$

(d) $\lim\limits_{x\to 1^+} \frac{x^2-3x+2}{x^3-4x} = \lim\limits_{x\to 1^+} \frac{(x-2)(x-1)}{x(x-2)(x+2)} = \lim\limits_{x\to 1^+} \frac{(x-1)}{x(x+2)} = \frac{0}{(1)(3)} = 0$

(e) $\lim\limits_{x\to 0^+} \frac{x-1}{x(x+2)} = -\infty \qquad \left(\frac{\text{negative}}{\text{positive}\cdot\text{positive}}\right)$

and $\lim\limits_{x\to 0^-} \frac{x-1}{x(x+2)} = \infty \quad \left(\frac{\text{negative}}{\text{negative}\cdot\text{positive}}\right)$
so the function has no limit as $x \to 0$.

59. (a) $\lim\limits_{t\to 0^+} \left[2 - \frac{3}{t^{1/3}}\right] = -\infty$ (b) $\lim\limits_{t\to 0^-} \left[2 - \frac{3}{t^{1/3}}\right] = \infty$

60. (a) $\lim\limits_{t\to 0^+} \left[\frac{1}{t^{3/5}} + 7\right] = \infty$ (b) $\lim\limits_{t\to 0^-} \left[\frac{1}{t^{3/5}} + 7\right] = -\infty$

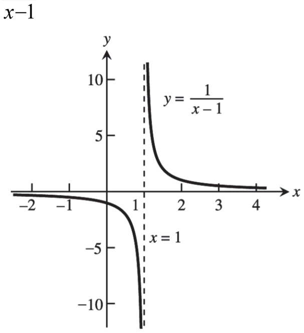
61. (a) $\lim\limits_{x\to 0^+} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$ (b) $\lim\limits_{x\to 0^-} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$

(c) $\lim\limits_{x\to 1^+} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$ (d) $\lim\limits_{x\to 1^-} \left[\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right] = \infty$

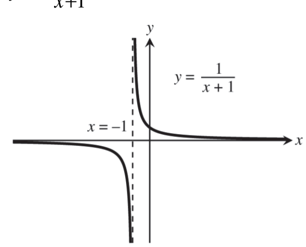
62. (a) $\lim\limits_{x\to 0^+} \left[\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right] = \infty$ (b) $\lim\limits_{x\to 0^-} \left[\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right] = -\infty$

(c) $\lim\limits_{x\to 1^+} \left[\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right] = -\infty$ (d) $\lim\limits_{x\to 1^-} \left[\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right] = -\infty$

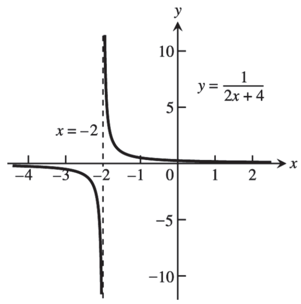
63. $y = \frac{1}{x-1}$



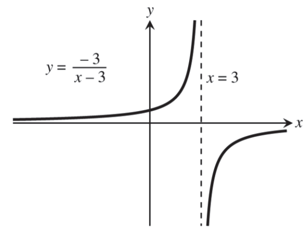
64. $y = \frac{1}{x+1}$



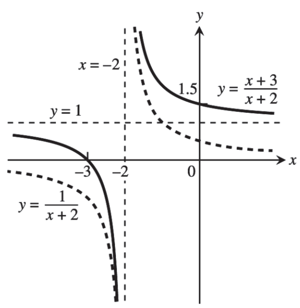
65. $y = \frac{1}{2x+4}$



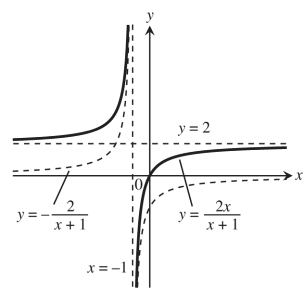
66. $y = \frac{-3}{x-3}$



67. $y = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$



68. $y = \frac{2x}{x+1} = 2 - \frac{2}{x+1}$



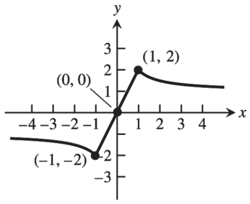
69. domain $= (-\infty, \infty)$; y in range and $y = 4 + \frac{3x^2}{x^2+1}$, $0 \le \frac{3x^2}{x^2+1} < 3$ and $\lim\limits_{x\to\pm\infty} \frac{3x^2}{x^2+1} = 3 \Rightarrow$ range $= [4, 7)$; horizontal asymptote is $y = 7$.

70. domain $= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$; y in range and $y = \frac{2x}{x^2-1}$; if $x = 0$, then $y = 0$, $\lim\limits_{x\to\pm\infty} \frac{2x}{x^2-1} = 0$, $\lim\limits_{x\to 1^+} \frac{2x}{x^2-1} = +\infty$, and $\lim\limits_{x\to 1^-} \frac{2x}{x^2-1} = -\infty$; $\lim\limits_{x\to -1^+} \frac{2x}{x^2-1} = \infty$ and $\lim\limits_{x\to -1^-} \frac{2x}{x^2-1} = -\infty$ $\Rightarrow$ range $= (-\infty, \infty)$; horizontal asymptote is $y = 0$; vertical asymptotes are $x = -1$, $x = 1$

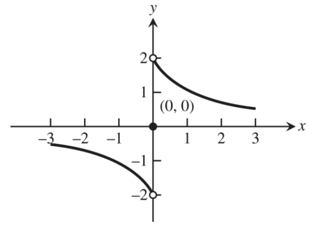
71. domain $= (-\infty, 0) \cup (0, \infty)$; y in the range and $y = \frac{\sqrt{x^2+4}}{x}$; $y' = \frac{-4}{x^2\sqrt{x^2+4}} < 0$; $\lim\limits_{x\to 0^+} \frac{\sqrt{x^2+4}}{x} = \infty$, $\lim\limits_{x\to 0^-} \frac{\sqrt{x^2+4}}{x} = -\infty$, $\lim\limits_{x\to\infty} \frac{\sqrt{x^2+4}}{x} = 1$, and $\lim\limits_{x\to-\infty} \frac{\sqrt{x^2+4}}{x} = -1$ $\Rightarrow$ range $= (-\infty, -1) \cup (1, \infty)$; horizontal asymptotes are $y = 1$, $y = -1$; vertical asymptote is $x = 0$

72. domain $= (-\infty, 2) \cup (2, \infty)$; y in the range and $y = \frac{x^3}{x^3-8}$; $y' = \frac{-24x^2}{\left(x^3-8\right)^2} \le 0$; $\lim\limits_{x\to 2^+} \frac{x^3}{x^3-8} = \infty$, $\lim\limits_{x\to 2^-} \frac{x^3}{x^3-8} = -\infty$, and $\lim\limits_{x\to\pm\infty} \frac{x^3}{x^3-8} = 1$ $\Rightarrow$ range $= (-\infty, 1) \cup (1, \infty)$; horizontal asymptote is $y = 1$; vertical asymptote is $x = 2$

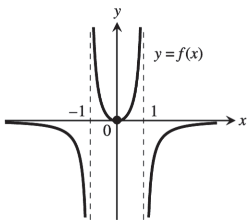
73. Here is one possibility.



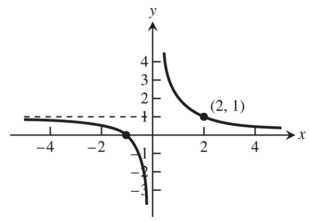
74. Here is one possibility.



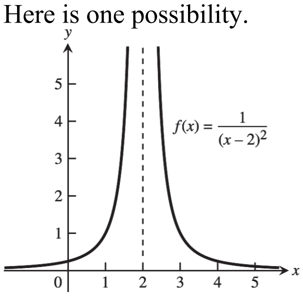
75. Here is one possibility.



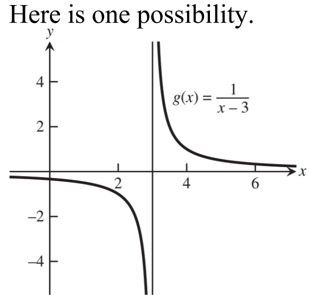
76. Here is one possibility.



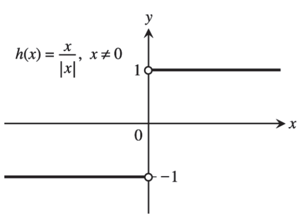
77. Here is one possibility.



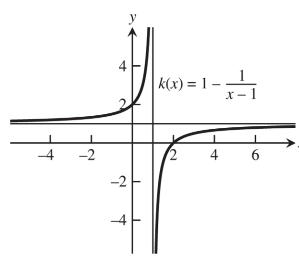
78. Here is one possibility.



79. Here is one possibility.



80. Here is one possibility.



81. Yes. If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 2$ then the ratio the polynomials' leading coefficients is 2, so $\lim\limits_{x\to-\infty} \frac{f(x)}{g(x)} = 2$ as well.

82. Yes, it can have a horizontal or oblique asymptote.

83. At most 1 horizontal asymptote: If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$, then the ratio of the polynomials' leading coefficients is L, so $\lim\limits_{x\to-\infty} \frac{f(x)}{g(x)} = L$ as well.

84. $\lim\limits_{x\to\infty}\left(\sqrt{x+9}-\sqrt{x+4}\right)=\lim\limits_{x\to\infty}\left[\sqrt{x+9}-\sqrt{x+4}\right]\cdot\left[\frac{\sqrt{x+9}+\sqrt{x+4}}{\sqrt{x+9}+\sqrt{x+4}}\right]=\lim\limits_{x\to\infty}\frac{(x+9)-(x+4)}{\sqrt{x+9}+\sqrt{x+4}}$

$=\lim\limits_{x\to\infty}\frac{5}{\sqrt{x+9}+\sqrt{x+4}}=\lim\limits_{x\to\infty}\frac{\frac{5}{\sqrt{x}}}{\sqrt{1+\frac{9}{x}}+\sqrt{1+\frac{4}{x}}}=\frac{0}{1+1}=0$

85. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+25}-\sqrt{x^2-1}\right)=\lim\limits_{x\to\infty}\left[\sqrt{x^2+25}-\sqrt{x^2-1}\right]\cdot\left[\frac{\sqrt{x^2+25}+\sqrt{x^2-1}}{\sqrt{x^2+25}+\sqrt{x^2-1}}\right]=\lim\limits_{x\to\infty}\frac{(x^2+25)-(x^2-1)}{\sqrt{x^2+25}+\sqrt{x^2-1}}$

$=\lim\limits_{x\to\infty}\frac{26}{\sqrt{x^2+25}+\sqrt{x^2-1}}=\lim\limits_{x\to\infty}\frac{\frac{26}{x}}{\sqrt{1+\frac{25}{x^2}}+\sqrt{1-\frac{1}{x^2}}}=\frac{0}{1+1}=0$

86. $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+3}+x\right)=\lim\limits_{x\to-\infty}\left[\sqrt{x^2+3}+x\right]\cdot\left[\frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x}\right]=\lim\limits_{x\to-\infty}\frac{(x^2+3)-(x^2)}{\sqrt{x^2+3}-x}=\lim\limits_{x\to-\infty}\frac{3}{\sqrt{x^2+3}-x}$

$=\lim\limits_{x\to-\infty}\frac{\frac{3}{\sqrt{x^2}}}{\sqrt{1+\frac{3}{x^2}}-\frac{x}{\sqrt{x^2}}}=\lim\limits_{x\to-\infty}\frac{-\frac{3}{x}}{\sqrt{1+\frac{3}{x^2}}+1}=\frac{0}{1+1}=0$

87. $\lim\limits_{x\to-\infty}\left(2x+\sqrt{4x^2+3x-2}\right)=\lim\limits_{x\to-\infty}\left[2x+\sqrt{4x^2+3x-2}\right]\cdot\left[\frac{2x-\sqrt{4x^2+3x-2}}{2x-\sqrt{4x^2+3x-2}}\right]=\lim\limits_{x\to-\infty}\frac{(4x^2)-(4x^2+3x-2)}{2x-\sqrt{4x^2+3x-2}}$

$=\lim\limits_{x\to-\infty}\frac{-3x+2}{2x-\sqrt{4x^2+3x-2}}=\lim\limits_{x\to-\infty}\frac{\frac{-3x+2}{\sqrt{x^2}}}{\frac{2x}{\sqrt{x^2}}-\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}=\lim\limits_{x\to-\infty}\frac{\frac{-3x+2}{-x}}{\frac{2x}{-x}-\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}$

$=\lim\limits_{x\to-\infty}\frac{3-\frac{2}{x}}{-2-\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}=\frac{3-0}{-2-2}=-\frac{3}{4}$

88. $\lim\limits_{x\to\infty}\left(\sqrt{9x^2-x}-3x\right)=\lim\limits_{x\to\infty}\left[\sqrt{9x^2-x}-3x\right]\cdot\left[\frac{\sqrt{9x^2-x}+3x}{\sqrt{9x^2-x}+3x}\right]=\lim\limits_{x\to\infty}\frac{(9x^2-x)-(9x^2)}{\sqrt{9x^2-x}+3x}=\lim\limits_{x\to\infty}\frac{-x}{\sqrt{9x^2-x}+3x}$

$=\lim\limits_{x\to\infty}\frac{-\frac{x}{x}}{\sqrt{\frac{9x^2}{x^2}-\frac{x}{x^2}}+\frac{3x}{x}}=\lim\limits_{x\to\infty}\frac{-1}{\sqrt{9-\frac{1}{x}}+3}=\frac{-1}{3+3}=-\frac{1}{6}$

89. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)=\lim\limits_{x\to\infty}\left[\sqrt{x^2+3x}-\sqrt{x^2-2x}\right]\cdot\left[\frac{\sqrt{x^2+3x}+\sqrt{x^2-2x}}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}\right]=\lim\limits_{x\to\infty}\frac{(x^2+3x)-(x^2-2x)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$

$=\lim\limits_{x\to\infty}\frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}=\lim\limits_{x\to\infty}\frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}}=\frac{5}{1+1}=\frac{5}{2}$

90. $\lim\limits_{x\to\infty}\sqrt{x^2+x}-\sqrt{x^2-x}=\lim\limits_{x\to\infty}\left[\sqrt{x^2+x}-\sqrt{x^2-x}\right]\cdot\left[\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}\right]=\lim\limits_{x\to\infty}\frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim\limits_{x\to\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$

$=\lim\limits_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}}=\frac{2}{1+1}=1$

91. For any $\epsilon>0$, take $N=1$. Then for all $x>N$ we have that $|f(x)-k|=|k-k|=0<\epsilon$.

92. For any $\epsilon>0$, take $N=1$. Then for all $y<-N$ we have that $|f(x)-k|=|k-k|=0<\epsilon$.

93. For every real number $-B<0$, we must find a $\delta>0$ such that for all x, $0<|x-0|<\delta\Rightarrow\frac{-1}{x^2}<-B$. Now, $-\frac{1}{x^2}<-B<0\Leftrightarrow\frac{1}{x^2}>B>0\Leftrightarrow x^2<\frac{1}{B}\Leftrightarrow|x|<\frac{1}{\sqrt{B}}$. Choose $\delta=\frac{1}{\sqrt{B}}$, then $0<|x|<\delta\Rightarrow|x|<\frac{1}{\sqrt{B}}$ $\Rightarrow\frac{-1}{x^2}<-B$ so that $\lim\limits_{x\to0}-\frac{1}{x^2}=-\infty$.

94. For every real number $B > 0$, we must find a $\delta > 0$ such that for all x, $0 < |x - 0| < \delta \Rightarrow \frac{1}{|x|} > B$. Now, $\frac{1}{|x|} > B > 0 \Leftrightarrow |x| < \frac{1}{B}$. Choose $\delta = \frac{1}{B}$. Then $0 < |x - 0| < \delta \Rightarrow |x| < \frac{1}{B} \Rightarrow \frac{1}{|x|} > B$ so that $\lim\limits_{x \to 0} \frac{1}{|x|} = \infty$.

95. For every real number $-B < 0$, we must find a $\delta > 0$ such that for all x, $0 < |x - 3| < \delta \Rightarrow \frac{-2}{(x-3)^2} < -B$. Now, $\frac{-2}{(x-3)^2} < -B < 0 \Leftrightarrow \frac{2}{(x-3)^2} > B > 0 \Leftrightarrow \frac{(x-3)^2}{2} < \frac{1}{B} \Leftrightarrow (x-3)^2 < \frac{2}{B} \Leftrightarrow 0 < |x - 3| < \sqrt{\frac{2}{B}}$. Choose $\delta = \sqrt{\frac{2}{B}}$, then $0 < |x - 3| < \delta \Rightarrow \frac{-2}{(x-3)^2} < -B < 0$ so that $\lim\limits_{x \to 3} \frac{-2}{(x-3)^2} = -\infty$.

96. For every real number $B > 0$, we must find a $\delta > 0$ such that for all x, $0 < |x - (-5)| < \delta \Rightarrow \frac{1}{(x+5)^2} > B$. Now, $\frac{1}{(x+5)^2} > B > 0 \Leftrightarrow (x+5)^2 < \frac{1}{B} \Leftrightarrow |x + 5| < \frac{1}{\sqrt{B}}$. Choose $\delta = \frac{1}{\sqrt{B}}$. Then $0 < |x - (-5)| < \delta$ $\Rightarrow |x + 5| < \frac{1}{\sqrt{B}} \Rightarrow \frac{1}{(x+5)^2} > B$ so that $\lim\limits_{x \to -5} \frac{1}{(x+5)^2} = \infty$.

97. (a) We say that $f(x)$ approaches infinity as x approaches c from the left, and write $\lim\limits_{x \to c} f(x) = \infty$, if for every positive number B, there exists a corresponding number $\delta > 0$ such that for all x, $c - \delta < x < c \Rightarrow f(x) > B$.

(b) We say that $f(x)$ approaches minus infinity as x approaches c from the right, and write $\lim\limits_{x \to c} f(x) = -\infty$, if for every positive number B (or negative number $-B$) there exists a corresponding number $\delta > 0$ such that for all x, $c < x < c \Rightarrow f(x) < -B$.

(c) We say that $f(x)$ approaches minus infinity as x approaches c from the left, and write $\lim\limits_{x \to c} f(x) = -\infty$, if for every positive number B (or negative number $-B$) there exists a corresponding number $\delta > 0$ such that for all x, $c - \delta < x < c \Rightarrow f(x) < -B$.

98. For $B > 0$, $\frac{1}{x} > B > 0 \Leftrightarrow x < \frac{1}{B}$. Choose $\delta = \frac{1}{B}$. Then $0 < x < \delta \Rightarrow 0 < x < \frac{1}{B} \Rightarrow \frac{1}{x} > B$ so that $\lim\limits_{x \to 0^+} \frac{1}{x} = \infty$.

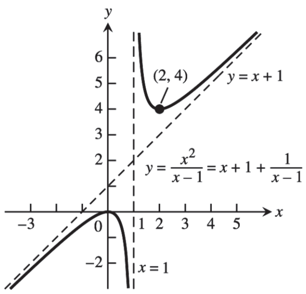
99. For $B > 0$, $\frac{1}{x} < -B < 0 \Leftrightarrow -\frac{1}{x} > B > 0 \Leftrightarrow -x < \frac{1}{B} \Leftrightarrow -\frac{1}{B} < x$. Choose $\delta = \frac{1}{B}$. Then $-\delta < x < 0 \Rightarrow -\frac{1}{B} < x$ $\Rightarrow \frac{1}{x} < -B$ so that $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$.

100. For $B > 0$, $\frac{1}{x-2} < -B \Leftrightarrow -\frac{1}{x-2} > B \Leftrightarrow -(x-2) < \frac{1}{B} \Leftrightarrow x - 2 > -\frac{1}{B} \Leftrightarrow x > 2 - \frac{1}{B}$. Choose $\delta = \frac{1}{B}$. Then $2 - \delta < x < 2 \Rightarrow -\delta < x - 2 < 0 \Rightarrow -\frac{1}{B} < x - 2 < 0 \Rightarrow \frac{1}{x-2} < -B < 0$ so that $\lim\limits_{x \to 2^-} \frac{1}{x-2} = -\infty$.

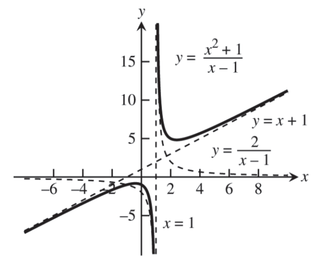
101. For $B > 0$, $\frac{1}{x-2} > B \Leftrightarrow 0 < x - 2 < \frac{1}{B}$. Choose $\delta = \frac{1}{B}$. Then $2 < x < 2 + \delta \Rightarrow 0 < x - 2 < \delta \Rightarrow 0 < x - 2 < \frac{1}{B}$ $\Rightarrow \frac{1}{x-2} > B > 0$ so that $\lim\limits_{x \to 2^+} \frac{1}{x-2} = \infty$.

102. For $B > 0$ and $0 < x < 1$, $\frac{1}{1-x^2} > B \Leftrightarrow 1 - x^2 < \frac{1}{B} \Leftrightarrow (1-x)(1+x) < \frac{1}{B}$. Now $\frac{1+x}{2} < 1$ since $x < 1$. Choose $\delta < \frac{1}{2B}$. Then $1 - \delta < x < 1 \Rightarrow -\delta < x - 1 < 0 \Rightarrow 1 - x < \delta < \frac{1}{2B} \Rightarrow (1-x)(1+x) < \frac{1}{B}\left(\frac{1+x}{2}\right) < \frac{1}{B} \Rightarrow \frac{1}{1-x^2} > B$ for $0 < x < 1$ and x near $1 \Rightarrow \lim\limits_{x \to 1^-} \frac{1}{1-x^2} = \infty$.

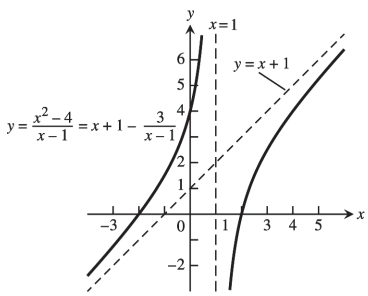
103. $y=\frac{x^2}{x-1}=x+1+\frac{1}{x-1}$



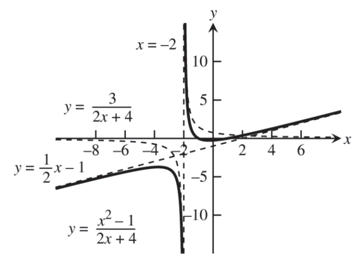
104. $y=\frac{x^2+1}{x-1}=x+1+\frac{2}{x-1}$



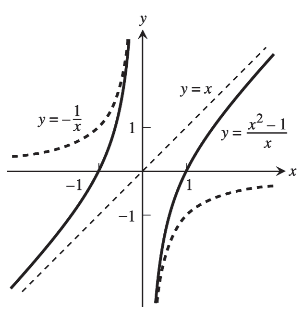
105. $y=\frac{x^2-4}{x-1}=x+1-\frac{3}{x-1}$



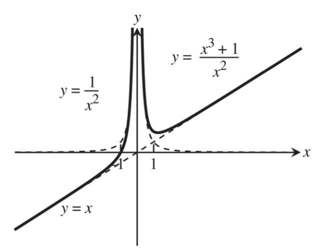
106. $y=\frac{x^2-1}{2x+4}=\frac{1}{2}x-1+\frac{3}{2x+4}$



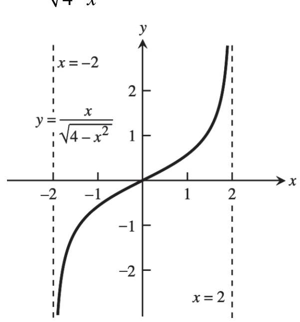
107. $y=\frac{x^2-1}{x}=x-\frac{1}{x}$



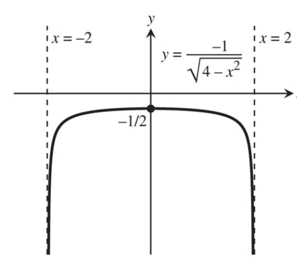
108. $y=\frac{x^3+1}{x^2}=x+\frac{1}{x^2}$



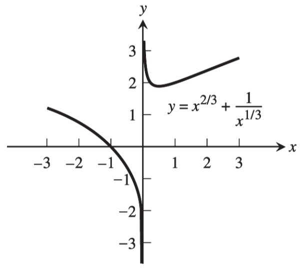
109. $y=\frac{x}{\sqrt{4-x^2}}$



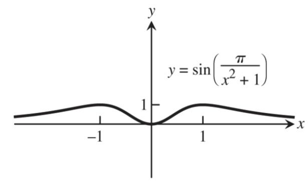
110. $y=\frac{-1}{\sqrt{4-x^2}}$



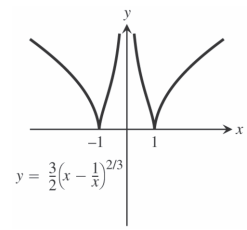
111. $y = x^{2/3} + \frac{1}{x^{1/3}}$



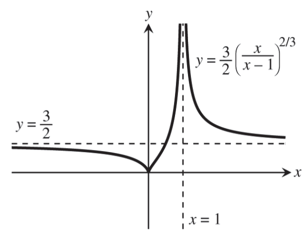
112. $y = \sin\left(\frac{\pi}{x^2+1}\right)$



113. (a) $y \to \infty$ (see accompanying graph)
(b) $y \to \infty$ (see accompanying graph)
(c) cusps at $x = \pm 1$ (see accompanying graph)

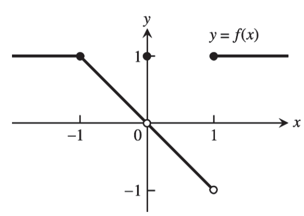


114. (a) $y \to 0$ and a cusp at $x = 0$ (see the accompanying graph)
(b) $y \to \frac{3}{2}$ (see accompanying graph)
(c) a vertical asymptote at $x = 1$ and contains the point $\left(-1, \frac{3}{2\sqrt[3]{4}}\right)$ (see accompanying graph)



CHAPTER 2 PRACTICE EXERCISES

1. At $x = -1$: $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = 1$
$\Rightarrow \lim_{x \to -1} f(x) = 1 = f(-1)$
$\Rightarrow f$ is continuous at $x = -1$.



At $x = 0$: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$
$\Rightarrow \lim_{x \to 0} f(x) = 0$.
But $f(0) = 1 \neq \lim_{x \to 0} f(x)$
$\Rightarrow f$ is discontinuous at $x = 0$.
If we define $f(0) = 0$, then the discontinuity at $x = 0$ is removable.

At $x = 1$: $\lim_{x \to 1^-} f(x) = -1$ and $\lim_{x \to 1^+} f(x) = 1$
$\Rightarrow \lim_{x \to 1} f(x)$ does not exist
$\Rightarrow f$ is discontinuous at $x = 1$.

2. At $x=-1$: $\lim\limits_{x\to -1^-} f(x)=0$ and $\lim\limits_{x\to -1^+} f(x)=-1$

$\Rightarrow \lim\limits_{x\to -1} f(x)$ does not exist

$\Rightarrow f$ is discontinuous at $x=-1$.

At $x=0$: $\lim\limits_{x\to 0^-} f(x)=-\infty$ and $\lim\limits_{x\to 0^+} f(x)=\infty$

$\Rightarrow \lim\limits_{x\to 0} f(x)$ does not exist

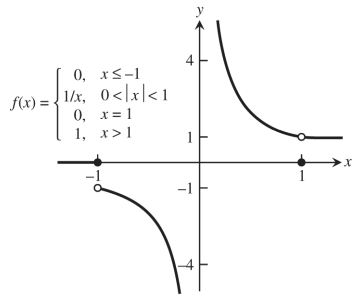
$\Rightarrow f$ is discontinuous at $x=0$.

At $x=1$: $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)=1 \Rightarrow \lim\limits_{x\to 1} f(x)=1$.

But $f(1)=0\neq \lim\limits_{x\to 1} f(x)$

$\Rightarrow f$ is discontinuous at $x=1$.

If we define $f(1)=1$, then the discontinuity at $x=1$ is removable.



3. (a) $\lim\limits_{t\to t_0}(3f(t))=3\lim\limits_{t\to t_0} f(t)=3(-7)=-21$

(b) $\lim\limits_{t\to t_0}(f(t))^2=\left(\lim\limits_{t\to t_0} f(t)\right)^2=(-7)^2=49$

(c) $\lim\limits_{t\to t_0}(f(t)\cdot g(t))=\lim\limits_{t\to t_0} f(t)\cdot\lim\limits_{t\to t_0} g(t)=(-7)(0)=0$

(d) $\lim\limits_{t\to t_0}\frac{f(t)}{g(t)-7}=\frac{\lim\limits_{t\to t_0} f(t)}{\lim\limits_{t\to t_0}(g(t)-7)}=\frac{\lim\limits_{t\to t_0} f(t)}{\lim\limits_{t\to t_0} g(t)-\lim\limits_{t\to t_0} 7}=\frac{-7}{0-7}=1$

(e) $\lim\limits_{t\to t_0}\cos(g(t))=\cos\left(\lim\limits_{t\to t_0} g(t)\right)=\cos 0=1$

(f) $\lim\limits_{t\to t_0}|f(t)|=\left|\lim\limits_{t\to t_0} f(t)\right|=|-7|=7$

(g) $\lim\limits_{t\to t_0}(f(t)+g(t))=\lim\limits_{t\to t_0} f(t)+\lim\limits_{t\to t_0} g(t)=-7+0=-7$

(h) $\lim\limits_{t\to t_0}\left(\frac{1}{f(t)}\right)=\frac{1}{\lim\limits_{t\to t_0} f(t)}=\frac{1}{-7}=-\frac{1}{7}$

4. (a) $\lim\limits_{x\to 0}-g(x)=-\lim\limits_{x\to 0} g(x)=-\sqrt{2}$

(b) $\lim\limits_{x\to 0}(g(x)\cdot f(x))=\lim\limits_{x\to 0} g(x)\cdot\lim\limits_{x\to 0} f(x)=\left(\sqrt{2}\right)\left(\frac{1}{2}\right)=\frac{\sqrt{2}}{2}$

(c) $\lim\limits_{x\to 0}(f(x)+g(x))=\lim\limits_{x\to 0} f(x)+\lim\limits_{x\to 0} g(x)=\frac{1}{2}+\sqrt{2}$

(d) $\lim\limits_{x\to 0}\frac{1}{f(x)}=\frac{1}{\lim\limits_{x\to 0} f(x)}=\frac{1}{\frac{1}{2}}=2$

(e) $\lim\limits_{x\to 0}(x+f(x))=\lim\limits_{x\to 0} x+\lim\limits_{x\to 0} f(x)=0+\frac{1}{2}=\frac{1}{2}$

(f) $\lim\limits_{x\to 0}\frac{f(x)\cdot\cos x}{x-1}=\frac{\lim\limits_{x\to 0} f(x)\cdot\lim\limits_{x\to 0}\cos x}{\lim\limits_{x\to 0} x-\lim\limits_{x\to 0} 1}=\frac{\left(\frac{1}{2}\right)(1)}{0-1}=-\frac{1}{2}$

5. Since $\lim\limits_{x\to 0} x=0$ we must have that $\lim\limits_{x\to 0}(4-g(x))=0$. Otherwise, if $\lim\limits_{x\to 0}(4-g(x))$ is a finite positive number, we would have $\lim\limits_{x\to 0^-}\left[\frac{4-g(x)}{x}\right]=-\infty$ and $\lim\limits_{x\to 0^+}\left[\frac{4-g(x)}{x}\right]=\infty$ so the limit could not equal 1 as $x\to 0$. Similar reasoning holds if $\lim\limits_{x\to 0}(4-g(x))$ is a finite negative number. We conclude that $\lim\limits_{x\to 0} g(x)=4$.

6. $2 = \lim\limits_{x\to -4}\left[x \lim\limits_{x\to 0} g(x)\right] = \lim\limits_{x\to -4} x \cdot \lim\limits_{x\to -4}\left[\lim\limits_{x\to 0} g(x)\right] = -4 \lim\limits_{x\to -4}\left[\lim\limits_{x\to 0} g(x)\right] = -4 \lim\limits_{x\to 0} g(x)$ (since $\lim\limits_{x\to 0} g(x)$ is a constant) $\Rightarrow \lim\limits_{x\to 0} g(x) = \frac{2}{-4} = -\frac{1}{2}$.

7. (a) $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} x^{1/3} = c^{1/3} = f(c)$ for every real number $c \Rightarrow f$ is continuous on $(-\infty, \infty)$.
   (b) $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} x^{3/4} = c^{3/4} = g(c)$ for every nonnegative real number $c \Rightarrow g$ is continuous on $[0, \infty)$.
   (c) $\lim\limits_{x\to c} h(x) = \lim\limits_{x\to c} x^{-2/3} = \frac{1}{c^{2/3}} = h(c)$ for every nonzero real number $c \Rightarrow h$ is continuous on $(-\infty, 0)$ and $(-\infty, \infty)$.
   (d) $\lim\limits_{x\to c} k(x) = \lim\limits_{x\to c} x^{-1/6} = \frac{1}{c^{1/6}} = k(c)$ for every positive real number $c \Rightarrow k$ is continuous on $(0, \infty)$

8. (a) $\bigcup\limits_{n\in I}\left(\left(n-\frac{1}{2}\right)\pi, \left(n+\frac{1}{2}\right)\pi\right)$, where I = the set of all integers.
   (b) $\bigcup\limits_{n\in I}(n\pi, (n+1)\pi)$, where I = the set of all integers.
   (c) $(-\infty, \pi)\cup(\pi, \infty)$
   (d) $(-\infty, 0)\cup(0, \infty)$

9. (a) $\lim\limits_{x\to 0} \frac{x^2-4x+4}{x^3+5x^2-14x} = \lim\limits_{x\to 0} \frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim\limits_{x\to 0} \frac{x-2}{x(x+7)}$, $x \neq 2$; the limit does not exist because $\lim\limits_{x\to 0^-} \frac{x-2}{x(x+7)} = \infty$ and $\lim\limits_{x\to 0^+} \frac{x-2}{x(x+7)} = -\infty$
   (b) $\lim\limits_{x\to 2} \frac{x^2-4x+4}{x^3+5x^2-14x} = \lim\limits_{x\to 2} \frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim\limits_{x\to 2} \frac{x-2}{x(x+7)}$, $x \neq 2$, and $\lim\limits_{x\to 2} \frac{x-2}{x(x+7)} = \frac{0}{2(9)} = 0$

10. (a) $\lim\limits_{x\to 0} \frac{x^2+x}{x^5+2x^4+x^3} = \lim\limits_{x\to 0} \frac{x(x+1)}{x^3(x^2+2x+1)} = \lim\limits_{x\to 0} \frac{x+1}{x^2(x+1)(x+1)} = \lim\limits_{x\to 0} \frac{1}{x^2(x+1)}$, $x \neq 0$ and $x \neq -1$.
    Now $\lim\limits_{x\to 0^-} \frac{1}{x^2(x+1)} = \infty$ and $\lim\limits_{x\to 0^+} \frac{1}{x^2(x+1)} = \infty \Rightarrow \lim\limits_{x\to 0} \frac{x^2+x}{x^5+2x^4+x^3} = \infty$.
    (b) $\lim\limits_{x\to -1} \frac{x^2+x}{x^5+2x^4+x^3} = \lim\limits_{x\to -1} \frac{x(x+1)}{x^3(x^2+2x+1)} = \lim\limits_{x\to -1} \frac{1}{x^2(x+1)}$, $x \neq 0$ and $x \neq -1$. The limit does not exist because $\lim\limits_{x\to -1^-} \frac{1}{x^2(x+1)} = -\infty$ and $\lim\limits_{x\to -1^+} \frac{1}{x^2(x+1)} = \infty$.

11. $\lim\limits_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim\limits_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim\limits_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$

12. $\lim\limits_{x\to a} \frac{x^2-a^2}{x^4-a^4} = \lim\limits_{x\to a} \frac{(x^2-a^2)}{(x^2+a^2)(x^2-a^2)} = \lim\limits_{x\to a} \frac{1}{x^2+a^2} = \frac{1}{2a^2}$

13. $\lim\limits_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim\limits_{h\to 0} \frac{(x^2+2hx+h^2)-x^2}{h} = \lim\limits_{h\to 0} (2x+h) = 2x$

14. $\lim\limits_{x\to 0} \frac{(x+h)^2-x^2}{h} = \lim\limits_{x\to 0} \frac{(x^2+2hx+h^2)-x^2}{h} = \lim\limits_{x\to 0} (2x+h) = h$

15. $\lim\limits_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim\limits_{x\to 0} \frac{2-(2+x)}{2x(2+x)} = \lim\limits_{x\to 0} \frac{-1}{4+2x} = -\frac{1}{4}$

16. $\lim\limits_{x\to 0} \frac{(2+x)^3-8}{x} = \lim\limits_{x\to 0} \frac{(x^3+6x^2+12x+8)-8}{x} = \lim\limits_{x\to 0} (x^2+6x+12) = 12$

17. $\lim\limits_{x\to 1} \frac{x^{1/3}-1}{\sqrt{x}-1} = \lim\limits_{x\to 1} \frac{(x^{1/3}-1)}{(\sqrt{x}-1)} \cdot \frac{(x^{2/3}+x^{1/3}+1)(\sqrt{x}+1)}{(\sqrt{x}+1)(x^{2/3}+x^{1/3}+1)} = \lim\limits_{x\to 1} \frac{(x-1)(\sqrt{x}+1)}{(x-1)(x^{2/3}+x^{1/3}+1)} = \lim\limits_{x\to 1} \frac{\sqrt{x}+1}{x^{2/3}+x^{1/3}+1} = \frac{1+1}{1+1+1} = \frac{2}{3}$

18. $\lim\limits_{x\to 64}\frac{x^{2/3}-16}{\sqrt{x}-8}=\lim\limits_{x\to 64}\frac{(x^{1/3}-4)(x^{1/3}+4)}{\sqrt{x}-8}=\lim\limits_{x\to 64}\frac{(x^{1/3}-4)(x^{1/3}+4)}{\sqrt{x}-8}\cdot\frac{(x^{2/3}+4x^{1/3}+16)(\sqrt{x}+8)}{(\sqrt{x}+8)(x^{2/3}+4x^{1/3}+16)}$
$=\lim\limits_{x\to 64}\frac{(x-64)\,(x^{1/3}+4)\,(\sqrt{x}+8)}{(x-64)\,(x^{2/3}+4x^{1/3}+16)}=\lim\limits_{x\to 64}\frac{(x^{1/3}+4)\,(\sqrt{x}+8)}{x^{2/3}+4x^{1/3}+16}=\frac{(4+4)\,(8+8)}{16+16+16}=\frac{8}{3}$

19. $\lim\limits_{x\to 0}\frac{\tan 2x}{\tan \pi x}=\lim\limits_{x\to 0}\frac{\sin\ 2x}{\cos\ 2x}\cdot\frac{\cos \pi x}{\sin \pi x}=\lim\limits_{x\to 0}\left(\frac{\sin\ 2x}{2x}\right)\left(\frac{\cos \pi x}{\cos\ 2x}\right)\left(\frac{\pi x}{\sin \pi x}\right)\left(\frac{2x}{\pi x}\right)=1\cdot 1\cdot 1\cdot\frac{2}{\pi}=\frac{2}{\pi}$

20. $\lim\limits_{x\to\pi^-}\csc\ x=\lim\limits_{x\to\pi^-}\frac{1}{\sin\ x}=\infty$

21. $\lim\limits_{x\to\pi}\sin\left(\frac{x}{2}+\sin\ x\right)=\sin\left(\frac{\pi}{2}+\sin\ \pi\right)=\sin\left(\frac{\pi}{2}\right)=1$

22. $\lim\limits_{x\to\pi}\cos^2(x-\tan x)=\cos^2\ (\pi-\tan\pi)\ =\ \cos^2(\pi)=(-1)^2\ =1$

23. $\lim\limits_{x\to 0}\frac{8x}{3\sin\ x-x}=\lim\limits_{x\to 0}\frac{8}{3\frac{\sin x}{x}-1}=\frac{8}{3(1)-1}=4$

24. $\lim\limits_{x\to 0}=\frac{\cos\ 2x-1}{\sin x}=\lim\limits_{x\to 0}\left(\frac{\cos\ 2x-1}{\sin x}\cdot\frac{\cos\ 2x\ +1}{\cos\ 2x+1}\right)=\lim\limits_{x\to 0}\frac{\cos^2\ 2x-1}{\sin x(\cos\ 2x+1)}=\lim\limits_{x\to 0}\frac{-\sin^2 2x}{\sin\ x(\cos\ 2x+1)}$
$=\lim\limits_{x\to 0}\frac{-4\sin x\cos^2 x}{\cos\ 2x+1}=\frac{-4(0)(1)^2}{1+1}=0$

25. $\lim\limits_{x\to 0^+}[4\,g(x)]^{1/3}=2\Rightarrow\left[\lim\limits_{x\to 0^+}4\,g(x)\right]^{1/3}=2\Rightarrow\lim\limits_{x\to 0^+}4\,g(x)=8$, since $2^3=8$. Then $\lim\limits_{x\to 0^+}g(x)=2$.

26. $\lim\limits_{x\to\sqrt{5}}\frac{1}{x+g(x)}=2\Rightarrow\lim\limits_{x\to\sqrt{5}}(x+g(x))=\frac{1}{2}\Rightarrow\sqrt{5}+\lim\limits_{x\to\sqrt{5}}g(x)=\frac{1}{2}\Rightarrow\lim\limits_{x\to\sqrt{5}}g(x)=\frac{1}{2}-\sqrt{5}$

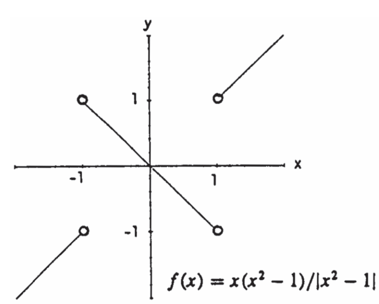
27. $\lim\limits_{x\to 1}\frac{3x^2+1}{g(x)}=\infty\Rightarrow\lim\limits_{x\to 1}g(x)=0$ since $\lim\limits_{x\to 1}(3x^2+1)=4$

28. $\lim\limits_{x\to -2}\frac{5-x^2}{\sqrt{g(x)}}=0\Rightarrow\lim\limits_{x\to -2}g(x)=\infty$ since $\lim\limits_{x\to -2}(5-x^2)=1$

29.(a) $f(-1)=-1$ and $f(2)=5\Rightarrow f$ has a root between -1 and 2 by the Intermediate Value Theorem.
(b), (c) root is 1.32471795724

30. (a) $f(-2)=-2$ and $f(0)=2\Rightarrow f$ has a root between -2 and 0 by the Intermediate Value Theorem.
(b), (c) root is -1.76929235424

31. At $x=-1$: $\lim\limits_{x\to -1^-}f(x)=\lim\limits_{x\to -1^-}\frac{x(x^2-1)}{|x^2-1|}$
$=\lim\limits_{x\to -1^-}\frac{x(x^2-1)}{x^2-1}=\lim\limits_{x\to -1^-}x=-1$, and
$\lim\limits_{x\to -1^+}f(x)=\lim\limits_{x\to -1^+}\frac{x(x^2-1)}{|x^2-1|}=\lim\limits_{x\to -1^+}\frac{x(x^2-1)}{-(x^2-1)}$
$=\lim\limits_{x\to -1}(-x)=-(-1)=1$. Since $\lim\limits_{x\to -1^-}f(x)\neq\lim\limits_{x\to -1^+}f(x)\Rightarrow\lim\limits_{x\to -1}f(x)$ does not exist, the function f cannot be extended to a continuous function at $x=-1$.



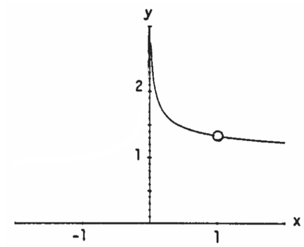
At $x=1$: $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \frac{x(x^2-1)}{|x^2-1|} = \lim\limits_{x\to 1^-} \frac{x(x^2-1)}{-(x^2-1)} = \lim\limits_{x\to 1^-} (-x) = -1$, and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \frac{x(x^2-1)}{|x^2-1|}$

$= \lim\limits_{x\to 1^+} \frac{x(x^2-1)}{x^2-1} = \lim\limits_{x\to 1^+} x = 1.$

Again $\lim\limits_{x\to 1} f(x)$ does not exist so f cannot be extended to a continuous function at $x=1$ either.

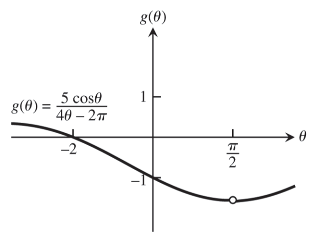
32. The discontinuity at $x=0$ of $f(x) = \sin\left(\frac{1}{x}\right)$ is nonremovable because $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist.

33. Yes, f does have a continuous extension at $a=1$: define $f(1) = \lim\limits_{x\to 1} \frac{x-1}{x-\sqrt[4]{x}} = \frac{4}{3}$.

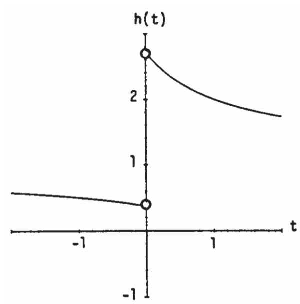


$f(x) = \frac{x-1}{x-\sqrt[4]{x}}, \quad a = 1$

34. Yes, g does have a continuous extension at $a = \frac{\pi}{2}$: $g\left(\frac{\pi}{2}\right) = \lim\limits_{\theta\to\frac{\pi}{2}} \frac{5\cos\theta}{4\theta-2\pi} = -\frac{5}{4}$.

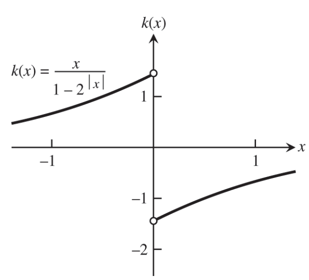


35. From the graph we see that $\lim\limits_{t\to 0^-} h(t) \neq \lim\limits_{t\to 0^+} h(t)$ so h cannot be extended to a continuous function at $a=0$.



$h(t) = (1+|t|)^{1/t}, \quad a = 0$

36. From the graph we see that $\lim\limits_{x\to 0^-} k(x) \neq \lim\limits_{x\to 0^+} k(x)$ so k cannot be extended to a continuous function at $a=0$.



37. $\lim\limits_{x\to\infty} \frac{2x+3}{5x+7} = \lim\limits_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2+0}{5+0} = \frac{2}{5}$

38. $\lim\limits_{x\to-\infty} \frac{2x^2+3}{5x^2+7} = \lim\limits_{x\to-\infty} \frac{2+\frac{3}{x^2}}{5+\frac{7}{x^2}} = \frac{2+0}{5+0} = \frac{2}{5}$

39. $\lim\limits_{x\to-\infty} \frac{x^2-4x+8}{3x^3} = \lim\limits_{x\to-\infty} \left(\frac{1}{3x} - \frac{4}{3x^2} + \frac{8}{3x^3}\right) = 0-0+0 = 0$

40. $\lim\limits_{x\to\infty} \frac{1}{x^2-7x+1} = \lim\limits_{x\to\infty} \frac{\frac{1}{x^2}}{1-\frac{7}{x}+\frac{1}{x^2}} = \frac{0}{1-0+0} = 0$

41. $\lim\limits_{x\to-\infty} \frac{x^2-7x}{x+1} = \lim\limits_{x\to-\infty} \frac{x-7}{1+\frac{1}{x}} = -\infty$

42. $\lim\limits_{x\to\infty} \frac{x^4+x^3}{12x^3+128} = \lim\limits_{x\to-\infty} \frac{x+1}{12+\frac{128}{x^3}} = \infty$

43. $\lim\limits_{x\to\infty} \frac{\sin x}{\lfloor x \rfloor} \le \lim\limits_{x\to\infty} \frac{1}{\lfloor x \rfloor} = 0$ since $\lfloor x \rfloor \to \infty$ as $x \to \infty \Rightarrow \lim\limits_{x\to\infty} \frac{\sin x}{\lfloor x \rfloor} = 0.$

44. $\lim\limits_{\theta\to\infty} \frac{\cos\,\theta-1}{\theta} \le \lim\limits_{\theta\to\infty} \frac{2}{\theta} = 0 \Rightarrow \lim\limits_{\theta\to\infty} \frac{\cos\theta-1}{\theta} = 0.$

45. $\lim\limits_{x\to\infty} \frac{x+\sin\,x+2\sqrt{x}}{x+\sin\,x} = \lim\limits_{x\to\infty} \frac{1+\frac{\sin x}{x}+\frac{2}{\sqrt{x}}}{1+\frac{\sin x}{x}} = \frac{1+0+0}{1+0} = 1$

46. $\lim\limits_{x\to\infty} \frac{x^{2/3}+x^{-1}}{x^{2/3}+\cos^2 x} = \lim\limits_{x\to\infty} \left(\frac{1+x^{-5/3}}{1+\frac{\cos^2 x}{x^{2/3}}}\right) = \frac{1+0}{1+0} = 1$

47. (a) $y = \frac{x^2+4}{x-3}$ is undefined at $x=3$: $\lim\limits_{x\to3^-} \frac{x^2+4}{x-3} = -\infty$ and $\lim\limits_{x\to3^+} \frac{x^2+4}{x-3} = +\infty$, thus $x=3$ is a vertical asymptote.

(b) $y = \frac{x^2-x-2}{x^2-2x+1}$ is undefined at $x=1$: $\lim\limits_{x\to1^-} \frac{x^2-x-2}{x^2-2x+1} = -\infty$ and $\lim\limits_{x\to1^+} \frac{x^2-x-2}{x^2-2x+1} = -\infty$, thus $x=1$ is a vertical asymptote.

(c) $y = \frac{x^2+x-6}{x^2+2x-8}$ is undefined at $x=2$ and -4: $\lim\limits_{x\to2} \frac{x^2+x-6}{x^2+2x-8} = \lim\limits_{x\to2} \frac{x+3}{x+4} = \frac{5}{6}$; $\lim\limits_{x\to-4^-} \frac{x^2+x-6}{x^2+2x-8} = \lim\limits_{x\to-4^-} \frac{x+3}{x+4} = \infty$ $\lim\limits_{x\to-4^+} \frac{x^2+x-6}{x^2+2x-8} = \lim\limits_{x\to-4^+} \frac{x+3}{x+4} = -\infty$. Thus $x=-4$ is a vertical asymptote.

48. (a) $y = \frac{1-x^2}{x^2+1}$: $\lim\limits_{x\to\infty} \frac{1-x^2}{x^2+1} = \lim\limits_{x\to\infty} \frac{\frac{1}{x^2}-1}{1+\frac{1}{x^2}} = \frac{-1}{1} = -1$ and $\lim\limits_{x\to-\infty} \frac{1-x^2}{x^2+1} = \lim\limits_{x\to-\infty} \frac{\frac{1}{x^2}-1}{1+\frac{1}{x^2}} = \frac{-1}{1} = -1$, thus $y=-1$ is a horizontal asymptote.

(b) $y = \frac{\sqrt{x}+4}{\sqrt{x+4}}$: $\lim\limits_{x\to\infty} \frac{\sqrt{x}+4}{\sqrt{x+4}} = \lim\limits_{x\to\infty} \frac{1+\frac{4}{\sqrt{x}}}{\sqrt{1+\frac{4}{x}}} = \frac{1+0}{\sqrt{1+0}} = 1$, thus $y=1$ is a horizontal asymptote.

(c) $y = \frac{\sqrt{x^2+4}}{x}$: $\lim\limits_{x\to\infty} \frac{\sqrt{x^2+4}}{x} = \lim\limits_{x\to\infty} \frac{\sqrt{1+\frac{4}{x^2}}}{1} = \frac{\sqrt{1+0}}{1} = 1$ and $\lim\limits_{x\to-\infty} \frac{\sqrt{x^2+4}}{x} = \lim\limits_{x\to-\infty} \frac{\sqrt{1+\frac{4}{x^2}}}{\frac{x}{\sqrt{x^2}}}$

$$= \lim_{x\to-\infty} \frac{\sqrt{1+\frac{4}{x^2}}}{\frac{x}{-x}} = \lim_{x\to-\infty} \frac{\sqrt{1+\frac{4}{x^2}}}{-1} = \frac{\sqrt{1+0}}{-1} = \frac{1}{-1} = -1,$$

thus $y=1$ and $y=-1$ are horizontal asymptotes.

(d) $y = \sqrt{\frac{x^2+9}{9x^2+1}}$: $\lim\limits_{x\to\infty} \sqrt{\frac{x^2+9}{9x^2+1}} = \lim\limits_{x\to\infty} \sqrt{\frac{1+\frac{9}{x^2}}{9+\frac{1}{x^2}}} = \sqrt{\frac{1+0}{9+0}} = \frac{1}{3}$ and $\lim\limits_{x\to-\infty} \sqrt{\frac{x^2+9}{9x^2+1}} = \lim\limits_{x\to-\infty} \sqrt{\frac{1+\frac{9}{x^2}}{9+\frac{1}{x^2}}} = \sqrt{\frac{1+0}{9+0}} = \frac{1}{3}$, thus $y = \frac{1}{3}$ is a horizontal asymptote.

49. domain $=[-4,2)\cup(2,4]$; y in range and $y=\frac{\sqrt{16-x^2}}{x-2}$, if $x=\pm 4$, then $y=0$, $\lim\limits_{x\to 2^+}\frac{\sqrt{16-x^2}}{x-2}=\infty$, and $\lim\limits_{x\to 2^-}\frac{\sqrt{16-x^2}}{x-2}=-\infty$, $\Rightarrow$ range $=(-\infty,\infty)$

50. Since $\lim\limits_{x\to b^{\pm}}\frac{\sqrt{ax^2+4}}{x-b}=\pm\infty\Rightarrow$ vertical asymptote is $x=b$; $\lim\limits_{x\to\infty}\frac{\sqrt{ax^2+4}}{x-b}=\lim\limits_{x\to\infty}\frac{|x|}{x-b}\cdot\sqrt{a+\frac{4}{x^2}}$ $=\lim\limits_{x\to\infty}\frac{x}{x-b}\sqrt{a+\frac{4}{x^2}}=\sqrt{a}\Rightarrow$ horizontal asymptote is $y=\sqrt{a}$, $\lim\limits_{x\to-\infty}\frac{\sqrt{ax^2+4}}{x-b}$ $=\lim\limits_{x\to-\infty}\frac{|x|}{x-b}\cdot\sqrt{a+\frac{4}{x^2}}=\lim\limits_{x\to-\infty}\frac{-x}{x-b}\cdot\sqrt{a+\frac{4}{x^2}}=-\sqrt{a}\Rightarrow$ horizontal asymptote is $y=-\sqrt{a}$

CHAPTER 2 ADDITIONAL AND ADVANCED EXERCISES

1. $\lim\limits_{v\to c^-}L=\lim\limits_{v\to c^-}L_0\sqrt{1-\frac{v^2}{c^2}}=L_0\sqrt{1-\frac{\lim\limits_{v\to c^-}v^2}{c^2}}=L_0\sqrt{1-\frac{c^2}{c^2}}=0$

The left-hand limit was needed because the function L is undefined if $v>c$ (the rocket cannot move faster than the speed of light).

2. (a) $\left|\frac{\sqrt{x}}{2}-1\right|<0.2\Rightarrow -0.2<\frac{\sqrt{x}}{2}-1<0.2\Rightarrow 0.8<\frac{\sqrt{x}}{2}<1.2\Rightarrow 1.6<\sqrt{x}<2.4\Rightarrow 2.56<x<5.76.$

(b) $\left|\frac{\sqrt{x}}{2}-1\right|<0.1\Rightarrow -0.1<\frac{\sqrt{x}}{2}-1<0.1\Rightarrow 0.9<\frac{\sqrt{x}}{2}<1.1\Rightarrow 1.8<\sqrt{x}<2.2\Rightarrow 3.24<x<4.84.$

3. $|10+(t-70)\times 10^{-4}-10|<0.0005\ \Rightarrow|(t-70)\times 10^{-4}|<0.0005\Rightarrow -0.0005<(t-70)\times 10^{-4}<0.0005$ $\Rightarrow -5<t-70<5\Rightarrow 65°<t<75°\Rightarrow$ Within 5° F.

4. We want to know in what interval to hold values of h to make V satisfy the inequality $|V-1000|=|36\pi h-1000|\le 10$. To find out, we solve the inequality:
$|36\pi h-1000|\le 10\Rightarrow -10\le 36\pi h-1000\le 10\ \Rightarrow 990\le 36\pi h\le 1010\Rightarrow\frac{990}{36\pi}\le h\le\frac{1010}{36\pi}\Rightarrow 8.8\le h\le 8.9$
where 8.8 was rounded up, to be safe, and 8.9 was rounded down, to be safe.
The interval in which we should hold h is about $8.9-8.8=0.1$ cm wide (1 mm). With stripes 1 mm wide, we can expect to measure a liter of water with an accuracy of 1%, which is more than enough accuracy for cooking.

5. Show $\lim\limits_{x\to 1}f(x)=\lim\limits_{x\to 1}(x^2-7)=-6=f(1)$.
Step 1: $|(x^2-7)+6|<\epsilon\Rightarrow -\epsilon<x^2-1<\epsilon\Rightarrow 1-\epsilon<x^2<1+\epsilon\Rightarrow\sqrt{1-\epsilon}<x<\sqrt{1+\epsilon}.$
Step 2: $|x-1|<\delta\Rightarrow -\delta<x-1<\delta\Rightarrow -\delta+1<x<\delta+1.$
Then $-\delta+1=\sqrt{1-\epsilon}$ or $\delta+1=\sqrt{1+\epsilon}$. Choose $\delta=\min\left\{1-\sqrt{1-\epsilon},\sqrt{1+\epsilon}-1\right\}$, then $0<|x-1|<\delta\Rightarrow$ $|(x^2-7)-6|<\epsilon$ and $\lim\limits_{x\to -1}f(x)=-6$. By the continuity text, $f(x)$ is continuous at $x=1$.

6. Show $\lim\limits_{x\to\frac{1}{4}}g(x)=\lim\limits_{x\to\frac{1}{4}}\frac{1}{2x}=2=g\left(\frac{1}{4}\right)$.
Step 1: $\left|\frac{1}{2x}-2\right|<\epsilon\Rightarrow -\epsilon<\frac{1}{2x}-2<\epsilon\Rightarrow 2-\epsilon<\frac{1}{2x}<2+\epsilon\Rightarrow\frac{1}{4-2\epsilon}>x>\frac{1}{4+2\epsilon}.$
Step 2: $\left|x-\frac{1}{4}\right|<\delta\Rightarrow -\delta<x-\frac{1}{4}<\delta\Rightarrow -\delta+\frac{1}{4}<x<\delta+\frac{1}{4}.$
Then $-\delta+\frac{1}{4}=\frac{1}{4+2\epsilon}\Rightarrow\delta=\frac{1}{4}-\frac{1}{4+2\epsilon}=\frac{\epsilon}{4(2-\epsilon)}$, or $\delta+\frac{1}{4}=\frac{1}{4-2\epsilon}\Rightarrow\delta=\frac{1}{4-2\epsilon}-\frac{1}{4}=\frac{\epsilon}{4(2-\epsilon)}.$
Choose $\delta=\frac{\epsilon}{4(2+\epsilon)}$, the smaller of the two values. Then $0<\left|x-\frac{1}{4}\right|<\delta\Rightarrow\left|\frac{1}{2x}-2\right|<\epsilon$ and $\lim\limits_{x\to\frac{1}{4}}\frac{1}{2x}=2.$
By the continuity test, $g(x)$ is continuous at $x=\frac{1}{4}$.

7. Show $\lim_{x\to 2} h(x) = \lim_{x\to 2}\sqrt{2x-3} = 1 = h(2)$.
Step 1: $\left|\sqrt{2x-3}-1\right| < \epsilon \Rightarrow -\epsilon < \sqrt{2x-3}-1 < \epsilon \Rightarrow 1-\epsilon < \sqrt{2x-3} < 1+\epsilon \Rightarrow \frac{(1-\epsilon)^2+3}{2} < x < \frac{(1+\epsilon)^2+3}{2}$.
Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta$ or $-\delta+2 < x < \delta+2$.
Then $-\delta+2 = \frac{(1-\epsilon)^2+3}{2} \Rightarrow \delta = 2-\frac{(1-\epsilon)^2+3}{2} = \frac{1-(1-\epsilon)^2}{2} = \epsilon-\frac{\epsilon^2}{2}$, or $\delta+2 = \frac{(1+\epsilon)^2+3}{2} \Rightarrow \delta = \frac{(1+\epsilon)^2+3}{2}-2 = \frac{(1+\epsilon)^2-1}{2} = \epsilon+\frac{\epsilon^2}{2}$. Choose $\delta = \epsilon-\frac{\epsilon^2}{2}$, the smaller of the two values. Then, $0 < |x-2| < \delta \Rightarrow \left|\sqrt{2x-3}-1\right| < \epsilon$, so $\lim_{x\to 2}\sqrt{2x-3} = 1$. By the continuity test, $h(x)$ is continuous at $x = 2$.

8. Show $\lim_{x\to 5} F(x) = \lim_{x\to 5}\sqrt{9-x} = 2 = F(5)$.
Step 1: $\left|\sqrt{9-x}-2\right| < \epsilon \Rightarrow -\epsilon < \sqrt{9-x}-2 < \epsilon \Rightarrow 9-(2-\epsilon)^2 > x > 9-(2+\epsilon)^2$.
Step 2: $0 < |x-5| < \delta \Rightarrow -\delta < x-5 < \delta \Rightarrow -\delta+5 < x < \delta+5$.
Then $-\delta+5 = 9-(2+\epsilon)^2 \Rightarrow \delta = (2+\epsilon)^2-4 = \epsilon^2+2\epsilon$, or $\delta+5 = 9-(2-\epsilon)^2 \Rightarrow \delta = 4-(2-\epsilon)^2 = \epsilon^2-2\epsilon$.
Choose $\delta = \epsilon^2-2\epsilon$, the smaller of the two values. Then, $0 < |x-5| < \delta \Rightarrow \left|\sqrt{9-x}-2\right| < \epsilon$, so $\lim_{x\to 5}\sqrt{9-x} = 2$.
By the continuity test, $F(x)$ is continuous at $x = 5$.

9. Suppose L_1 and L_2 are two different limits. Without loss of generality assume $L_2 > L_1$. Let $\epsilon = \frac{1}{3}(L_2-L_1)$. Since $\lim_{x\to x_0} f(x) = L_1$ there is a $\delta_1 > 0$ such that $0 < |x-x_0| < \delta_1 \Rightarrow |f(x)-L_1| < \epsilon \Rightarrow -\epsilon < f(x)-L_1 < \epsilon$
$\Rightarrow -\frac{1}{3}(L_2-L_1)+L_1 < f(x) < \frac{1}{3}(L_2-L_1)+L_1 \Rightarrow 4L_1-L_2 < 3f(x) < 2L_1+L_2$. Likewise, $\lim_{x\to x_0} f(x) = L_2$ so there is a δ_2 such that $0 < |x-x_0| < \delta_2 \Rightarrow |f(x)-L_2| < \epsilon \Rightarrow -\epsilon < f(x)-L_2 < \epsilon$
$\Rightarrow -\frac{1}{3}(L_2-L_1)+L_2 < f(x) < \frac{1}{3}(L_2-L_1)+L_2 \Rightarrow 2L_2+L_1 < 3f(x) < 4L_2-L_1 \Rightarrow L_1-4L_2 < -3f(x) < -2L_2-L_1$.
If $\delta = \min\{\delta_1, \delta_2\}$ both inequalities must hold for $0 < |x-x_0| < \delta$:
$$\left.\begin{aligned}4L_1-L_2 < 3f(x) &< 2L_1+L_2\\ L_1-4L_2 < -3f(x) &< -2L_2-L_1\end{aligned}\right\} \Rightarrow 5(L_1-L_2) < 0 < L_1-L_2.$$
That is, $L_1-L_2 < 0$ <u>and</u> $L_1-L_2 > 0$, a contradiction.

10. Suppose $\lim_{x\to c} f(x) = L$. If $k = 0$, then $\lim_{x\to c} kf(x) = \lim_{x\to c} 0 = 0 = 0\cdot\lim_{x\to c} f(x)$ and we are done. If $k \neq 0$, then given any $\epsilon > 0$, there is a $\delta > 0$ so that $0 < |x-c| < \delta < \Rightarrow |f(x)-L| < \frac{\epsilon}{|k|} \Rightarrow |k||f(x)-L| < \epsilon \Rightarrow |k(f(x)-L)| < \epsilon$
$\Rightarrow |(kf(x))-(kL)| < \epsilon$. Thus $\lim_{x\to c} kf(x) = kL = k\left(\lim_{x\to c} f(x)\right)$.

11. (a) Since $x \to 0^+$, $0 < x^3 < x < 1 \Rightarrow (x^3-x) \to 0^- \Rightarrow \lim_{x\to 0^+} f(x^3-x) = \lim_{y\to 0^-} f(y) = B$ where $y = x^3-x$.
(b) Since $x \to 0^-$, $-1 < x < x^3 < 0 \Rightarrow (x^3-x) \to 0^+ \Rightarrow \lim_{x\to 0^-} f(x^3-x) = \lim_{y\to 0^+} f(y) = A$ where $y = x^3-x$.
(c) Since $x \to 0^+$, $0 < x^4 < x^2 < 1 \Rightarrow (x^2-x^4) \to 0^+ \Rightarrow \lim_{x\to 0^+} f(x^2-x^4) = \lim_{y\to 0^+} f(y) = A$ where $y = x^2-x^4$.
(d) Since $x \to 0^-$, $-1 < x < 0 \Rightarrow 0 < x^4 < x^2 < 1 \Rightarrow (x^2-x^4) \to 0^+ \Rightarrow \lim_{x\to 0^+} f(x^2-x^4) = A$ as in part (c).

12. (a) True, because if $\lim_{x\to a}(f(x)+g(x))$ exists then $\lim_{x\to a}(f(x)+g(x)) - \lim_{x\to a} f(x) = \lim_{x\to a}[(f(x)+g(x))-f(x)] = \lim_{x\to a} g(x)$ exists, contrary to assumption.
(b) False; for example take $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$. Then neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists, but $\lim_{x\to 0}(f(x)+g(x)) = \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x}\right) = \lim_{x\to 0} 0 = 0$ exists.

(c) True, because $g(x) = |x|$ is continuous $\Rightarrow g(f(x)) = |f(x)|$ is continuous (it is the composite of continuous functions).

(d) False; for example let $f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases} \Rightarrow f(x)$ is discontinuous at $x = 0$. However $|f(x)| = 1$ is continuous at $x = 0$.

13. Show $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \frac{x^2-1}{x+1} = \lim\limits_{x \to -1} \frac{(x+1)(x-1)}{(x+1)} = -2,\ x \ne -1.$

Define the continuous extension of $f(x)$ as $F(x) = \begin{cases} \frac{x^2-1}{x+1}, & x \ne -1 \\ -2\ , & x = -1 \end{cases}$. We now prove the limit of $f(x)$ as $x \to -1$ exists and has the correct value.

Step 1: $\left|\frac{x^2-1}{x+1} - (-2)\right| < \epsilon \Rightarrow -\epsilon < \frac{(x+1)(x-1)}{(x+1)} + 2 < \epsilon \Rightarrow -\epsilon < (x-1) + 2 < \epsilon,\ x \ne -1 \Rightarrow -\epsilon - 1 < x < \epsilon - 1.$

Step 2: $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1.$

Then $-\delta - 1 = -\epsilon - 1 \Rightarrow \delta = \epsilon$, or $\delta - 1 = \epsilon - 1 \Rightarrow \delta = \epsilon$. Choose $\delta = \epsilon$. Then $0 < |x-(-1)| < \delta$ $\Rightarrow \left|\frac{x^2-1}{x+1} - (-2)\right| < \epsilon \Rightarrow \lim\limits_{x \to -1} F(x) = -2$. Since the conditions of the continuity test are met by $F(x)$, then $f(x)$ has a continuous extension to $F(x)$ at $x = -1$.

14. Show $\lim\limits_{x \to 3} g(x) = \lim\limits_{x \to 3} \frac{x^2-2x-3}{2x-6} = \lim\limits_{x \to 3} \frac{(x-3)(x+1)}{2(x-3)} = 2,\ x \ne 3.$

Define the continuous extension of $g(x)$ as $G(x) = \begin{cases} \frac{x^2-2x-3}{2x-6}, & x \ne 3 \\ 2\ , & x = 3 \end{cases}$. We now prove the limit of $g(x)$ as $x \to 3$ exists and has the correct value.

Step 1: $\left|\frac{x^2-2x-3}{2x-6} - 2\right| < \epsilon \Rightarrow -\epsilon < \frac{(x-3)(x+1)}{2(x-3)} - 2 < \epsilon \Rightarrow -\epsilon < \frac{x+1}{2} - 2 < \epsilon,\ x \ne 3 \Rightarrow 3 - 2\epsilon < x < 3 + 2\epsilon.$

Step 2: $|x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow 3 - \delta < x < \delta + 3.$

Then, $3 - \delta = 3 - 2\epsilon \Rightarrow \delta = 2\epsilon$, or $\delta + 3 = 3 + 2\epsilon \Rightarrow \delta = 2\epsilon$. Choose $\delta = 2\epsilon$. Then $0 < |x-3| < \delta$ $\Rightarrow \left|\frac{x^2-2x-3}{2x-6} - 2\right| < \epsilon \Rightarrow \lim\limits_{x \to 3} \frac{(x-3)(x+1)}{2(x-3)} = 2$. Since the conditions of the continuity test hold for $G(x)$, $g(x)$ can be continuously extended to $G(x)$ at $x = 3$.

15. (a) Let $\epsilon > 0$ be given. If x is rational, then $f(x) = x \Rightarrow |f(x) - 0| = |x - 0| < \epsilon \Leftrightarrow |x - 0| < \epsilon$; i.e., choose $\delta = \epsilon$. Then $|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$ for x rational. If x is irrational, then $f(x) = 0 \Rightarrow |f(x) - 0| < \epsilon \Leftrightarrow 0 < \epsilon$ which is true no matter how close irrational x is to 0, so again we can choose $\delta = \epsilon$. In either case, given $\epsilon > 0$ there is a $\delta = \epsilon > 0$ such that $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$. Therefore, f is continuous at $x = 0$.

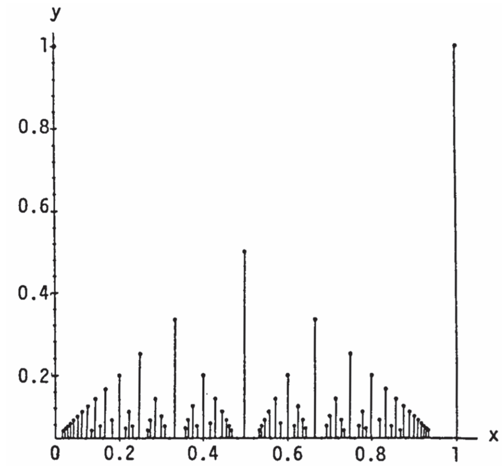
(b) Choose $x = c > 0$. Then within any interval $(c - \delta, c + \delta)$ there are both rational and irrational numbers. If c is rational, pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is an irrational number x in $(c - \delta, c + \delta) \Rightarrow |f(x) - f(c)| = |0 - c| = c > \frac{c}{2} = \epsilon$. That is, f is not continuous at any rational $c > 0$. On the other hand, suppose c is irrational $\Rightarrow f(c) = 0$. Again pick $\epsilon = \frac{c}{2}$. No matter how small we choose $\delta > 0$ there is a rational number x in $(c - \delta, c + \delta)$ with $|x - c| < \frac{c}{2} = \varepsilon \Leftrightarrow \frac{c}{2} < x < \frac{3c}{2}$. Then $|f(x) - f(c)| = |x - 0|$ $= |x| > \frac{c}{2} = \epsilon \Rightarrow f$ is not continuous at any irrational $c > 0$.

If $x = c < 0$, repeat the argument picking $\epsilon = \frac{|c|}{2} = \frac{-c}{2}$. Therefore f fails to be continuous at any nonzero value $x = c$.

16. (a) Let $c = \frac{m}{n}$ be a rational number in $[0, 1]$ reduced to lowest terms $\Rightarrow f(c) = \frac{1}{n}$. Pick $\epsilon = \frac{1}{2n}$. No matter how small $\delta > 0$ is taken, there is an irrational number x in the interval $(c - \delta, c + \delta) \Rightarrow |f(x) - f(c)|$ $= \left|0 - \frac{1}{n}\right| = \frac{1}{n} > \frac{1}{2n} = \epsilon$. Therefore f is discontinuous at $x = c$, a rational number.

(b) Now suppose c is an irrational number $\Rightarrow f(c)=0$. Let $\epsilon>0$ be given. Notice that $\frac{1}{2}$ is the only rational number reduced to lowest terms with denominator 2 and belonging to $[0,1]$; $\frac{1}{3}$ and $\frac{2}{3}$ the only rationals with denominator 3 belonging to $[0,1]$; $\frac{1}{4}$ and $\frac{3}{4}$ with denominator 4 in $[0,1]$; $\frac{1}{5},\frac{2}{5},\frac{3}{5}$ and $\frac{4}{5}$ with denominator 5 in $[0,1]$; etc. In general, choose N so that $\frac{1}{N}<\epsilon \Rightarrow$ there exist only finitely many rationals in $[0,1]$ having denominator $\le N$, say $r_1, r_2, \ldots, r_p$. Let $\delta=\min\{|c-r_i|: i=1,\ldots,p\}$. Then the interval $(c-\delta, c+\delta)$ contains no rational numbers with denominator $\le N$. Thus, $0<|x-c|<\delta \Rightarrow |f(x)-f(c)|=|f(x)-0|=|f(x)|\le\frac{1}{N}<\epsilon \Rightarrow f$ is continuous at $x=c$ irrational.

(c) The graph looks like the markings on a typical ruler when the points $(x, f(x))$ on the graph of $f(x)$ are connected to the x-axis with vertical lines.



$$f(x)=\begin{cases}1/n & \text{if } x=m/n \text{ is a rational number in lowest terms}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

17. Yes. Let R be the radius of the equator (earth) and suppose at a fixed instant of time we label noon as the zero point, 0, on the equator $\Rightarrow 0+\pi R$ represents the midnight point (at the same exact time). Suppose x_1 is a point on the equator "just after" noon $\Rightarrow x_1+\pi R$ is simultaneously "just after" midnight. It seems reasonable that the temperature T at a point just after noon is hotter than it would be at the diametrically opposite point just after midnight: That is, $T(x_1)-T(x_1+\pi R)>0$. At exactly the same moment in time pick x_2 to be a point just before midnight $\Rightarrow x_2+\pi R$ is just before noon. Then $T(x_2)-T(x_2+\pi R)<0$. Assuming the temperature function T is continuous along the equator (which is reasonable), the Intermediate Value Theorem says there is a point c between 0 (noon) and πR (simultaneously midnight) such that $T(c)-T(c+\pi R)=0$; i.e., there is always a pair of antipodal points on the earth's equator where the temperatures are the same.

18. $$\lim_{x\to c} f(x)g(x)=\lim_{x\to c}\tfrac{1}{4}\left[(f(x)+g(x))^2-(f(x)-g(x))^2\right]=\tfrac{1}{4}\left[\left(\lim_{x\to c}(f(x)+g(x))\right)^2-\left(\lim_{x\to c}(f(x)-g(x))\right)^2\right]$$
$$=\tfrac{1}{4}(3^2-(-1)^2)=2.$$

19. (a) At $x=0$: $\lim\limits_{a\to 0} r_+(a)=\lim\limits_{a\to 0}\frac{-1+\sqrt{1+a}}{a}=\lim\limits_{a\to 0}\left(\frac{-1+\sqrt{1+a}}{a}\right)\left(\frac{-1-\sqrt{1+a}}{-1-\sqrt{1+a}}\right)=\lim\limits_{a\to 0}\frac{1-(1+a)}{a(-1-\sqrt{1+a})}=\frac{-1}{-1-\sqrt{1+0}}=\frac{1}{2}$

At $x=-1$: $\lim\limits_{a\to -1^+} r_+(a)=\lim\limits_{a\to -1^+}\frac{1-(1+a)}{a(-1-\sqrt{1+a})}=\lim\limits_{a\to -1}\frac{-a}{a(-1-\sqrt{1+a})}=\frac{-1}{-1-\sqrt{0}}=1$

(b) At $x=0$: $\lim\limits_{a\to 0^-} r_-(a)=\lim\limits_{a\to 0^-}\frac{-1-\sqrt{1+a}}{a}=\lim\limits_{a\to 0^-}\left(\frac{-1-\sqrt{1+a}}{a}\right)\left(\frac{-1+\sqrt{1+a}}{-1+\sqrt{1+a}}\right)=\lim\limits_{a\to 0^-}\frac{1-(1+a)}{a(-1+\sqrt{1+a})}=\lim\limits_{a\to 0^-}\frac{-a}{a(-1+\sqrt{1+a})}$

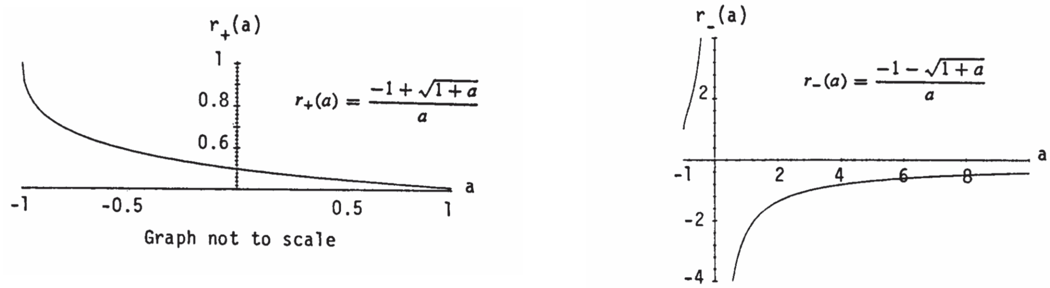
$=\lim\limits_{a\to 0^-}\frac{-1}{-1+\sqrt{1+a}}=\infty$ (because the denominator is always negative); $\lim\limits_{a\to 0^+} r_-(a)$

$=\lim\limits_{a\to 0^+}\frac{-1}{-1+\sqrt{1+a}}=-\infty$ (because the denominator is always positive).

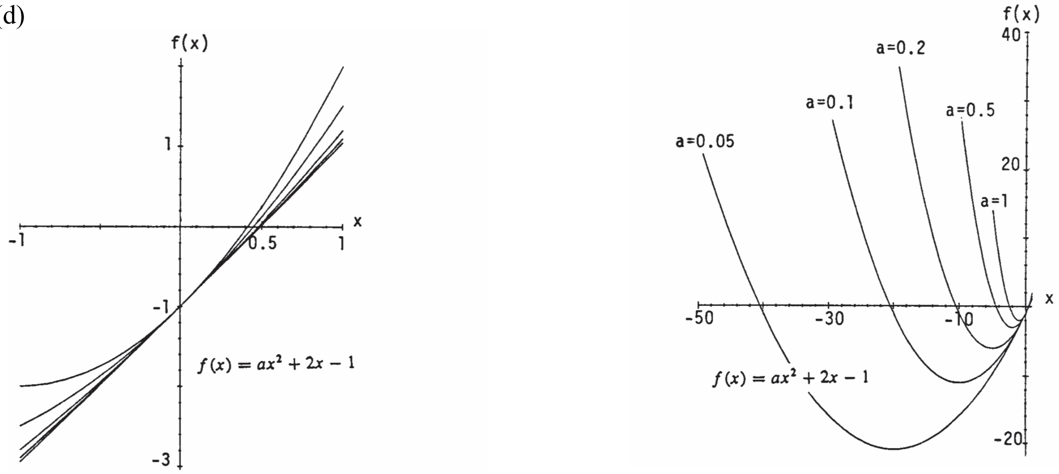
Therefore, $\lim\limits_{a\to 0} r_-(a)$ does not exist.

At $x=-1$: $\lim\limits_{a\to -1^+} r_-(a)=\lim\limits_{a\to -1^+}\frac{-1-\sqrt{1+a}}{a}=\lim\limits_{a\to -1^+}\frac{-1}{-1+\sqrt{1+a}}=1$

(c)



(d)



20. $f(x) = x + 2\cos x \Rightarrow f(0) = 0 + 2\cos 0 = 2 > 0$ and $f(-\pi) = -\pi + 2\cos(-\pi) = -\pi - 2 < 0$. Since $f(x)$ is continuous on $[-\pi, 0]$, by the Intermediate Value Theorem, $f(x)$ must take on every value between $[-\pi - 2, 2]$. Thus there is some number c in $[-\pi, 0]$ such that $f(c) = 0$; i.e., c is a solution to $x + 2\cos x = 0$.

21. (a) The function f is bounded on D if $f(x) \ge M$ and $f(x) \le N$ for all x in D. This means $M \le f(x) \le N$ for all x in D. Choose B to be max $\{|M|, |N|\}$. Then $|f(x)| \le B$. On the other hand, if $|f(x)| \le B$, then $-B \le f(x) \le B \Rightarrow f(x) \ge -B$ and $f(x) \le B \Rightarrow f(x)$ is bounded on D with $N = B$ an upper bound and $M = -B$ a lower bound.

(b) Assume $f(x) \le N$ for all x and that $L > N$. Let $\epsilon = \frac{L-N}{2}$. Since $\lim_{x \to x_0} f(x) = L$ there is a $\delta > 0$ such that $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon \Leftrightarrow L - \epsilon < f(x) < L + \epsilon \Leftrightarrow L - \frac{L-N}{2} < f(x) < L + \frac{L-N}{2} \Leftrightarrow \frac{L+N}{2} < f(x) < \frac{3L-N}{2}$. But $L > N \Rightarrow \frac{L+N}{2} > N \Rightarrow N < f(x)$ contrary to the boundedness assumption $f(x) \le N$. This contradiction proves $L \le N$.

(c) Assume $M \le f(x)$ for all x and that $L < M$. Let $\epsilon = \frac{M-L}{2}$. As in part (b), $0 < |x - x_0| < \delta \Rightarrow L - \frac{M-L}{2} < f(x) < L + \frac{M-L}{2} \Leftrightarrow \frac{3L-M}{2} < f(x) < \frac{M+L}{2} < M$, a contradiction.

22. (a) If $a \ge b$, then $a - b \ge 0 \Rightarrow |a - b| = a - b \Rightarrow \max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2} = \frac{a+b}{2} + \frac{a-b}{2} = \frac{2a}{2} = a$.

If $a \le b$, then $a - b \le 0 \Rightarrow |a - b| = -(a - b) = b - a \Rightarrow \max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2} = \frac{a+b}{2} + \frac{b-a}{2} = \frac{2b}{2} = b$.

(b) Let $\min\{a, b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$.

23. $\lim_{x \to 0} = \frac{\sin(1-\cos x)}{x} = \lim_{x \to 0} \frac{\sin(1-\cos x)}{1-\cos x} \cdot \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x \to 0} \frac{\sin(1-\cos x)}{1-\cos x} \cdot \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$

$= 1 \cdot \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} = 1 \cdot \left(\frac{0}{2}\right) = 0.$

24. $\lim\limits_{x\to 0^+}\frac{\sin x}{\sin\sqrt{x}} = \lim\limits_{x\to 0^+}\frac{\sin x}{x}\cdot\frac{\sqrt{x}}{\sin\sqrt{x}}\cdot\frac{x}{\sqrt{x}} = 1\cdot\lim\limits_{x\to 0^+}\frac{1}{\left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right)}\cdot\lim\limits_{x\to 0^+}\sqrt{x} = 1\cdot 1\cdot 0 = 0.$

25. $\lim\limits_{x\to 0}\frac{\sin(\sin x)}{x} = \lim\limits_{x\to 0}\frac{\sin(\sin x)}{\sin x}\cdot\frac{\sin x}{x} = \lim\limits_{x\to 0}\frac{\sin(\sin x)}{\sin x}\cdot\lim\limits_{x\to 0}\frac{\sin x}{x} = 1\cdot 1 = 1.$

26. $\lim\limits_{x\to 0}\frac{\sin(x^2+x)}{x} = \lim\limits_{x\to 0}\frac{\sin(x^2+x)}{x^2+x}\cdot(x+1) = \lim\limits_{x\to 0}\frac{\sin(x^2+x)}{x^2+x}\cdot\lim\limits_{x\to 0}(x+1) = 1\cdot 1 = 1.$

27. $\lim\limits_{x\to 2}\frac{\sin(x^2-4)}{x-2} = \lim\limits_{x\to 2}\frac{\sin(x^2-4)}{x^2-4}\cdot(x+2) = \lim\limits_{x\to 2}\frac{\sin(x^2-4)}{x^2-4}\cdot\lim\limits_{x\to 2}(x+2) = 1\cdot 4 = 4.$

28. $\lim\limits_{x\to 9}\frac{\sin(\sqrt{x}-3)}{x-9} = \lim\limits_{x\to 9}\frac{\sin(\sqrt{x}-3)}{\sqrt{x}-3}\cdot\frac{1}{\sqrt{x}+3} = \lim\limits_{x\to 9}\frac{\sin(\sqrt{x}-3)}{\sqrt{x}-3}\cdot\lim\limits_{x\to 9}\frac{1}{\sqrt{x}+3} = 1\cdot\frac{1}{6} = \frac{1}{6}.$

29. Since the highest power of x in the numerator is 1 more than the highest power of x in the denominator, there is an oblique asymptote. $y = \frac{2x^{3/2}+2x-3}{\sqrt{x}+1} = 2x - \frac{3}{\sqrt{x}+1}$, thus the oblique asymptote is $y = 2x$.

30. As $x\to\pm\infty$, $\frac{1}{x}\to 0 \Rightarrow \sin\left(\frac{1}{x}\right)\to 0 \Rightarrow 1+\sin\left(\frac{1}{x}\right)\to 1$, thus as $x\to\pm\infty$, $y = x + x\sin\left(\frac{1}{x}\right) = x\left(1+\sin\left(\frac{1}{x}\right)\right)\to x$; thus the oblique asymptote is $y = x$.

31. As $x\to\pm\infty$, $x^2+1\to x^2 \Rightarrow \sqrt{x^2+1}\to\sqrt{x^2}$; as $x\to-\infty$, $\sqrt{x^2} = -x$, and as $x\to+\infty$, $\sqrt{x^2} = x$; thus the oblique asymptotes are $y = x$ and $y = -x$.

32. As $x\to\pm\infty$, $x+2\to x \Rightarrow \sqrt{x^2+2x} = \sqrt{x(x+2)}\to\sqrt{x^2}$; as $x\to-\infty$, $\sqrt{x^2} = -x$, and as $x\to+\infty$, $\sqrt{x^2} = x$; asymptotes are $y = x$ and $y = -x$.

33. Assume $1<a<b$ and $\frac{a}{x}+x = \frac{1}{x-b} \Rightarrow a(x-b)+x^2(x-b) = x \Rightarrow f(x) = a(x-b)+x^2(x-b)-x = 0$; f is continuous for all x-values and $f(0) = -ab < 0$, $f(a+b) = a^2+(a+b)^2a-(a+b)$

$$= \underbrace{a^2-a}_{(+)} + \underbrace{(a+b)^2a-b}_{(+)} > 0.$$

Thus, by the Intermediate Value Theorem there is at least one number c, $0<c<a+b$, so that $f(c) = 0 \Rightarrow a(c-b)+c^2(c-b)-c = 0 \Rightarrow \frac{a}{c}+c = \frac{1}{c-b}$.

34. (a) $\lim\limits_{x\to 0}\frac{\sqrt{a+bx}-1}{x} = \frac{\text{"}\sqrt{a}-1\text{"}}{0}$, so $\sqrt{a}-1 = 0 \Rightarrow a = 1$, then $\lim\limits_{x\to 0}\frac{\sqrt{1+bx}-1}{x}\cdot\frac{\sqrt{1+bx}+1}{\sqrt{1+bx}+1} = \lim\limits_{x\to 0}\frac{(1+bx)-1}{x(\sqrt{1+bx}+1)}$

$= \lim\limits_{x\to 0}\frac{b}{\sqrt{1+bx}+1} = \frac{b}{2} = 2 \Rightarrow b = 4$

(b) $\lim\limits_{x\to 1}\frac{\tan(ax-a)+b-2}{x-1} = \frac{\text{"}\tan 0+b-2\text{"}}{0} = \frac{\text{"}b-2\text{"}}{0}$, so $b-2 = 0 \Rightarrow b = 2$, then $\lim\limits_{x\to 1}\frac{\tan(ax-a)}{x-1} = \lim\limits_{x\to 1} a\cdot\frac{\tan a(x-1)}{a(x-1)}$

$= \lim\limits_{x\to 1}\frac{a}{\cos a(x-1)}\cdot\frac{\sin a(x-1)}{a(x-1)} = \frac{a}{\cos 0}\cdot 1 = a = 3$

35. $\lim\limits_{x\to 1}\frac{x^{2/3}-1}{1-x^{1/2}} = \lim\limits_{x\to 1}\frac{(x^{1/6})^4-1^4}{1^3-(x^{1/6})^3} = \lim\limits_{x\to 1}\frac{(x^{1/6}-1)(x^{1/6}+1)((x^{1/6})^2+1)}{(1-x^{1/6})(1+(x^{1/6})+(x^{1/6})^2)} = \lim\limits_{x\to 1}\frac{-(x^{1/6}+1)(x^{1/3}+1)}{1+x^{1/6}+x^{1/3}} = \frac{-(2)(2)}{3} = \frac{-4}{3}$

36. $\lim\limits_{x\to 0^+} \frac{|3x+4|-|x|-4}{x} = \lim\limits_{x\to 0^+} \frac{(3x+4)-x-4}{x} = \lim\limits_{x\to 0^+} \frac{2x}{2} = 2$; assume $x > \frac{-3}{4} \Rightarrow \lim\limits_{x\to 0^-} \frac{|3x+4|-|x|-4}{x} = \lim\limits_{x\to 0^-} \frac{(3x+4)-(-x)-4}{x}$
$\lim\limits_{x\to 0^-} \frac{4x}{x} = 4 \Rightarrow \lim\limits_{x\to 0} \frac{|3x+4|-|x|-4}{x}$ does not exist.

37. (a) Domain $= \left\{0,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ldots\right\}$

(b) Consider any open interval (a, b) containing $c = 0$. Choose a positive integer N so that $\frac{1}{N} < b$. Then $x = \frac{1}{N}$ is in the domain and in the interval (a, b).

(c) $\lim\limits_{x\to 0} f(x) = 0$

38. (a) Domain $= \left\{0,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ldots\right\}$

(b) Consider any open interval (a, b) containing $c = 0$. Choose a positive integer N so that $\frac{1}{N} < b$. Then $x = \frac{1}{N}$ is in the domain and in the interval (a, b).

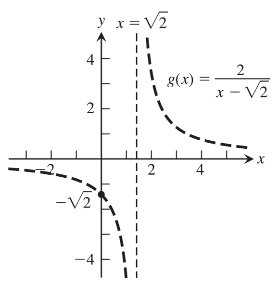
(c) $\lim\limits_{x\to 0} f(x) = 0$

39. (a) Domain
$= \left[\frac{1}{\pi},\ \infty\right) \cup \left[\frac{1}{3\pi},\ \frac{1}{2\pi}\right] \cup \left[\frac{1}{5\pi},\ \frac{1}{4\pi}\right] \cup \left[\frac{1}{7\pi},\ \frac{1}{6\pi}\right] \cup \cdots \cup \left[-\infty,\ \frac{-1}{\pi}\right) \cup \left[\frac{-1}{2\pi},\ \frac{-1}{3\pi}\right] \cup \left[\frac{-1}{4\pi},\ \frac{-1}{5\pi}\right] \cup \left[\frac{-1}{6\pi},\ \frac{-1}{7\pi}\right] \ldots$

(b) Consider any open interval (a, b) containing $c = 0$. Choose a positive integer N so that $\frac{1}{N\pi} < b$. Then $x = \frac{1}{N\pi}$ is in the domain and in the interval (a, b).

(c) $\lim\limits_{x\to 0} f(x) = 0$

40. (a)



(b) Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < |x| < \delta$ and x is in the domain of g, then $\left|\frac{2}{x-\sqrt{2}} + \sqrt{2}\right| < \varepsilon$.
Choose $\delta = \min\left\{\frac{1}{2}\varepsilon, \frac{1}{\sqrt{2}}\right\}$ so that $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} < \left|x - \sqrt{2}\right| < \frac{3}{\sqrt{2}}$, and $\left|\frac{2}{x-\sqrt{2}} + \sqrt{2}\right| = \frac{\sqrt{2}|x|}{|x-\sqrt{2}|} < \frac{\sqrt{2}\cdot\frac{1}{2}\varepsilon}{\frac{1}{\sqrt{2}}} = \varepsilon$.

(c) $\lim\limits_{x\to 0} g(x) = -\sqrt{2} = g(0)$ so g is continuous at $x = 0$.

(d) Function g is continuous at every point of its domain.

Chapter 2

Limits and Continuity

Section 2.1

Rates of Change and Tangents Lines to Curves

Average Speed

When $f(t)$ measures the distance traveled at time t,

$$\text{Average speed over } [t_1, t_2] = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

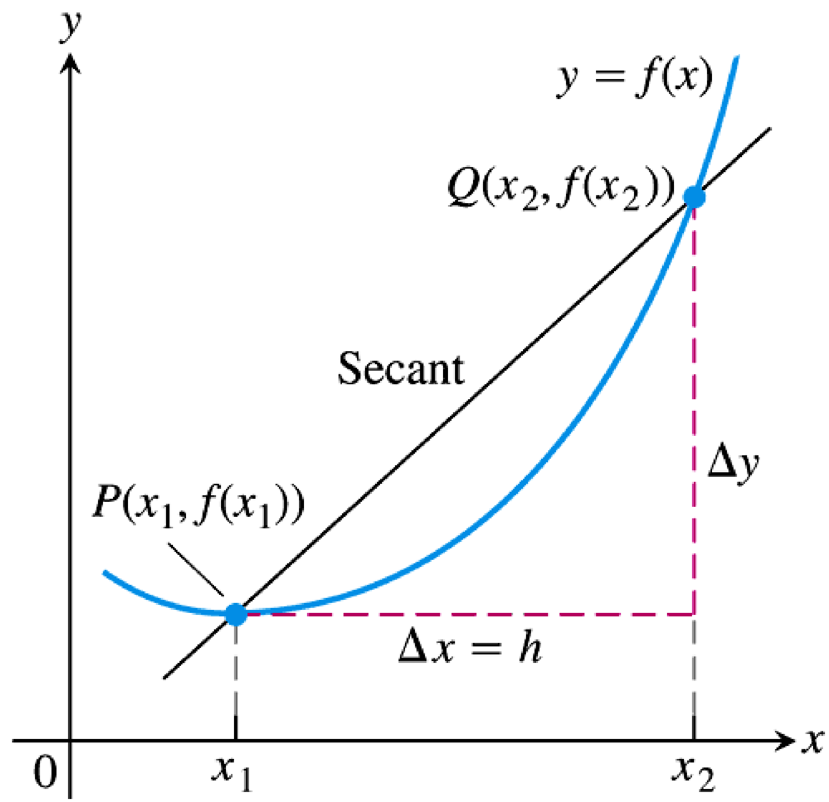


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y / \Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

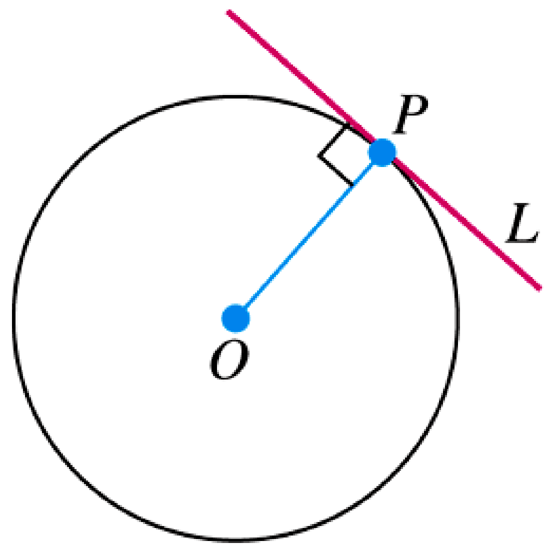


FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

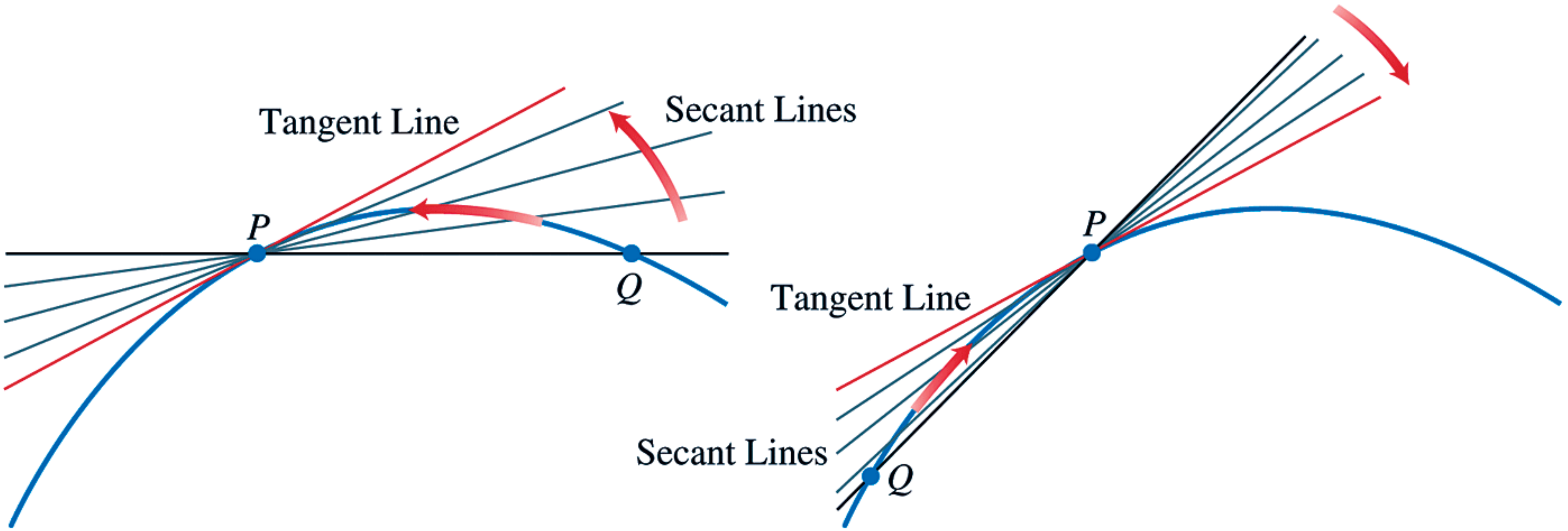


FIGURE 2.3 The tangent line to the curve at P is the line through P whose slope is the limit of the secant line slopes as $Q \to P$ from either side.

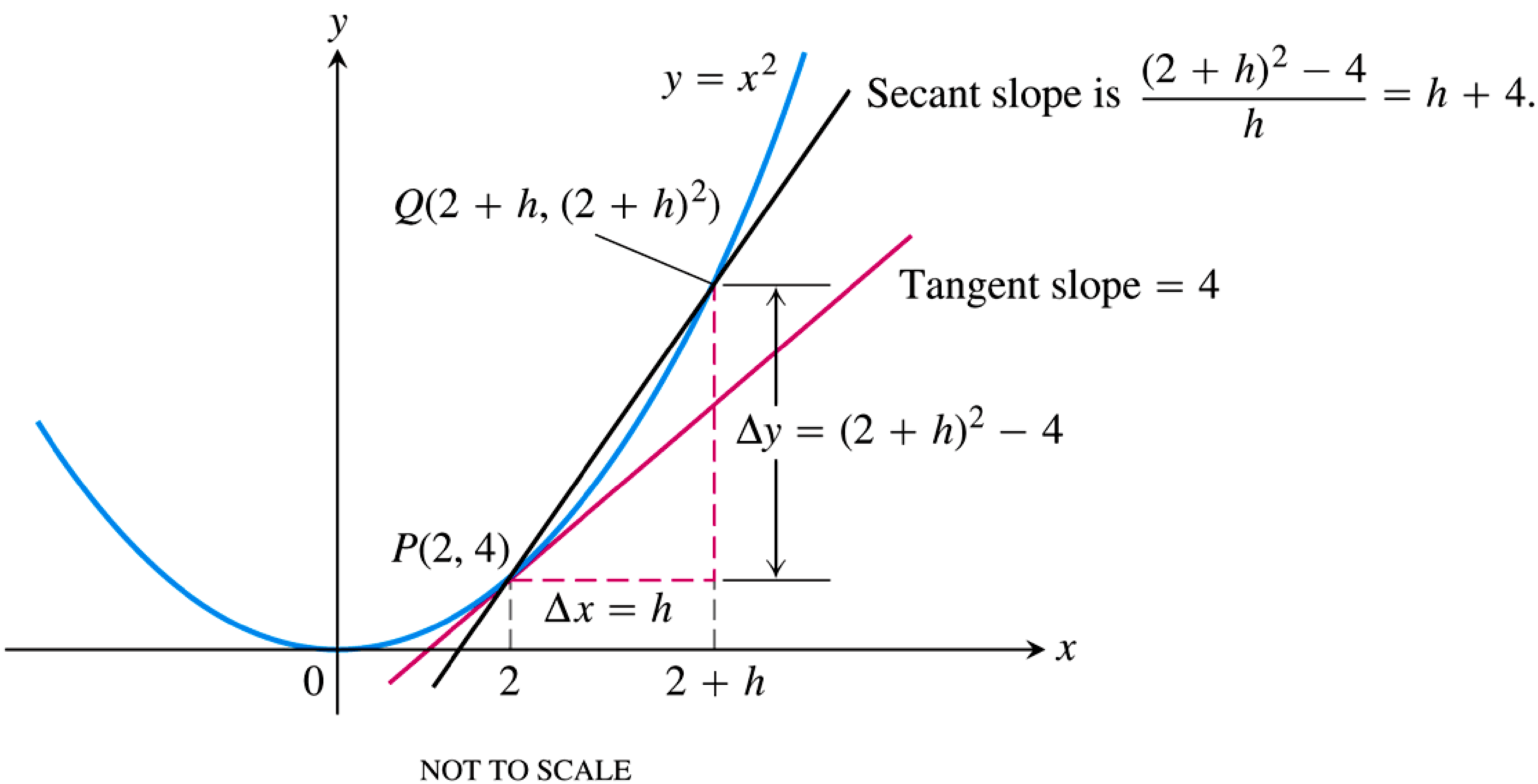


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant slopes (Example 3).

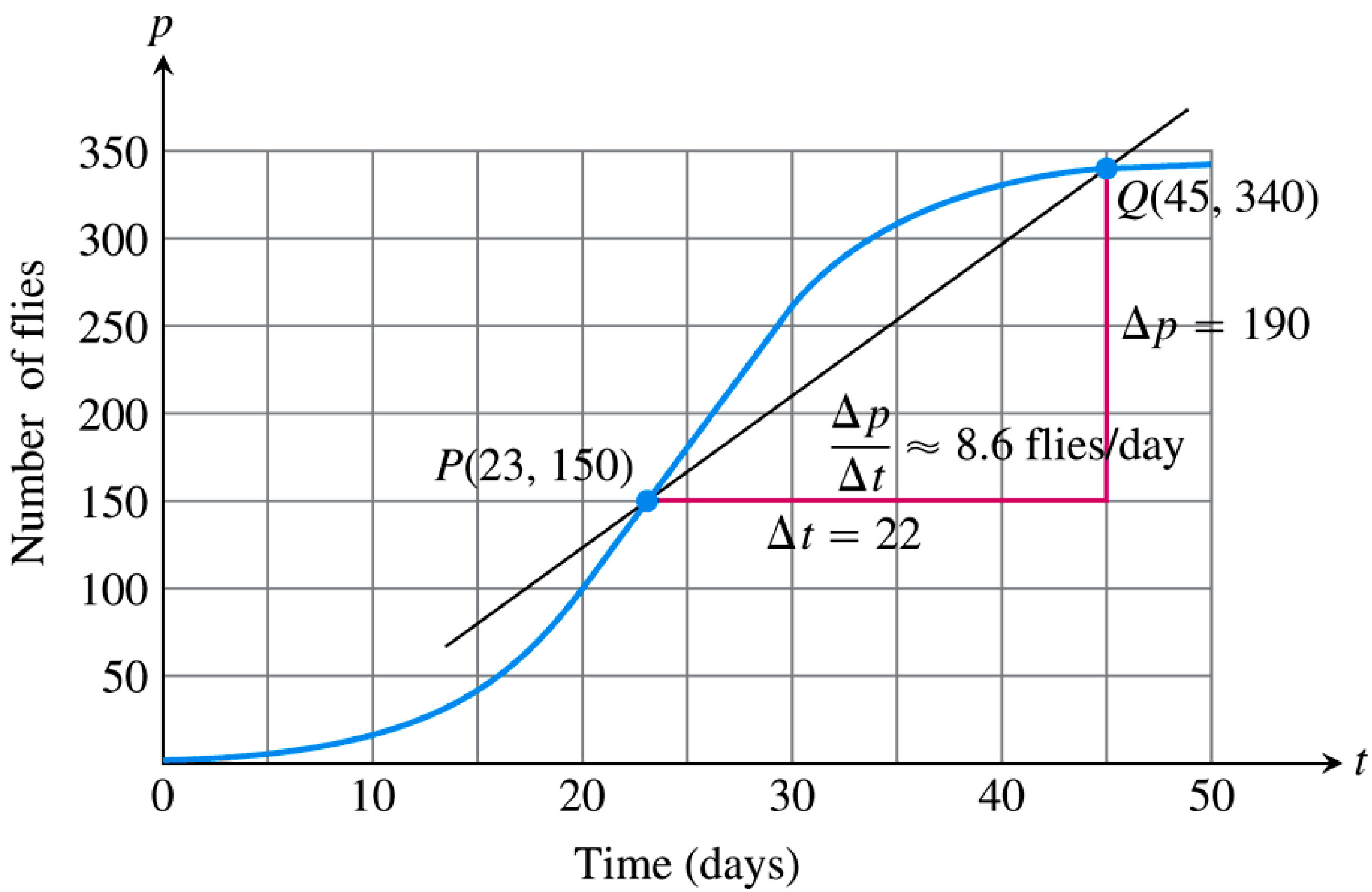


FIGURE 2.5 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p / \Delta t$ of the secant line (Example 4).

Q	Slope of $PQ = \Delta p/\Delta t$ (flies / day)
(45, 340)	$\dfrac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\dfrac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\dfrac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\dfrac{265 - 150}{30 - 23} \approx 16.4$

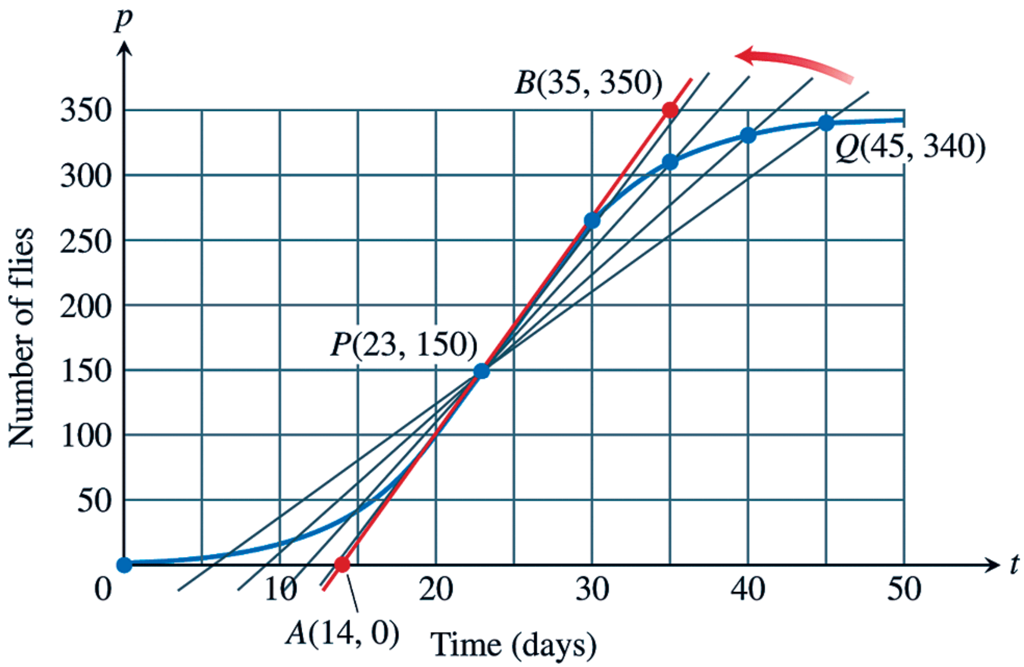


FIGURE 2.6 The positions and slopes of four secant lines through the point P on the fruit fly graph (Example 5).

Section 2.2

Limit of a Function and Limit Laws

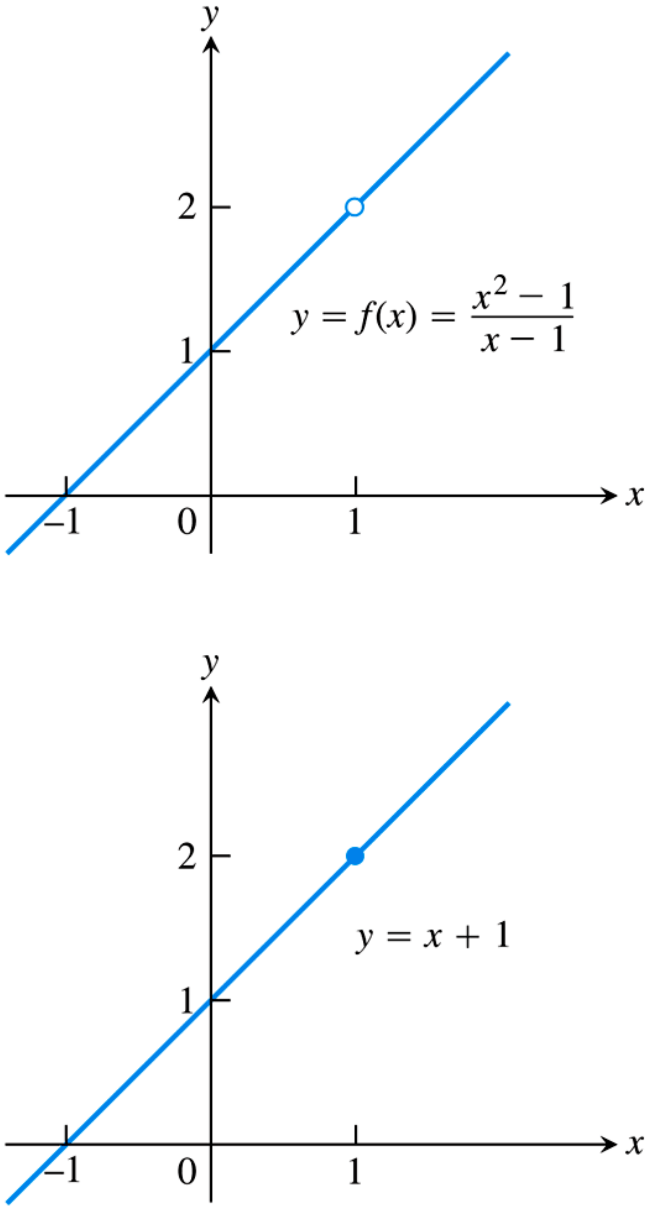
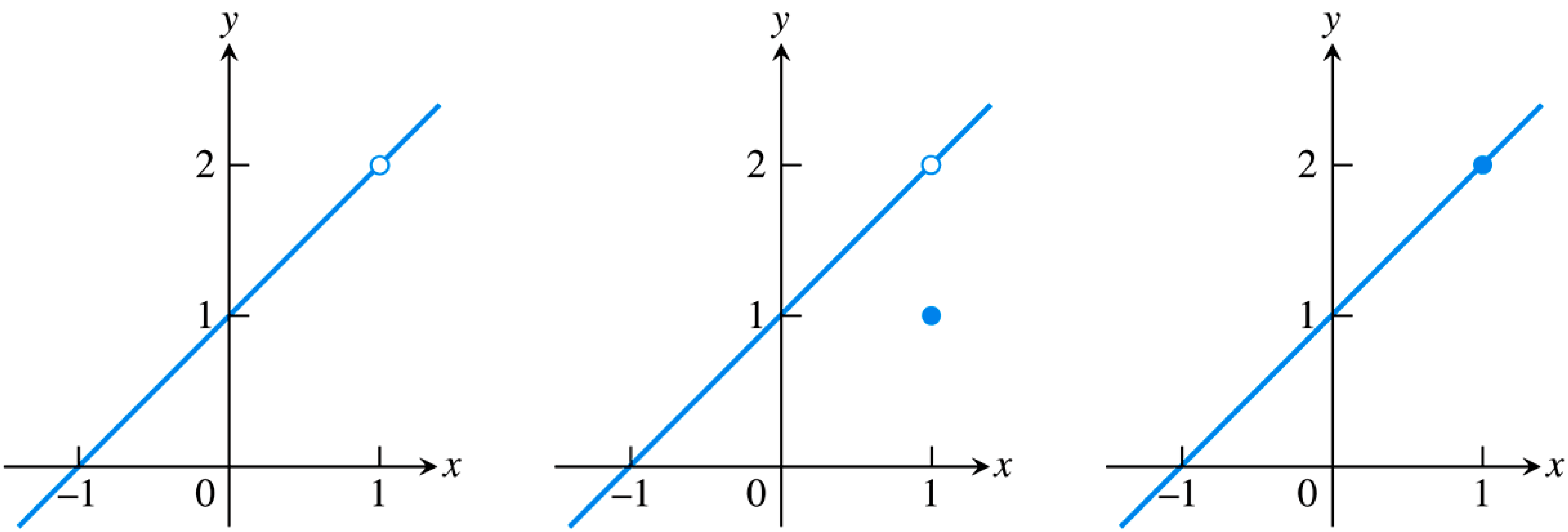


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \dfrac{x^2-1}{x-1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

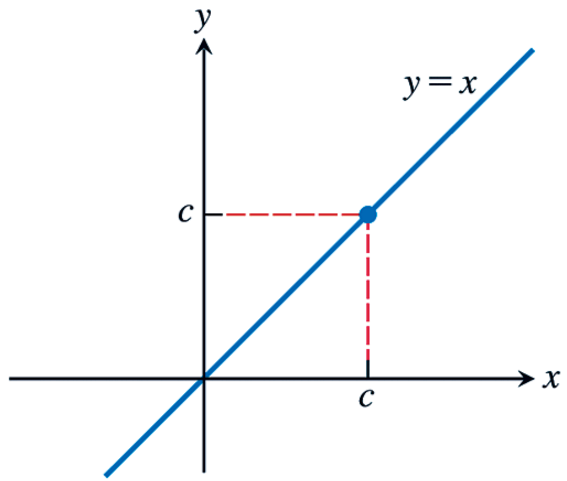


(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

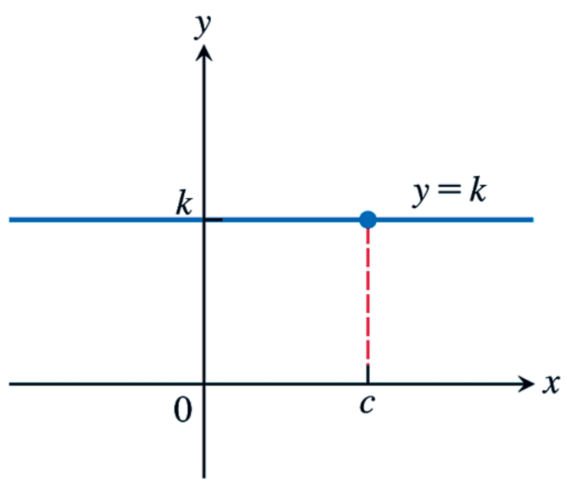
(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

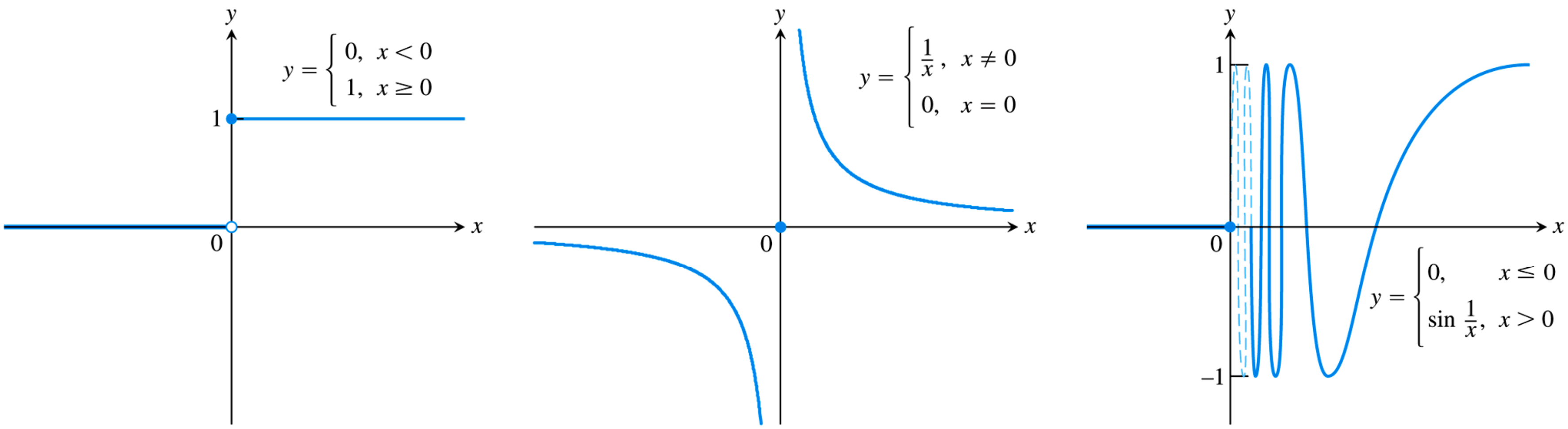


(a) Identity function



(b) Constant function

FIGURE 2.9 The functions in Example 3 have limits at all points c.



(a) Unit step function $U(x)$

(b) $g(x)$

(c) $f(x)$

FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

THEOREM 1—Limit Laws

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

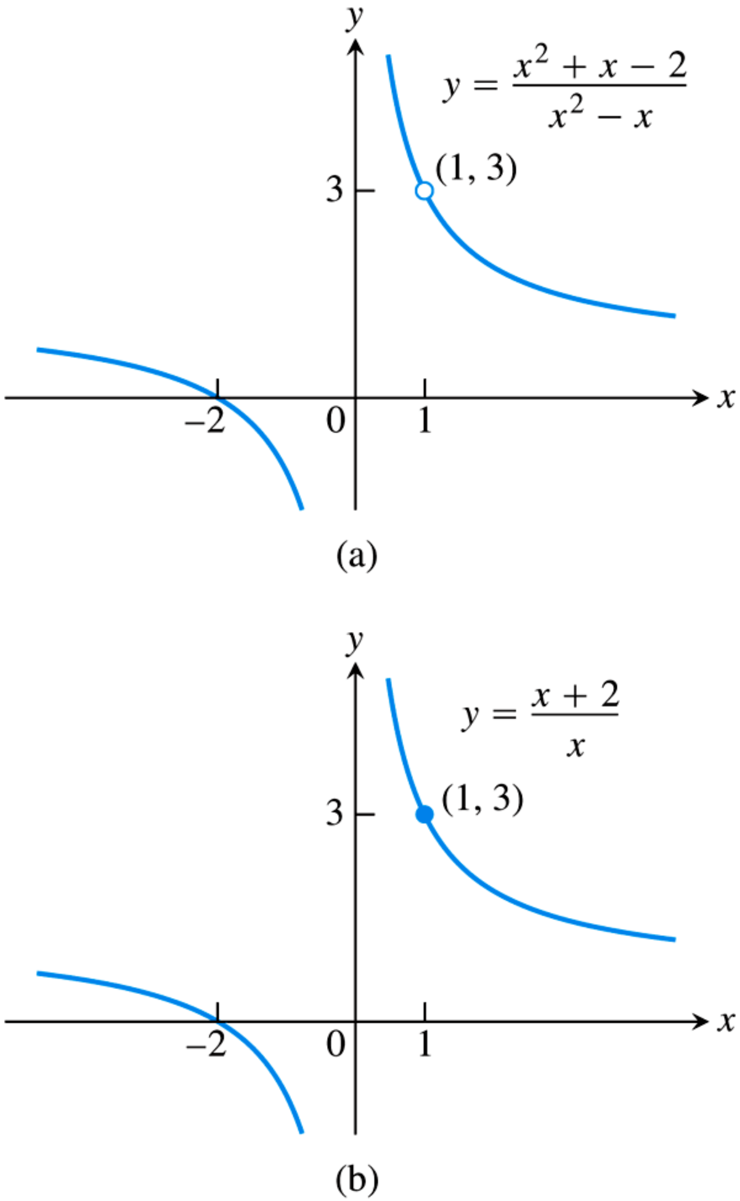


FIGURE 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \rightarrow 1$ (Example 7).

TABLE 2.3 Computed values of $f(x) = \dfrac{\sqrt{x^2 + 100} - 10}{x^2}$ near $x = 0$

x	$f(x)$	
± 1	0.049876	approaches 0.05?
± 0.5	0.049969	
± 0.1	0.049999	
± 0.01	0.050000	
± 0.0005	0.050000	approaches 0?
± 0.0001	0.000000	
± 0.00001	0.000000	
± 0.000001	0.000000	

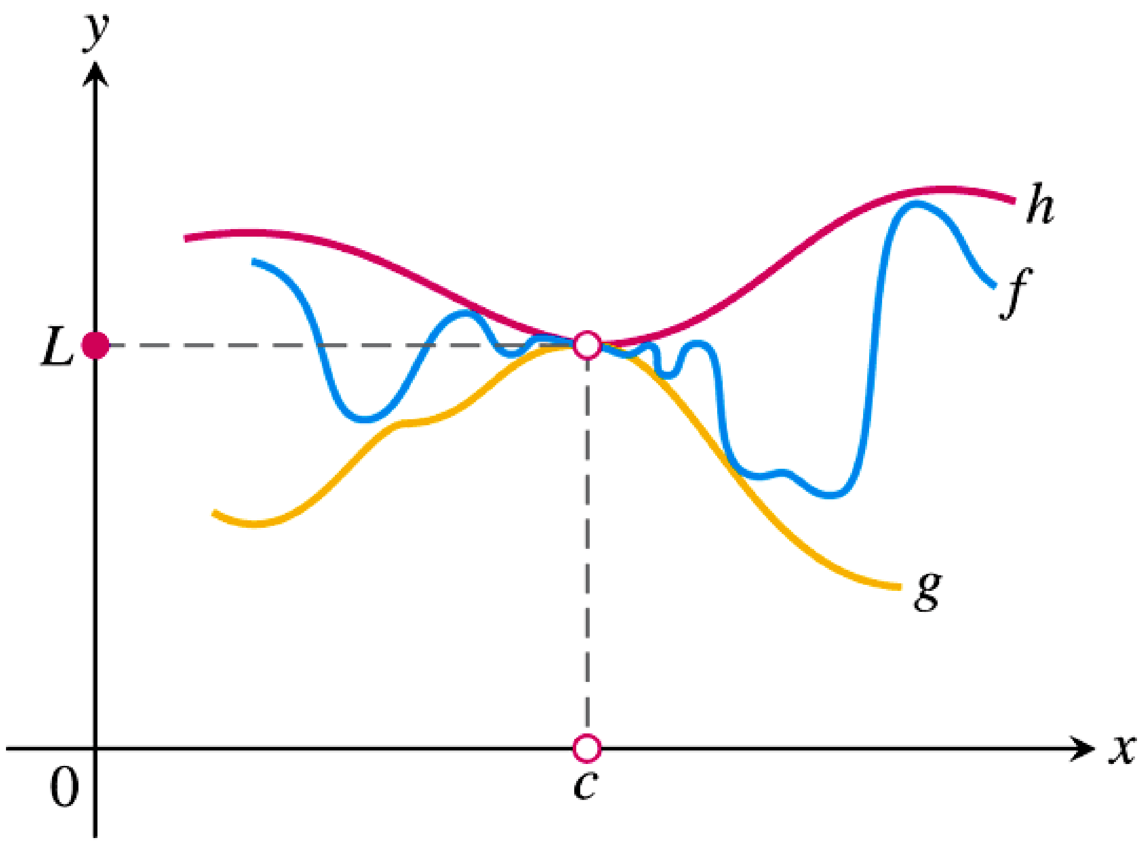


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

THEOREM 4—The Sandwich Theorem
Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

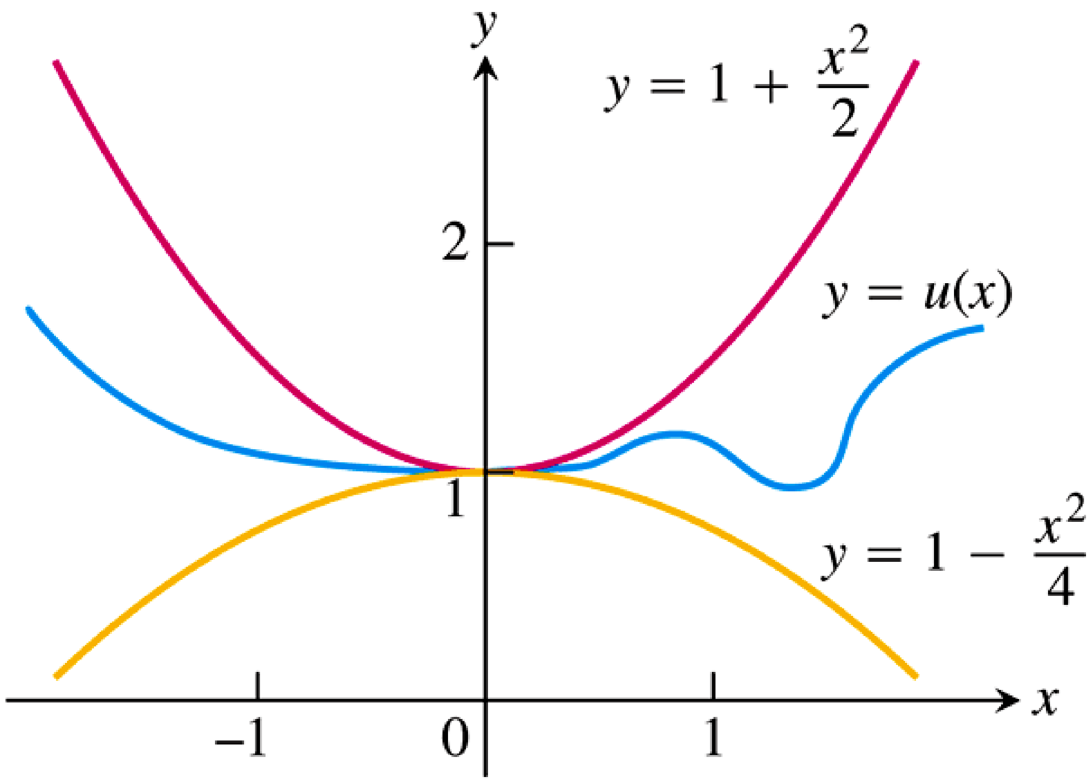


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 10).

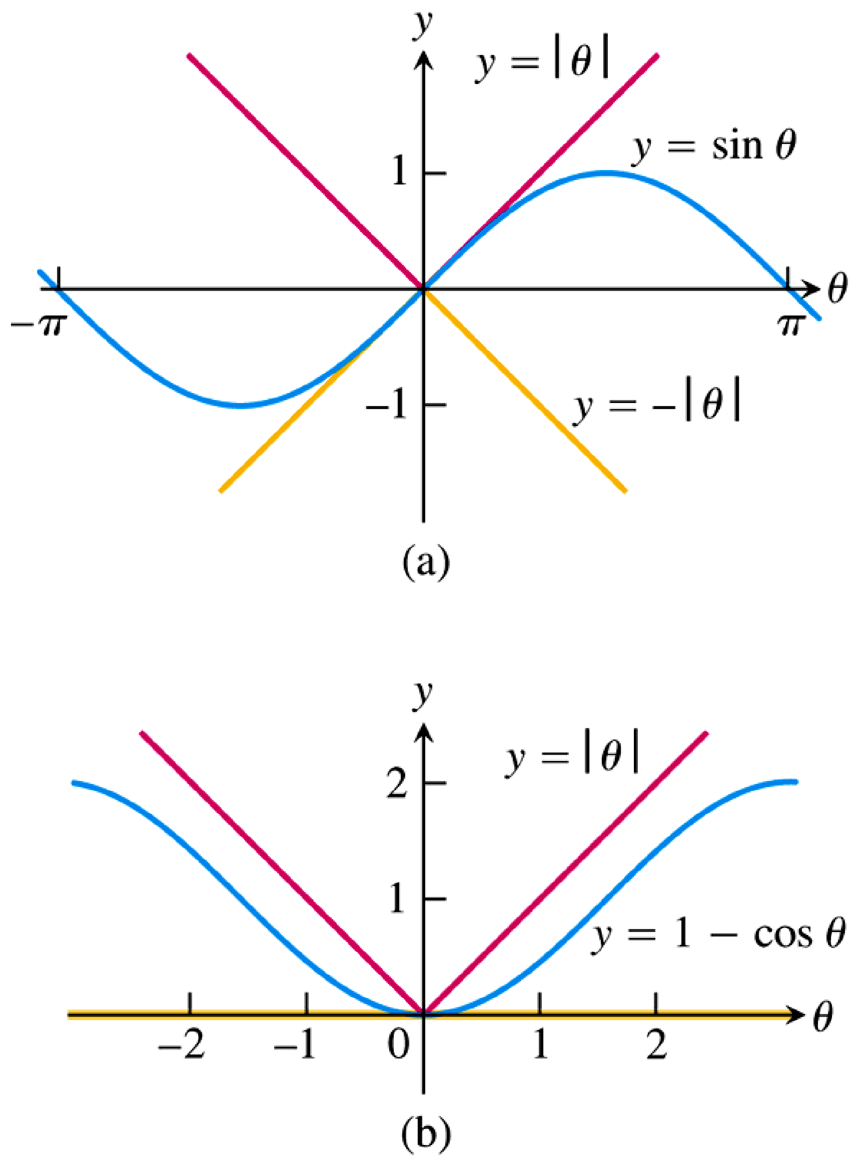


FIGURE 2.14 The Sandwich Theorem confirms the limits in Example 11.

Section 2.3

The Precise Definition of a Limit

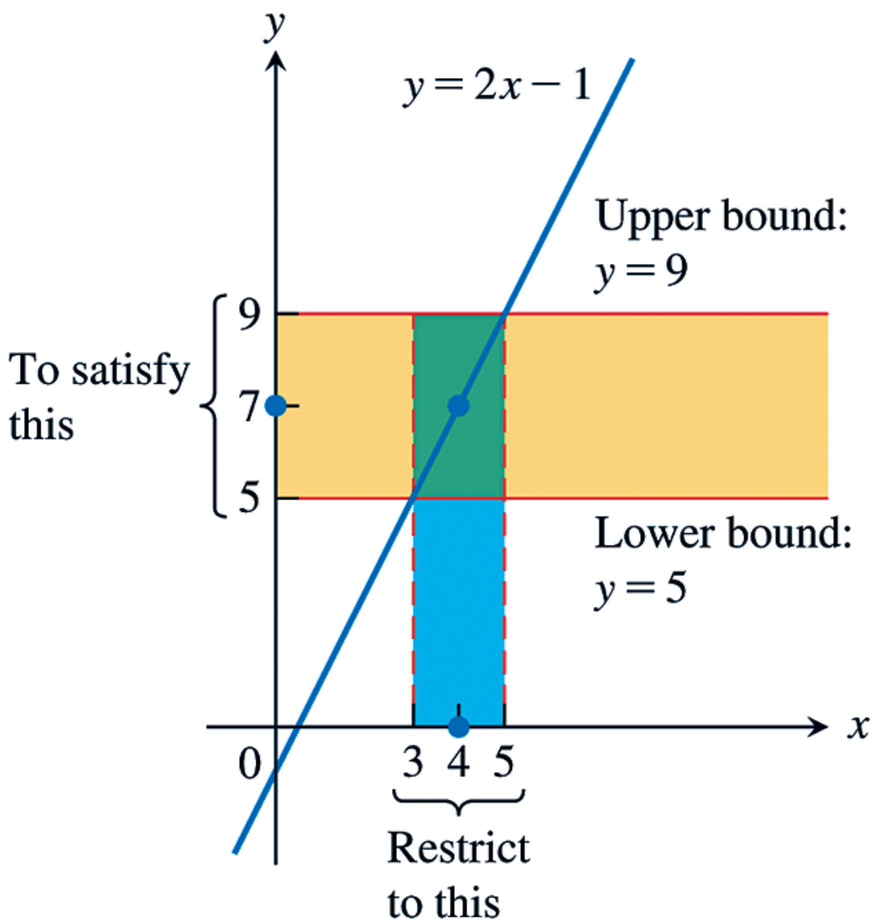


FIGURE 2.15 Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Example 1).

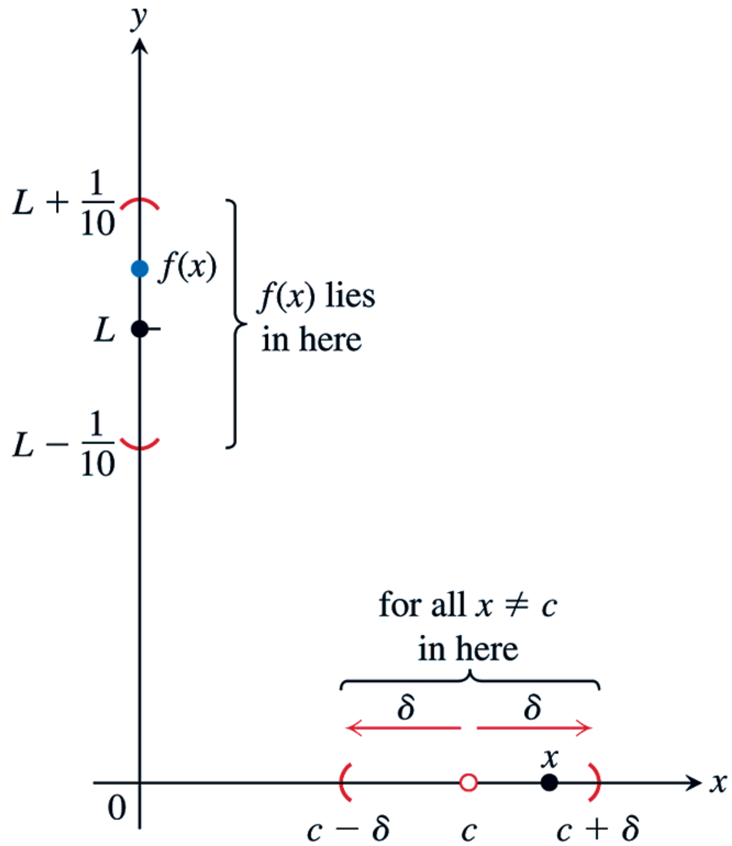


FIGURE 2.16 How should we define $\delta > 0$ so that keeping x within the interval $(c - \delta, c + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

DEFINITION Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x \to c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

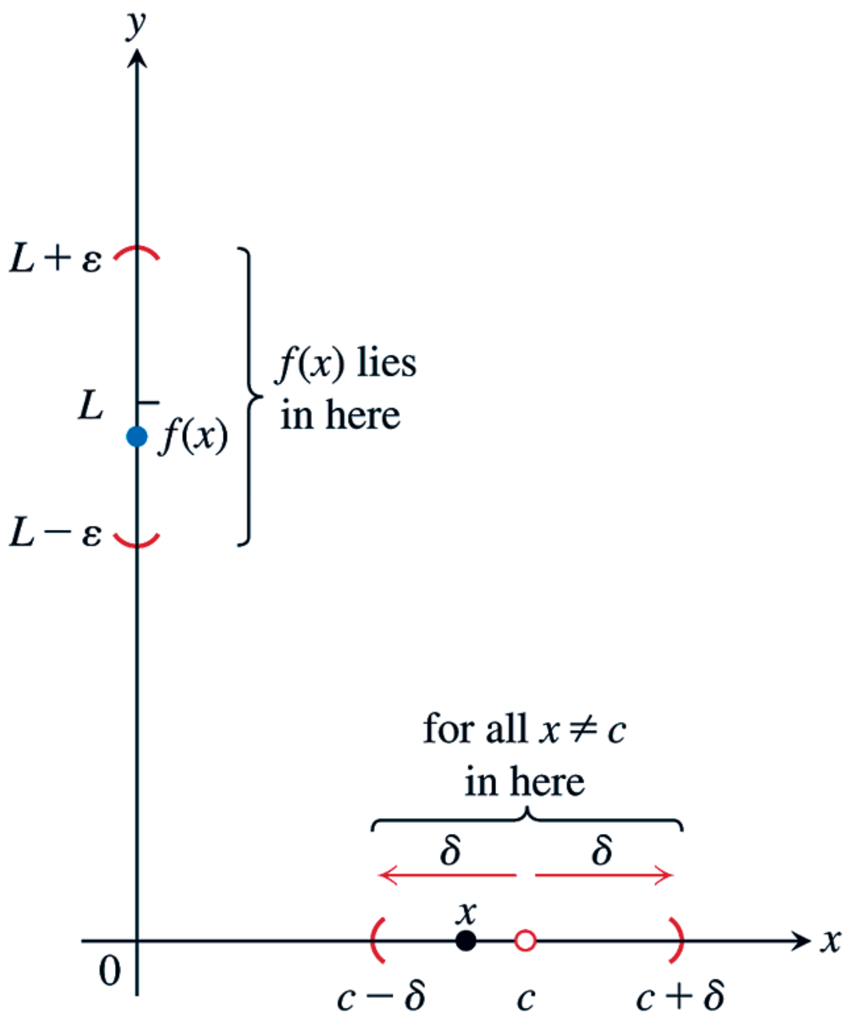
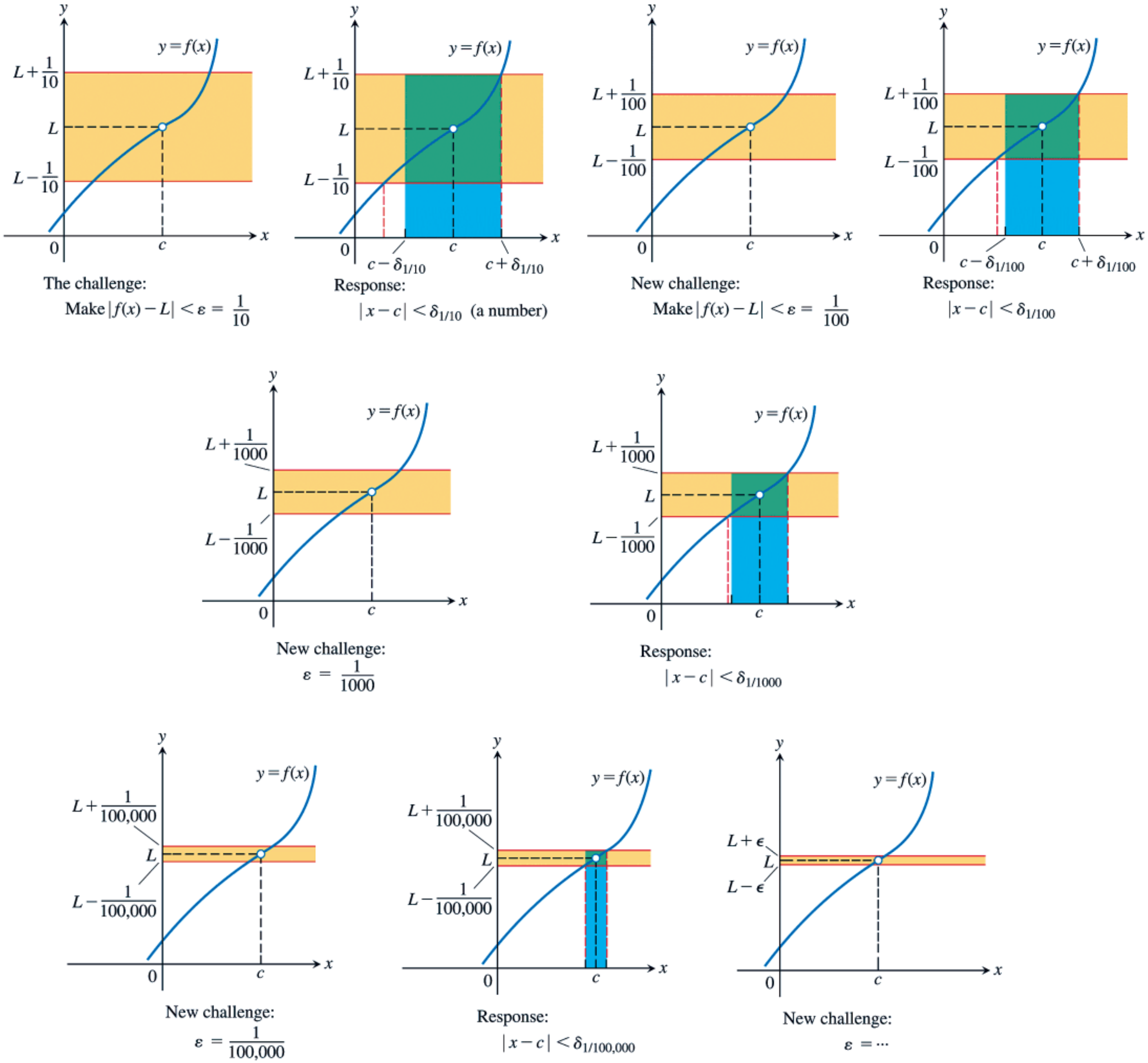


FIGURE 2.17 The relation of δ and ε in the definition of limit.



y = f(x)
L + 1/10
L
L − 1/10
0
c
The challenge:
Make |f(x) − L| < ε = 1/10
c − δ_{1/10}
c + δ_{1/10}
Response:
|x − c| < δ_{1/10} (a number)
L + 1/100
L − 1/100
New challenge:
Make |f(x) − L| < ε = 1/100
c − δ_{1/100}
c + δ_{1/100}
Response:
|x − c| < δ_{1/100}
L + 1/1000
L − 1/1000
New challenge:
ε = 1/1000
Response:
|x − c| < δ_{1/1000}
L + 1/100,000
L − 1/100,000
New challenge:
ε = 1/100,000
Response:
|x − c| < δ_{1/100,000}
L + ϵ
L − ϵ
New challenge:
ε = ···

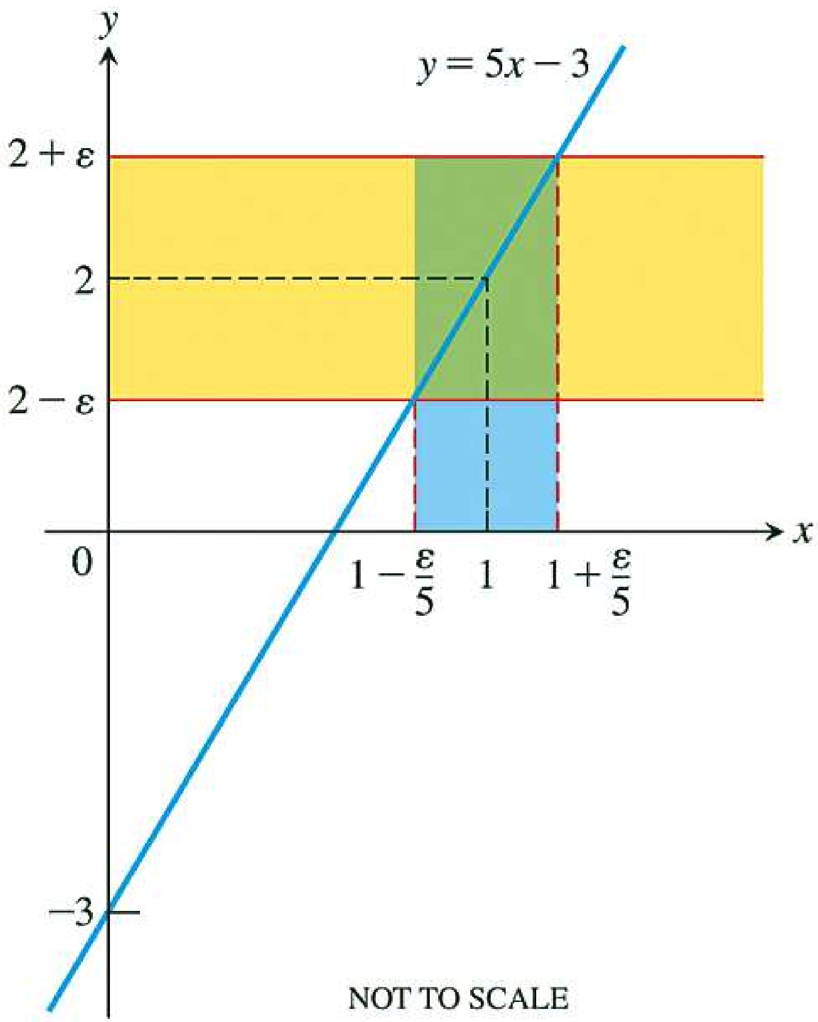


FIGURE 2.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \varepsilon/5$ guarantees that $|f(x) - 2| < \varepsilon$ (Example 2).

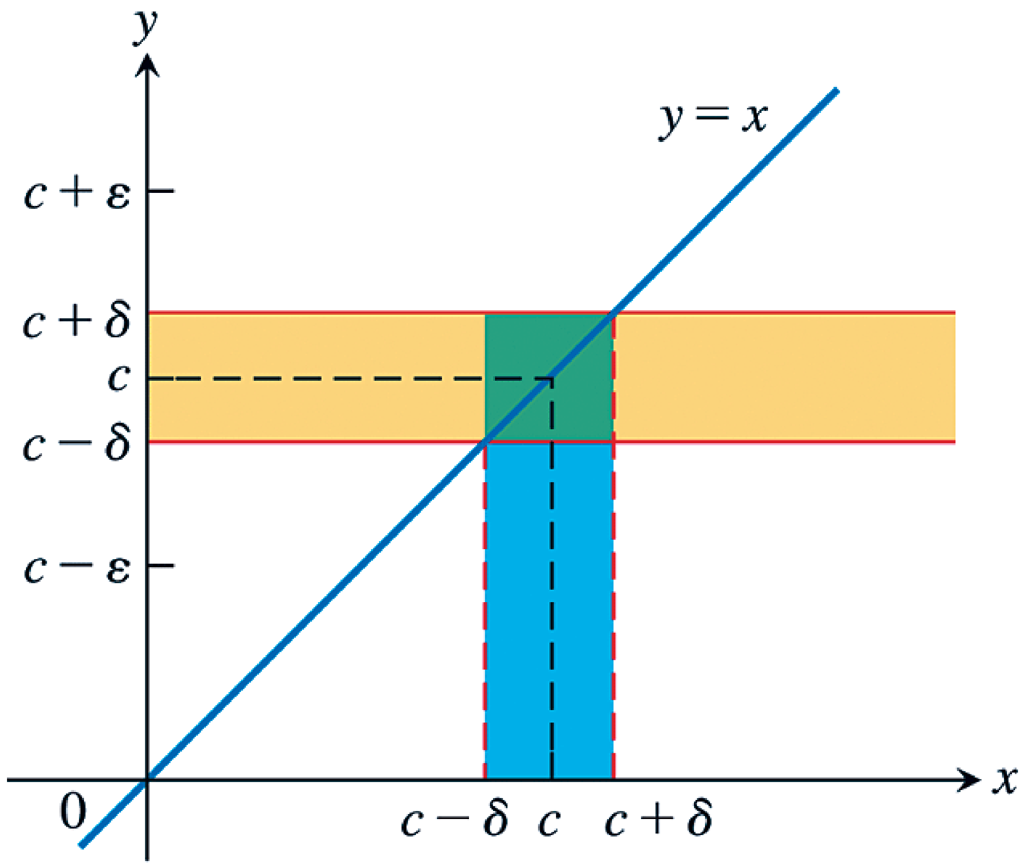


FIGURE 2.19 For the function $f(x) = x$, we find that $0 < |x - c| < \delta$ will guarantee $|f(x) - c| < \varepsilon$ whenever $\delta \le \varepsilon$ (Example 3a).

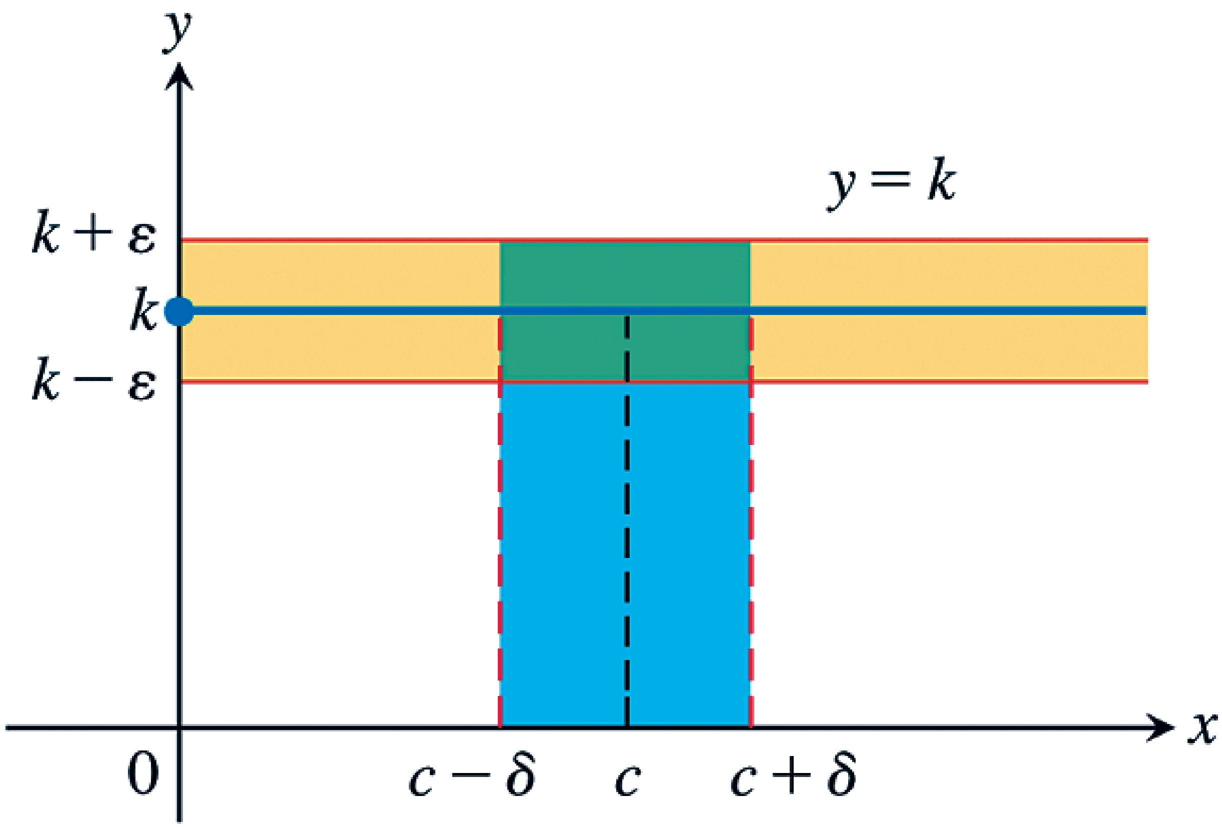


FIGURE 2.20 For the function $f(x) = k$, we find that $|f(x) - k| < \varepsilon$ for any positive δ (Example 3b).

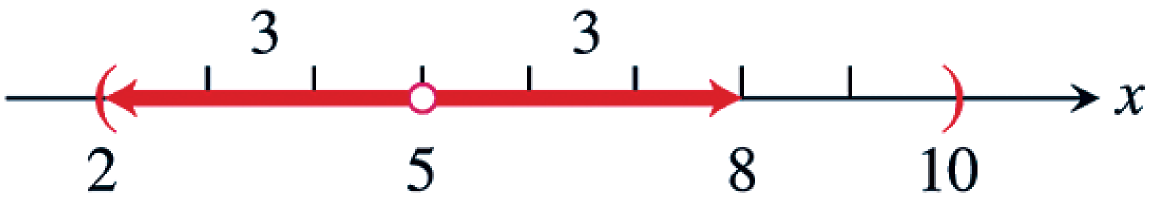


FIGURE 2.21 An open interval of radius 3 about $x = 5$ will lie inside the open interval (2, 10).

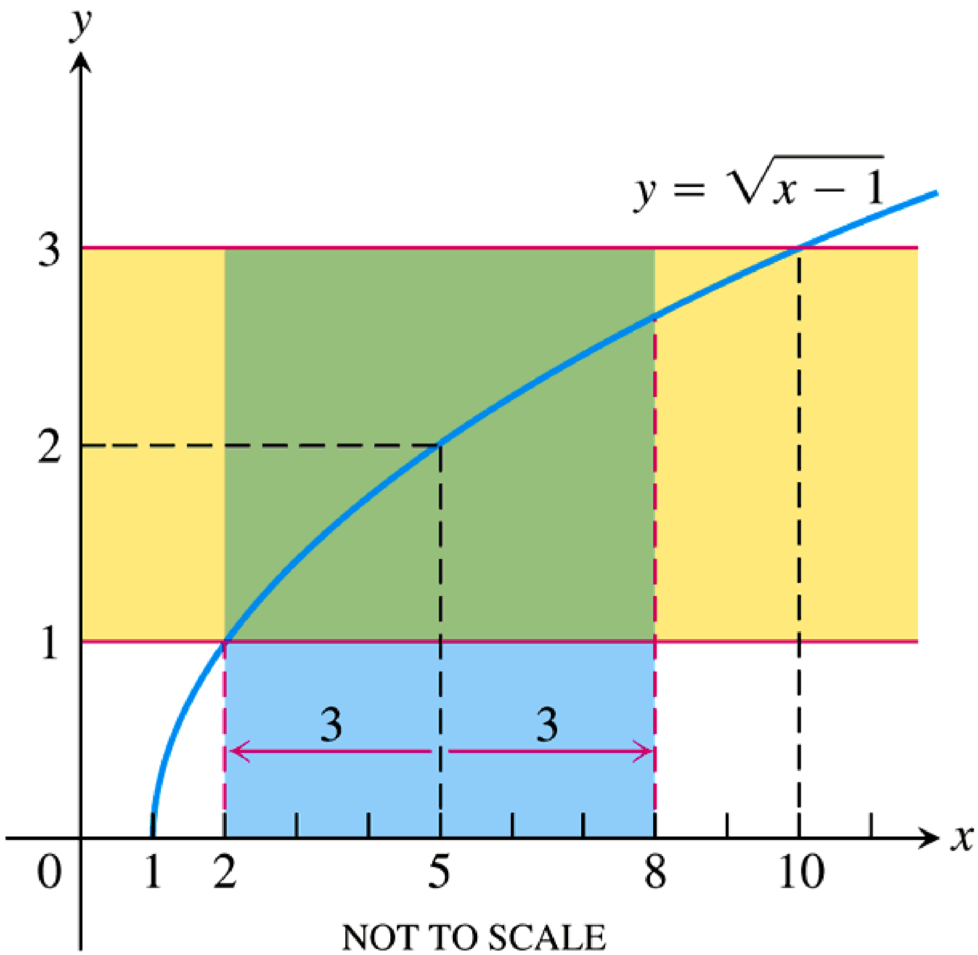


FIGURE 2.22 The function and intervals in Example 4.

How to Find Algebraically a δ for a Given f, L, c, and $\varepsilon > 0$

The process of finding a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \varepsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$. Note that we do not require the inequality to hold at $x = c$. It may hold there or it may not, but the value of f at $x = c$ does not influence the existence of a limit.
2. *Find a value of* $\delta > 0$ that places the open interval $(c - \delta, c + \delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq c$ in this δ-interval.

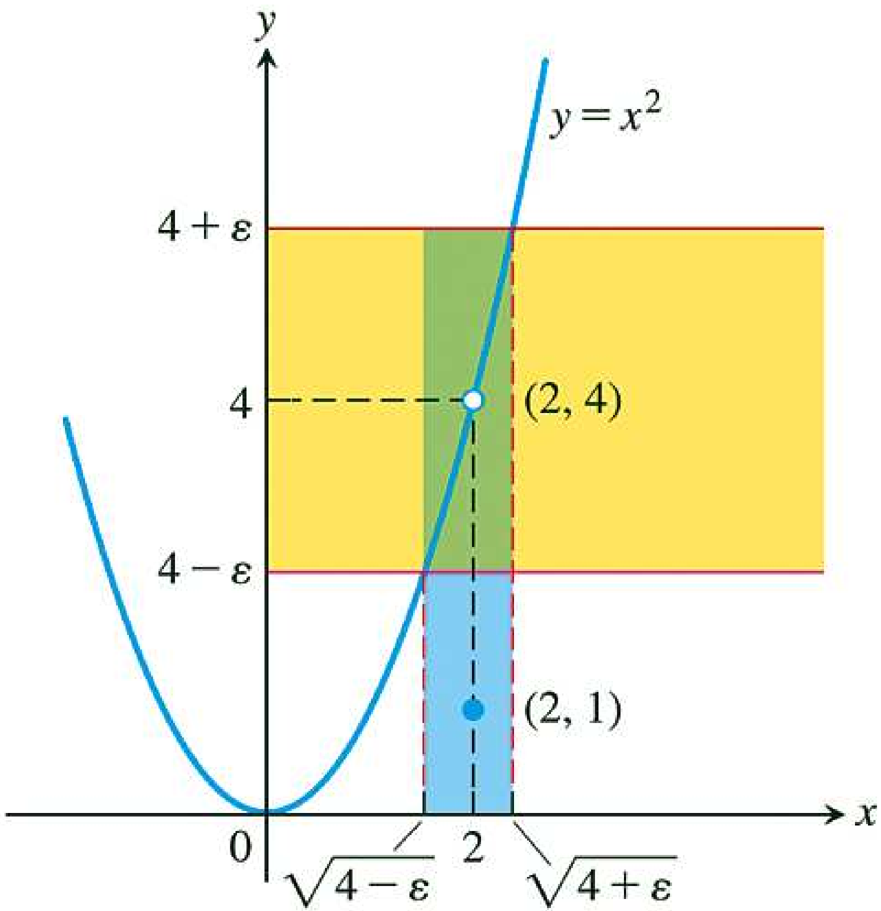


FIGURE 2.23 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \varepsilon$.

Section 2.4

One-Sided Limits

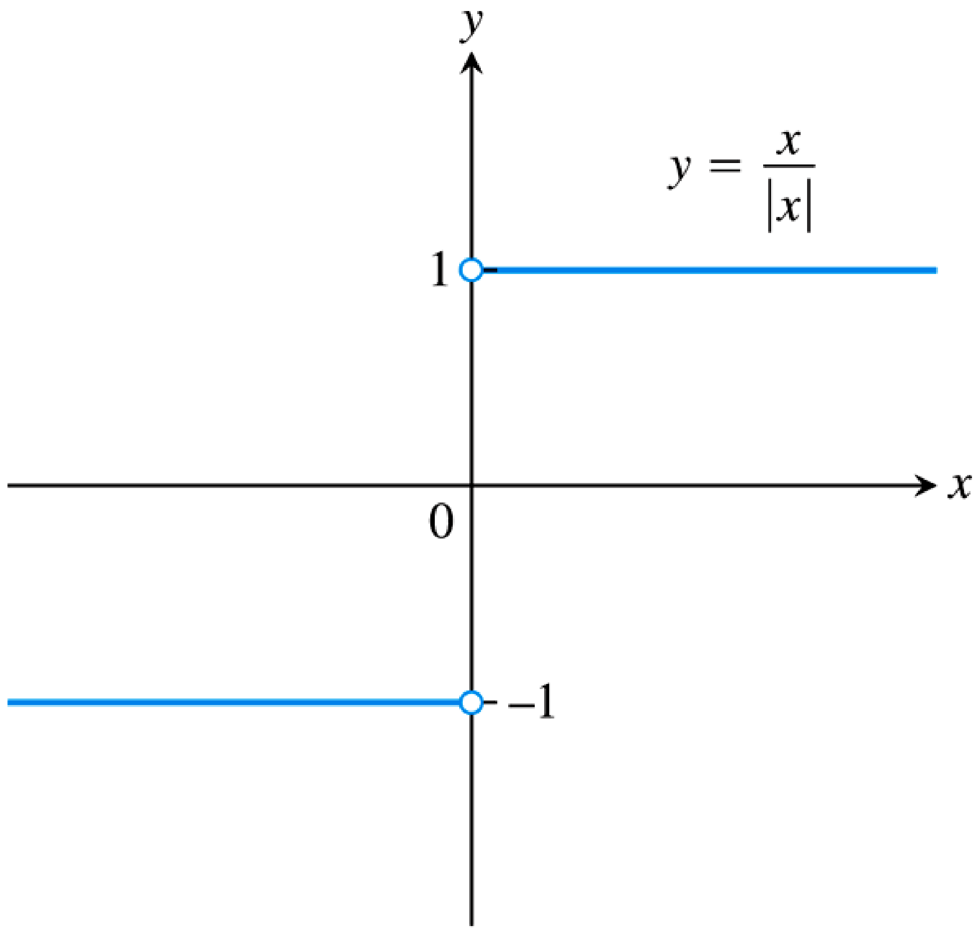
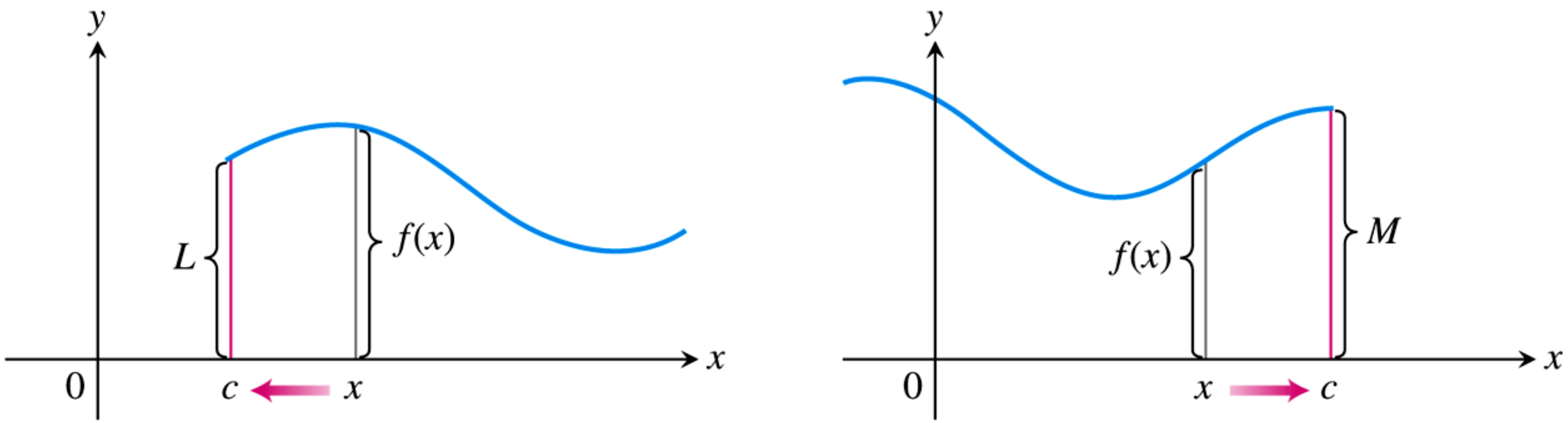


FIGURE 2.24 Different right-hand and left-hand limits at the origin.



(a) $\lim_{x \to c^+} f(x) = L$ (b) $\lim_{x \to c^-} f(x) = M$

FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

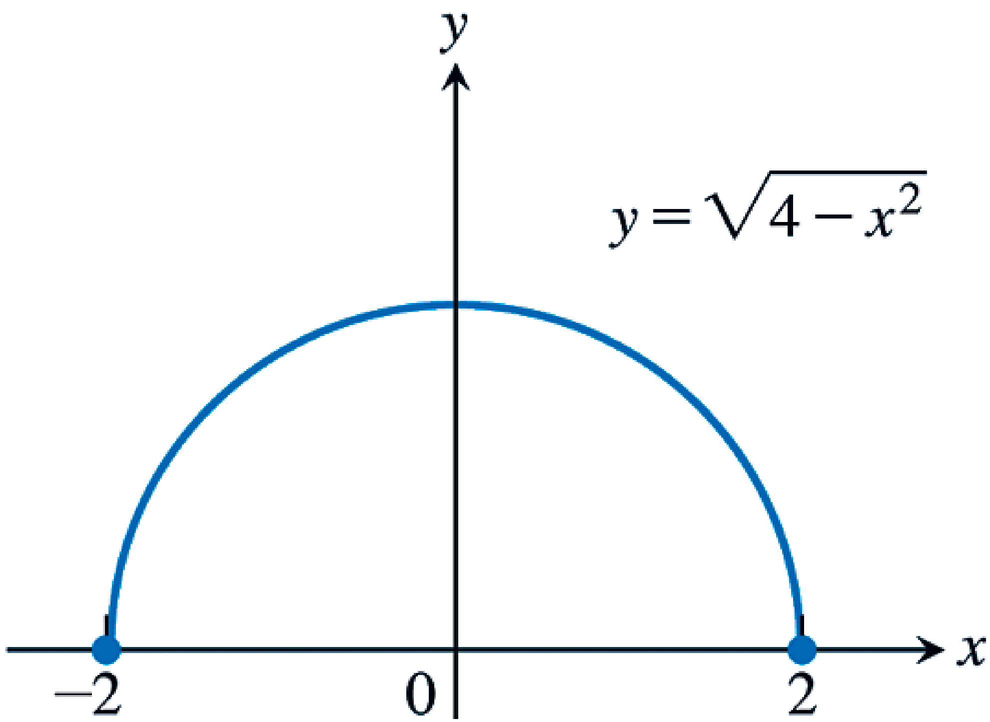


FIGURE 2.26 The function $f(x) = \sqrt{4 - x^2}$ has a right-hand limit 0 at $x = -2$ and a left-hand limit 0 at $x = 2$ (Example 1).

THEOREM 6 Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

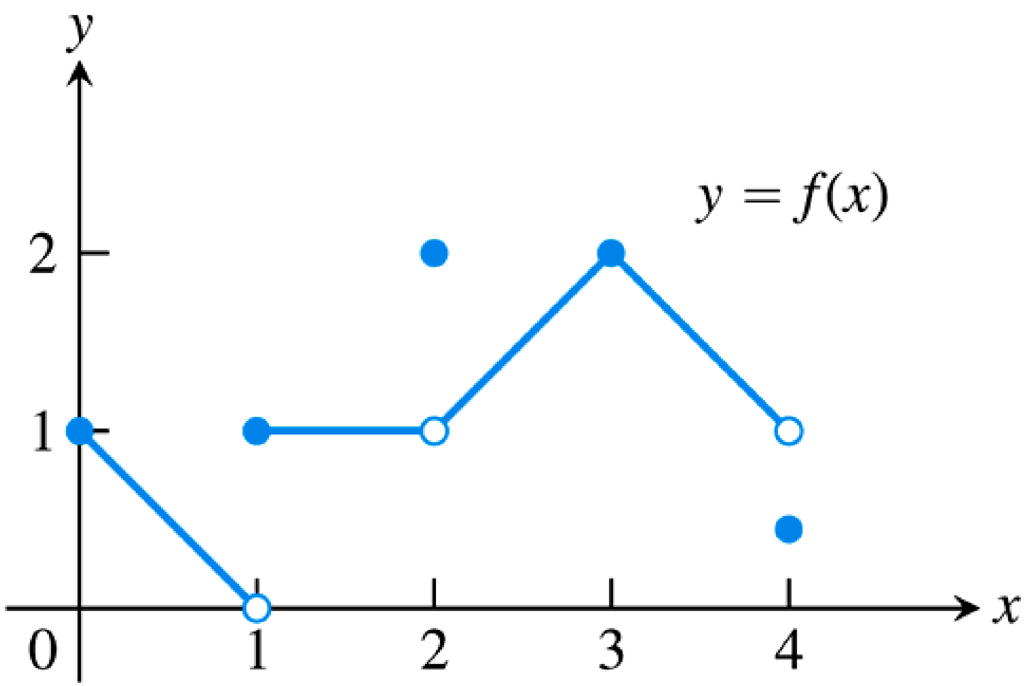


FIGURE 2.27 Graph of the function in Example 2.

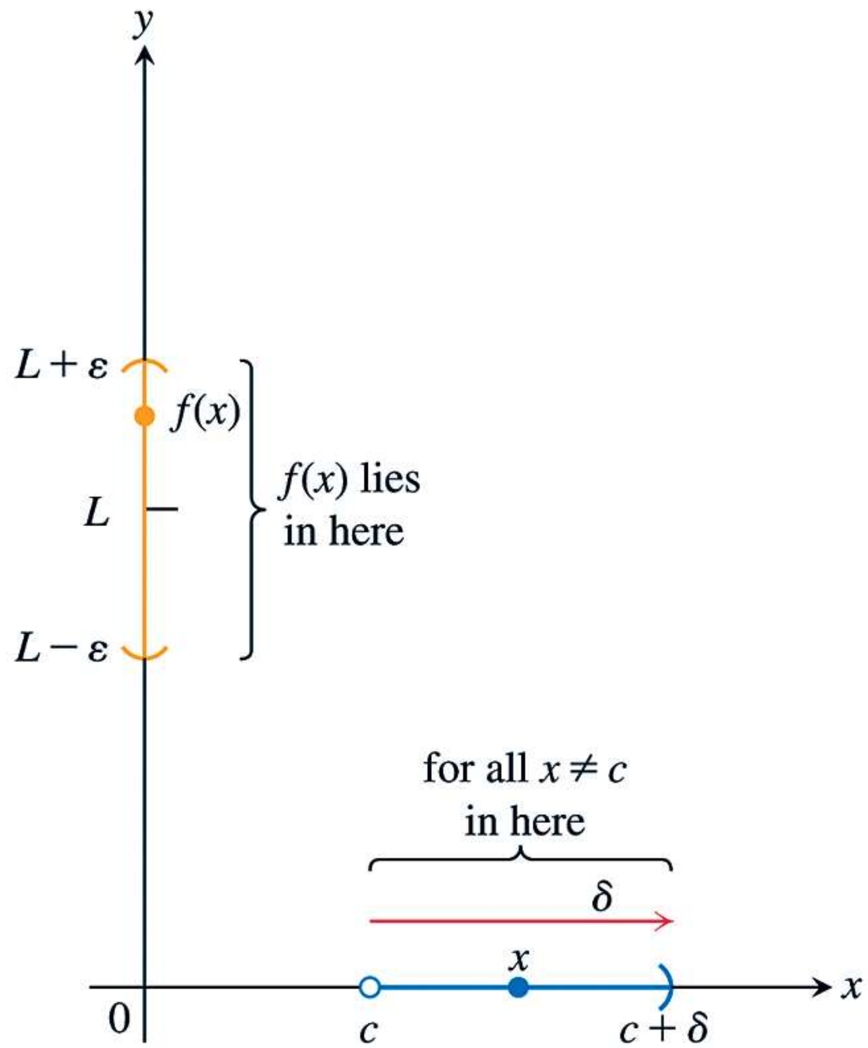


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

DEFINITIONS (a) Assume the domain of f contains an interval (c, d) to the right of c. We say that $f(x)$ has **right-hand limit L at c**, and write

$$\lim_{x \to c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$

(b) Assume the domain of f contains an interval (b, c) to the left of c. We say that f has **left-hand limit L at c**, and write

$$\lim_{x \to c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$

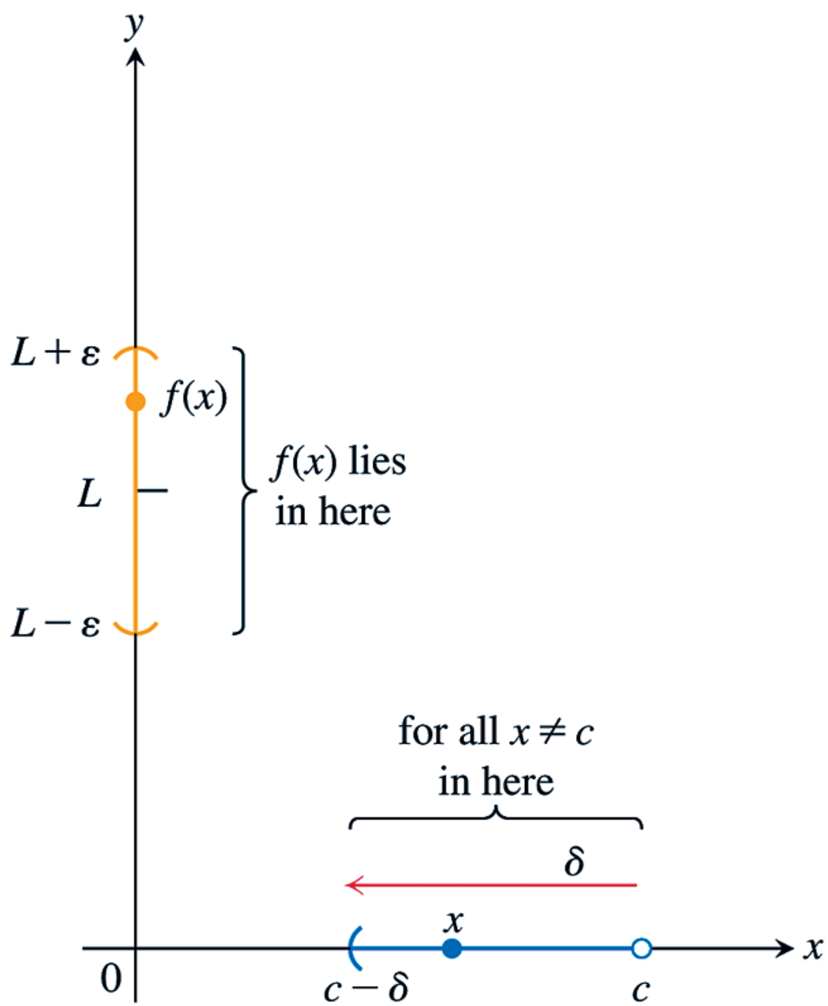


FIGURE 2.29 Intervals associated with the definition of left-hand limit.

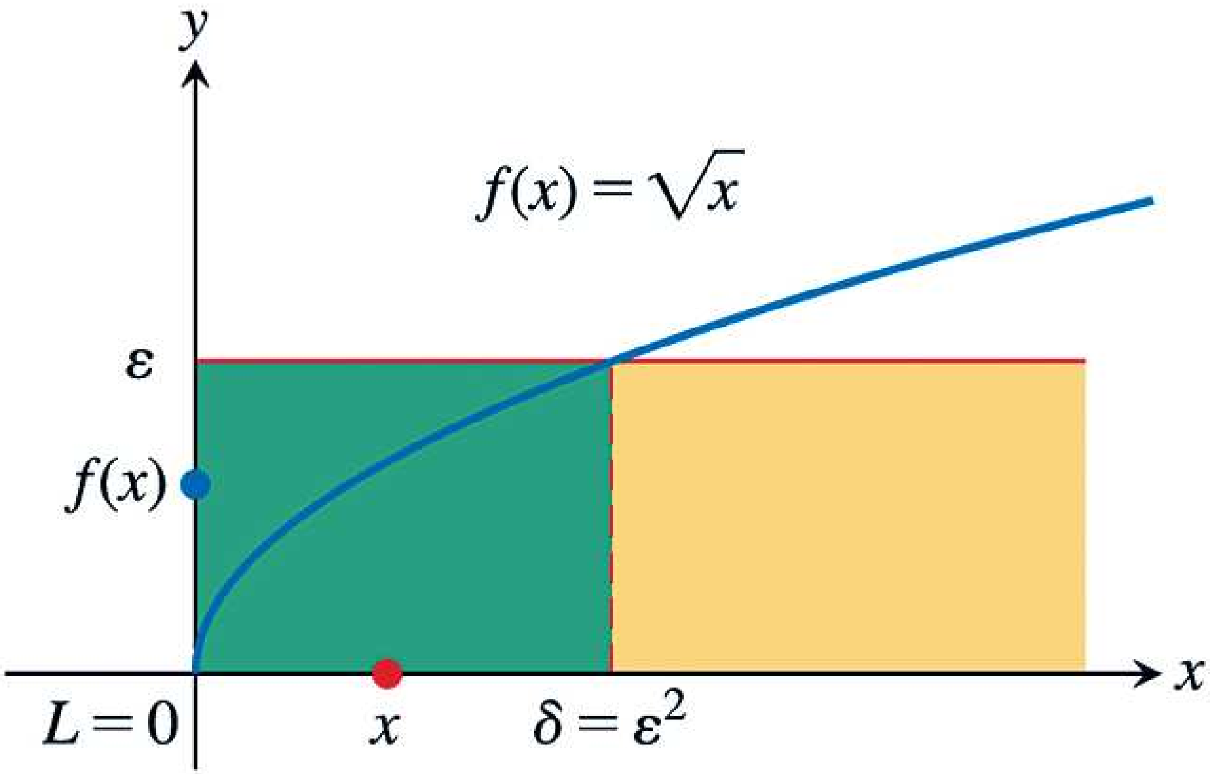


FIGURE 2.30 $\lim_{x \to 0^+} \sqrt{x} = 0$ in Example 3.

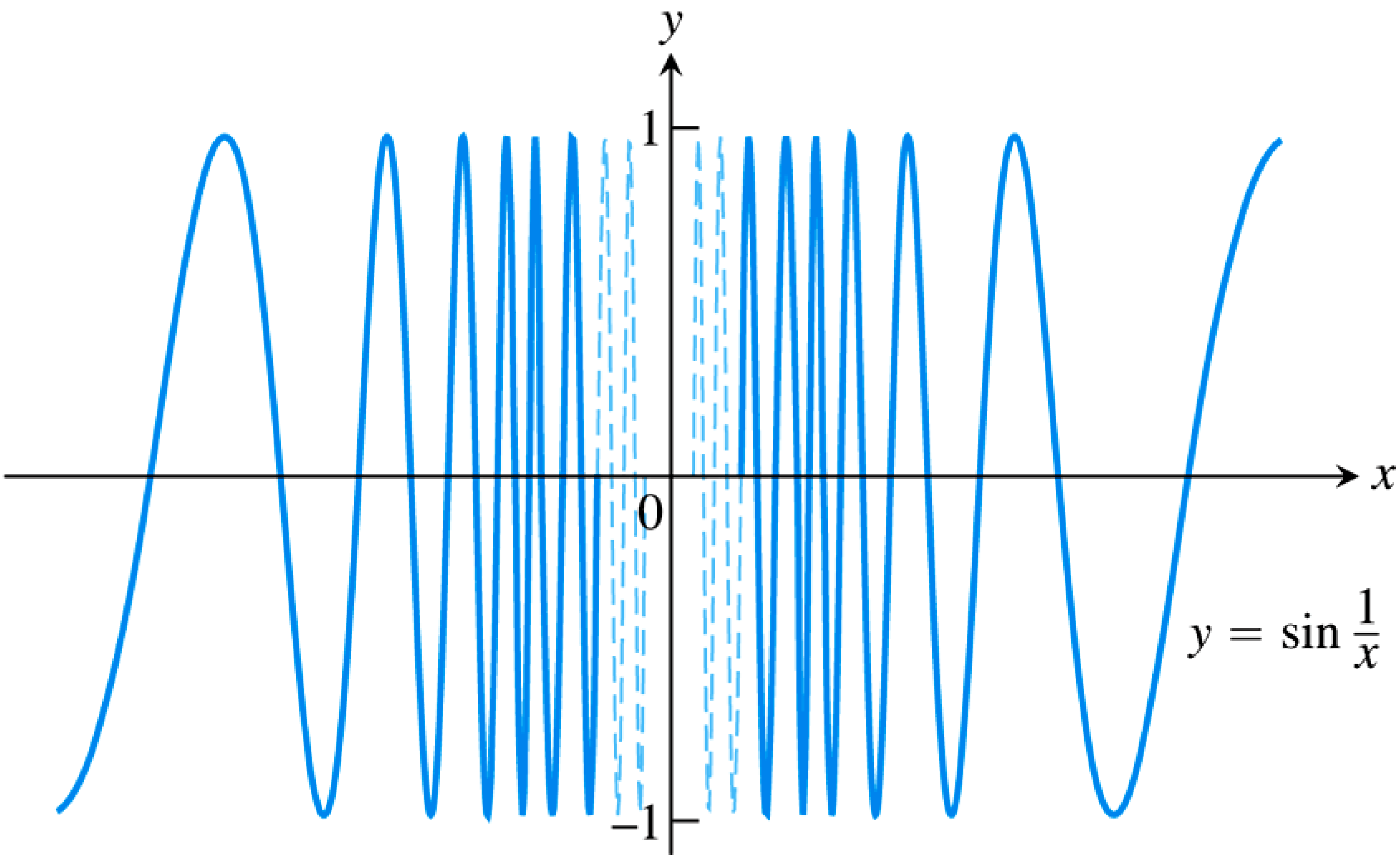


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y-axis.

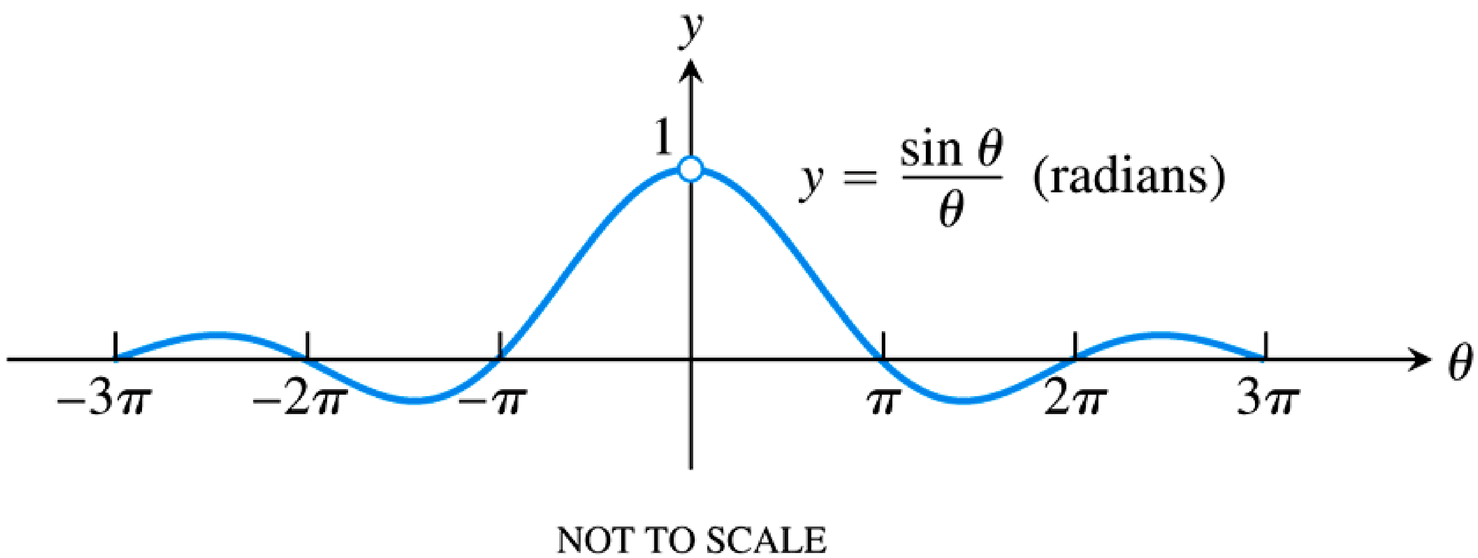


FIGURE 2.32 The graph of $f(\theta) = (\sin\theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

THEOREM 7—Limit of the Ratio $\sin\theta/\theta$ as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

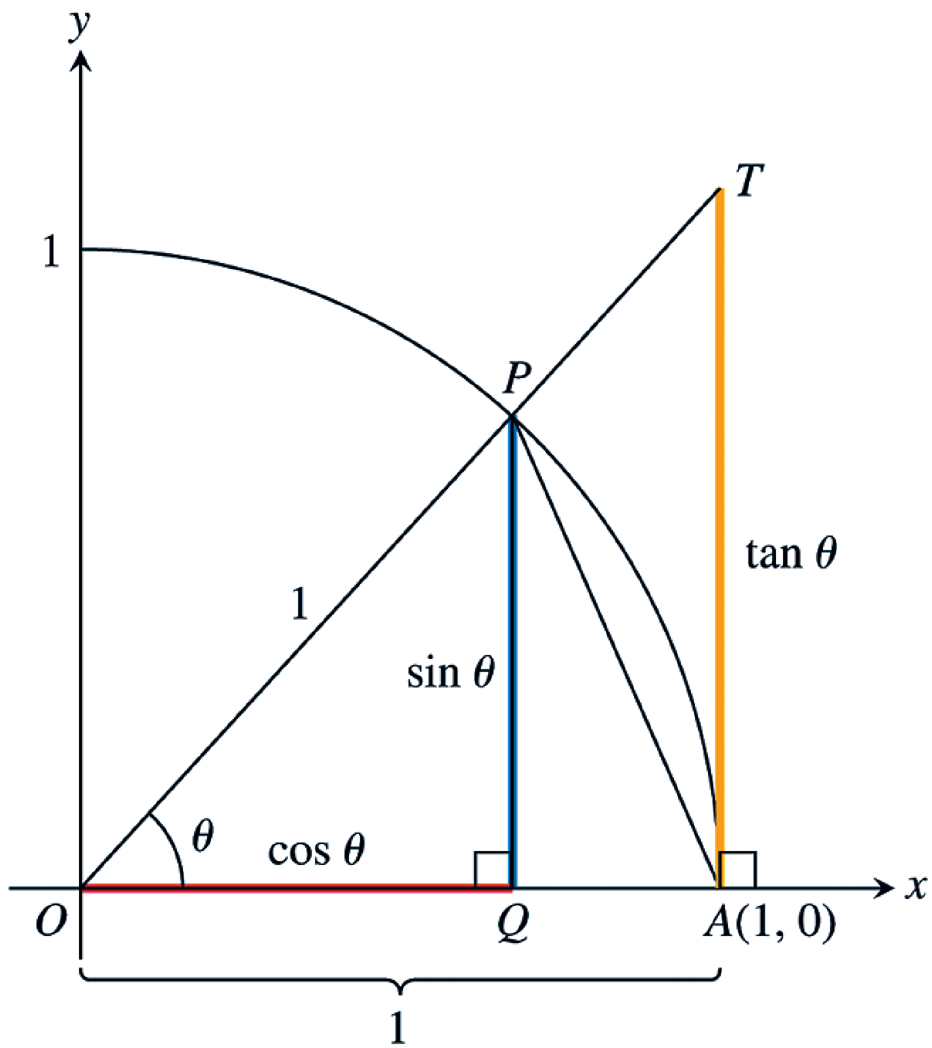


FIGURE 2.33 The ratio $TA/OA = \tan\theta$, and $OA = 1$, so $TA = \tan\theta$.

2.5

Continuity

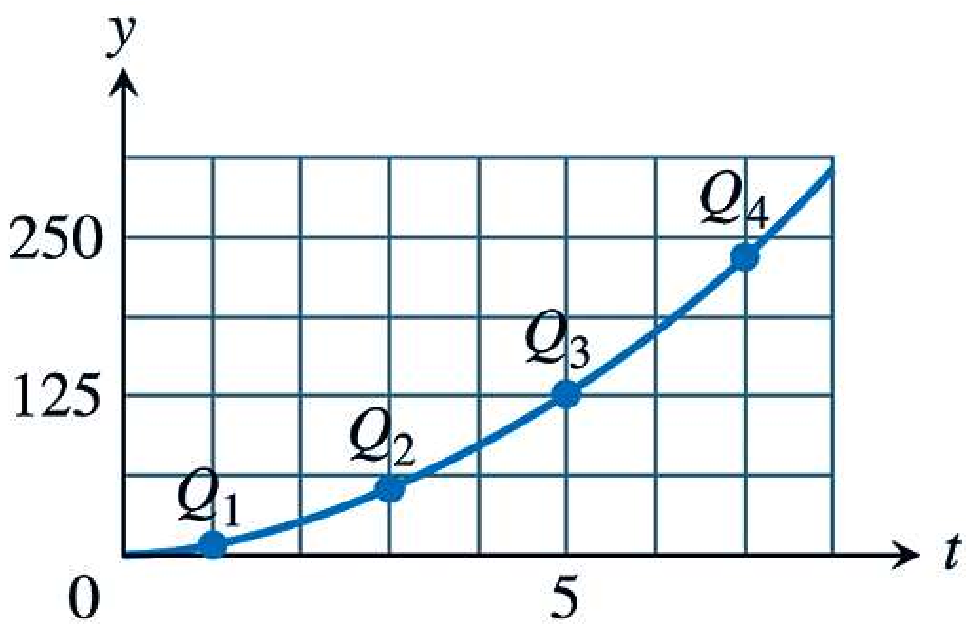


FIGURE 2.34 Connecting plotted points.

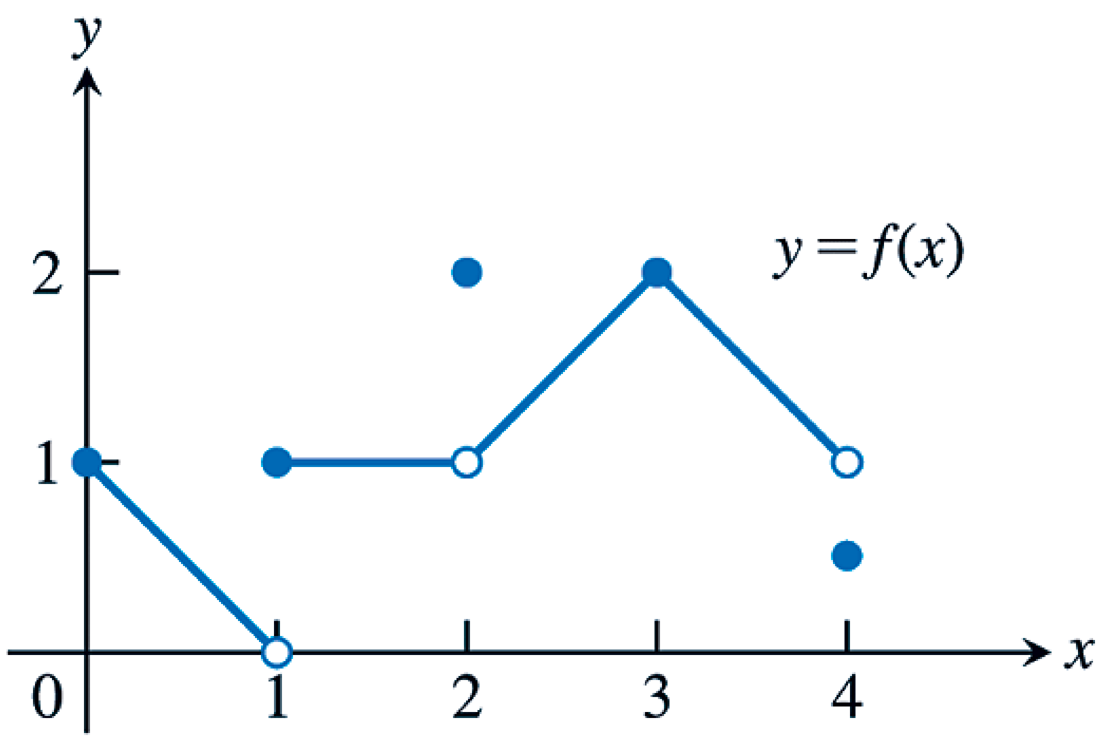


FIGURE 2.35 The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

DEFINITIONS Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.

The function f is **continuous at *c*** if

$$\lim_{x \to c} f(x) = f(c).$$

The function f is **right-continuous at *c* (or continuous from the right)** if

$$\lim_{x \to c^+} f(x) = f(c).$$

The function f is **left-continuous at *c* (or continuous from the left)** if

$$\lim_{x \to c^-} f(x) = f(c).$$

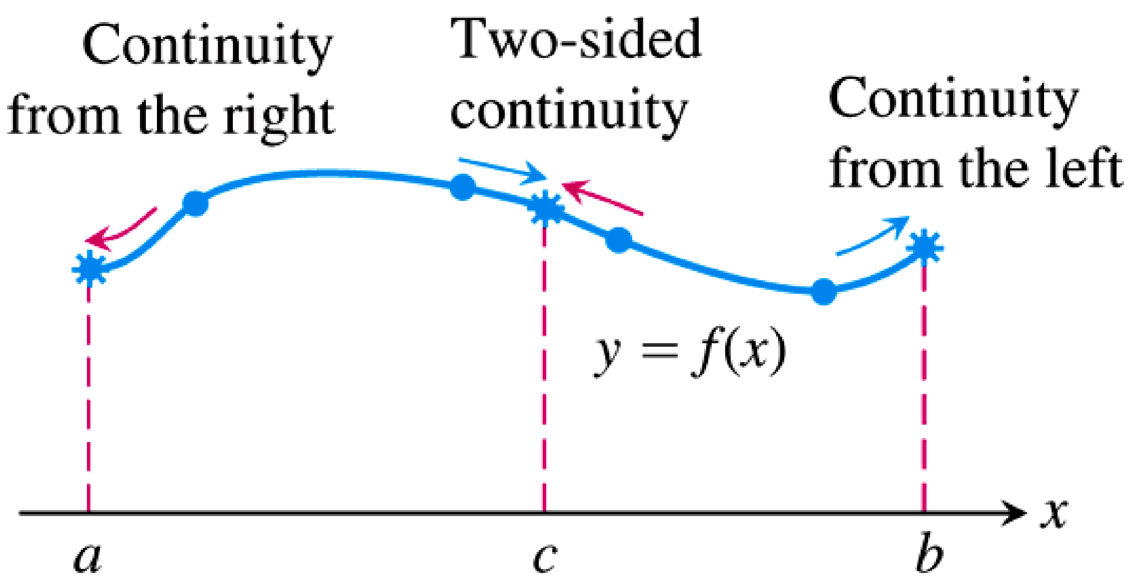


FIGURE 2.36 Continuity at points a, b, and c.

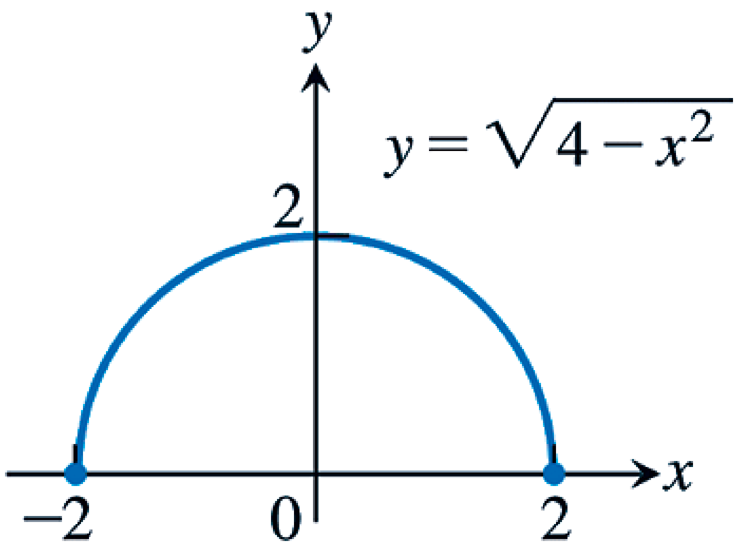


FIGURE 2.37 A function that is continuous over its domain (Example 2).

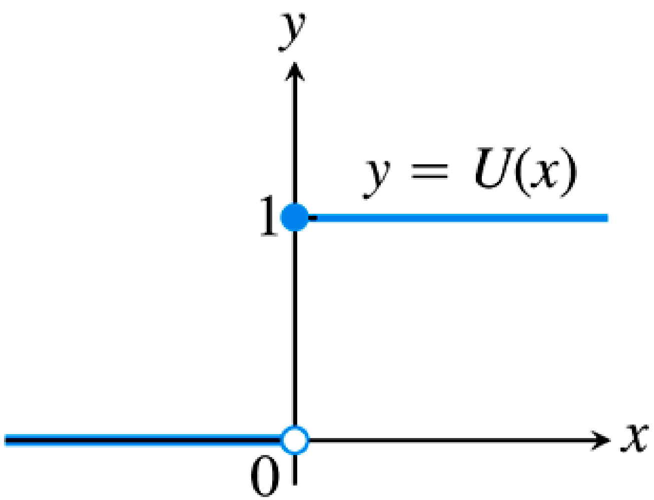


FIGURE 2.38 A function that has a jump discontinuity at the origin (Example 3).

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

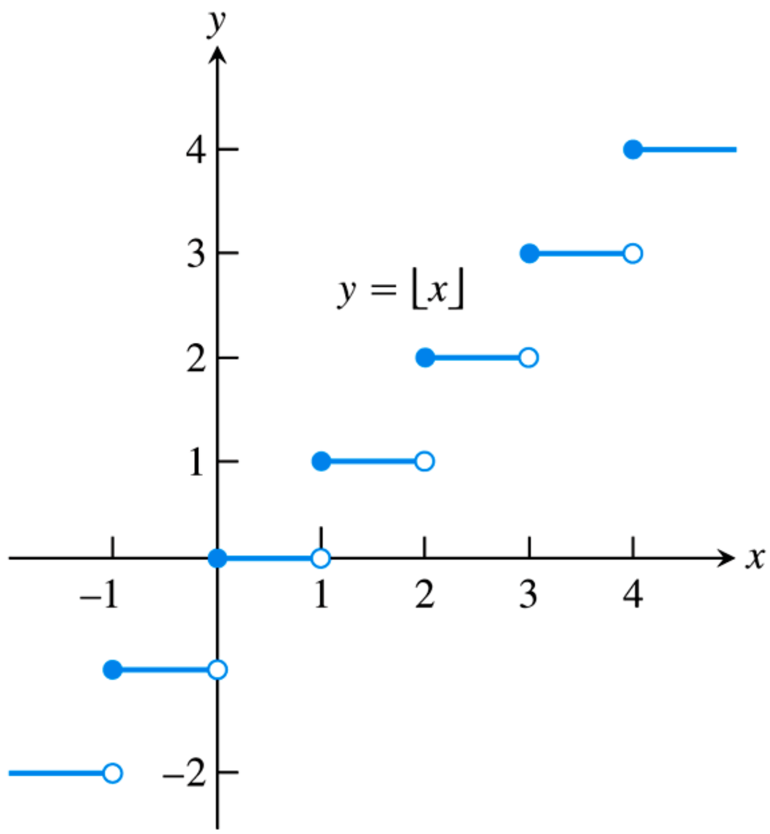


FIGURE 2.39 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

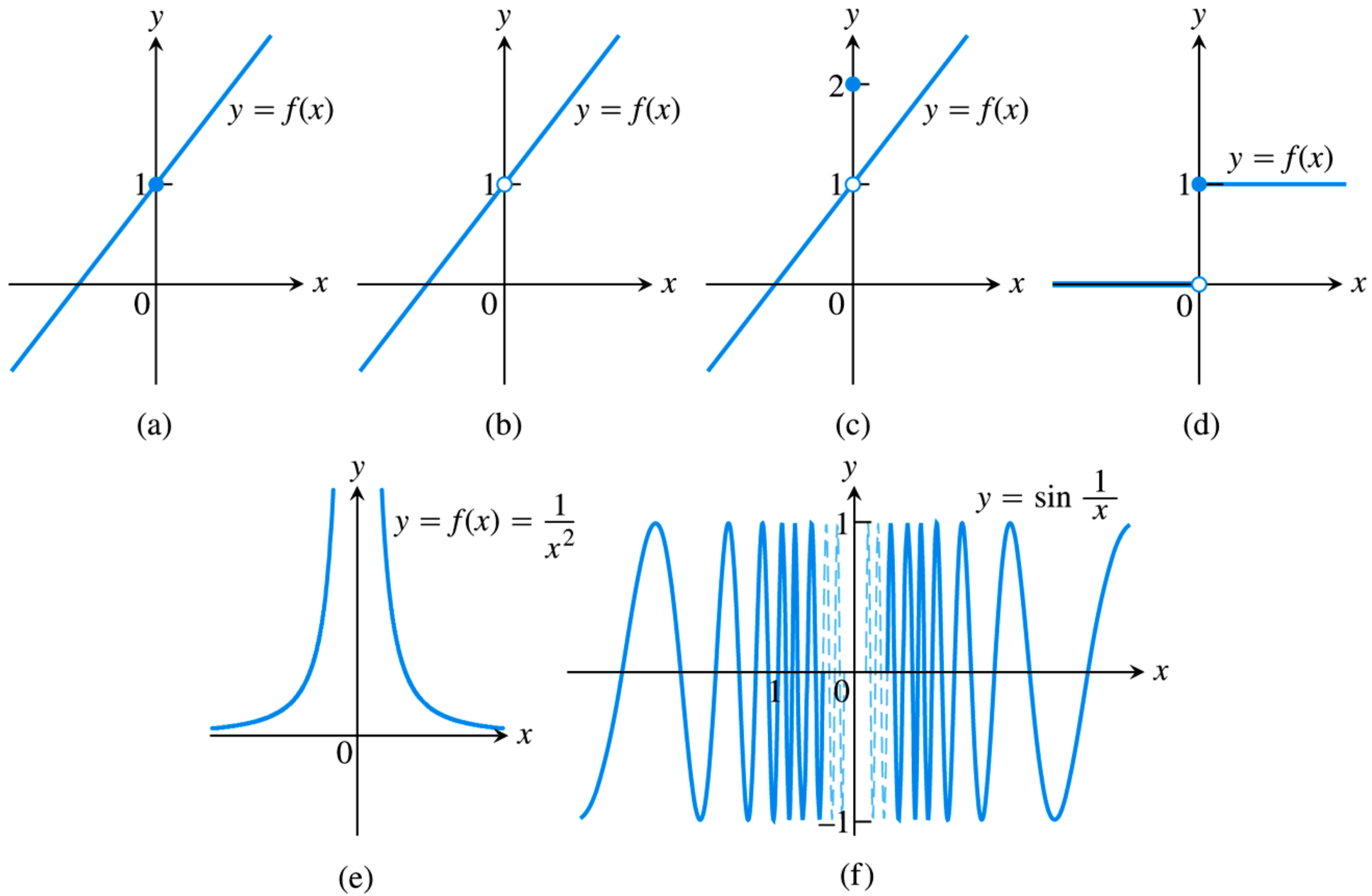


FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

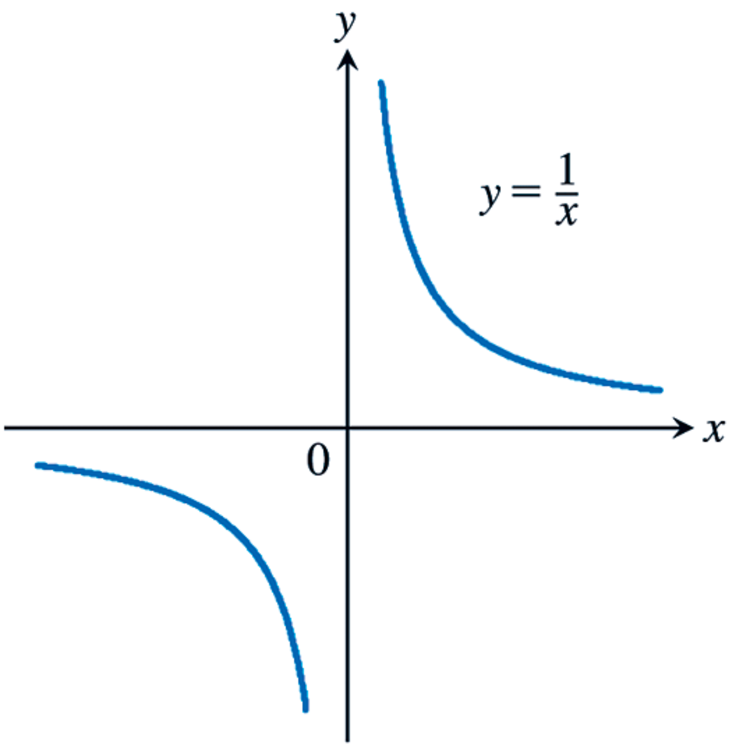


FIGURE 2.41 The function $f(x) = 1/x$ is continuous over its natural domain. It is not defined at the origin, so it is not continuous on any interval containing $x = 0$ (Example 5).

THEOREM 8—Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an interval containing c, where n is a positive integer

THEOREM 9—Compositions of Continuous Functions
If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

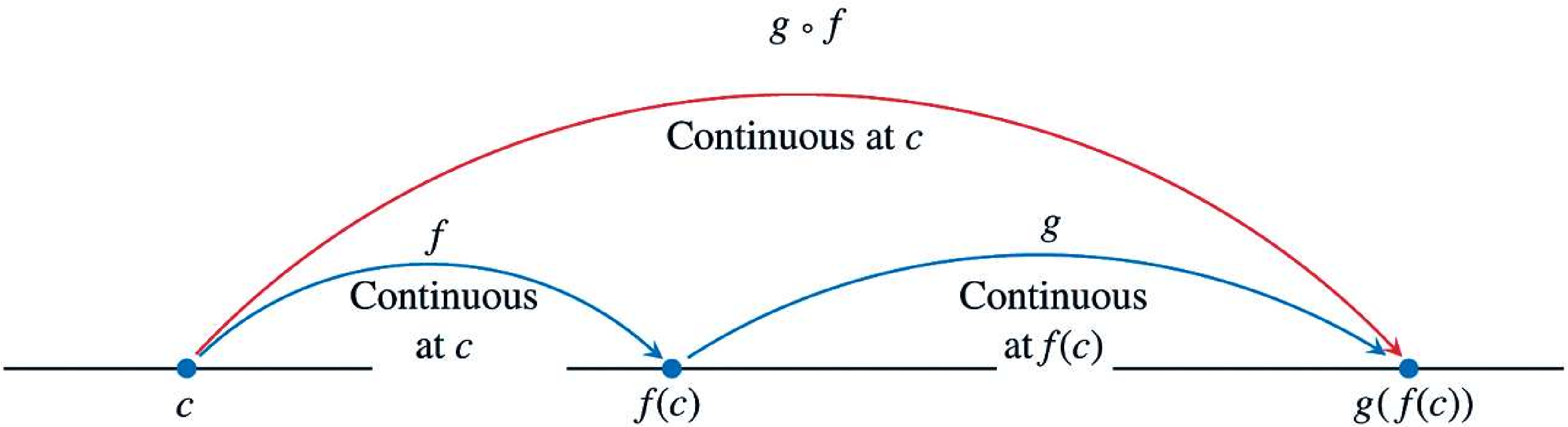


FIGURE 2.42 Compositions of continuous functions are continuous.

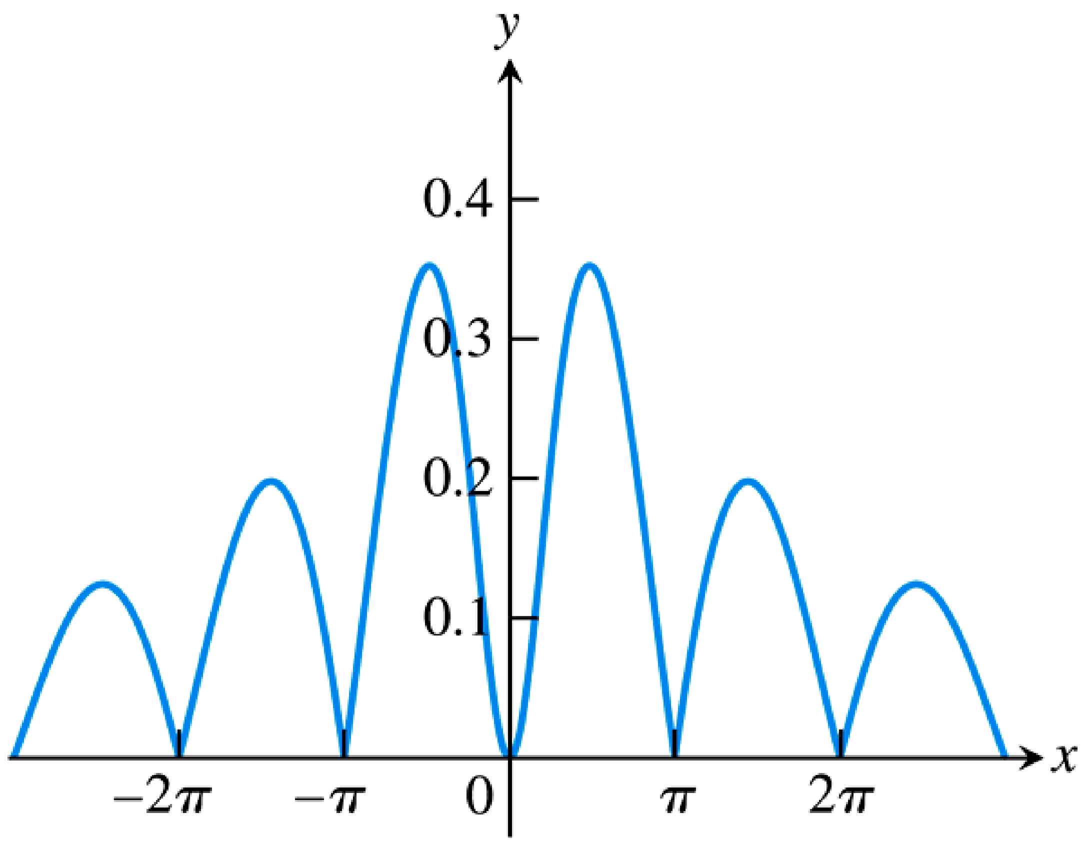


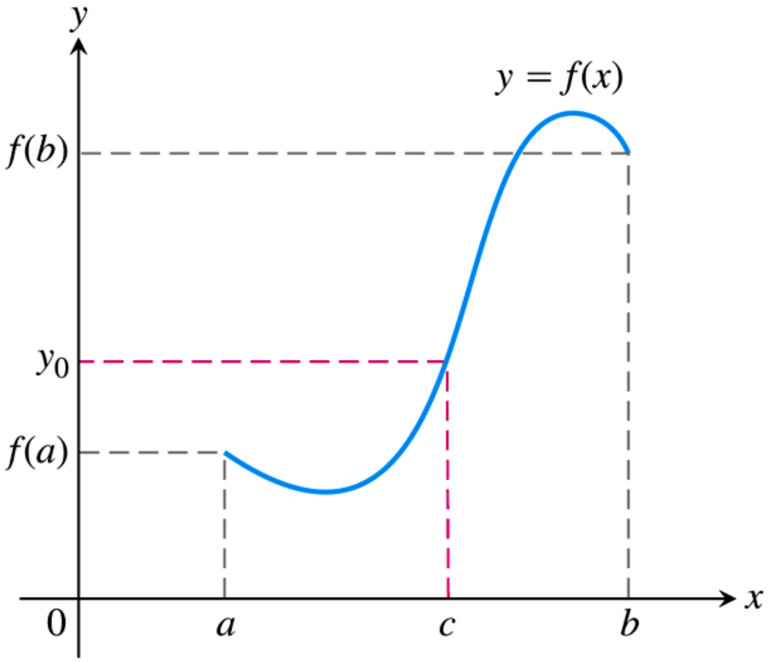
FIGURE 2.43 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 8d).

THEOREM 10—Limits of Continuous Functions

If $\lim_{x \to c} f(x) = b$ and g is continuous at the point b, then

$$\lim_{x \to c} g(f(x)) = g(b).$$

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



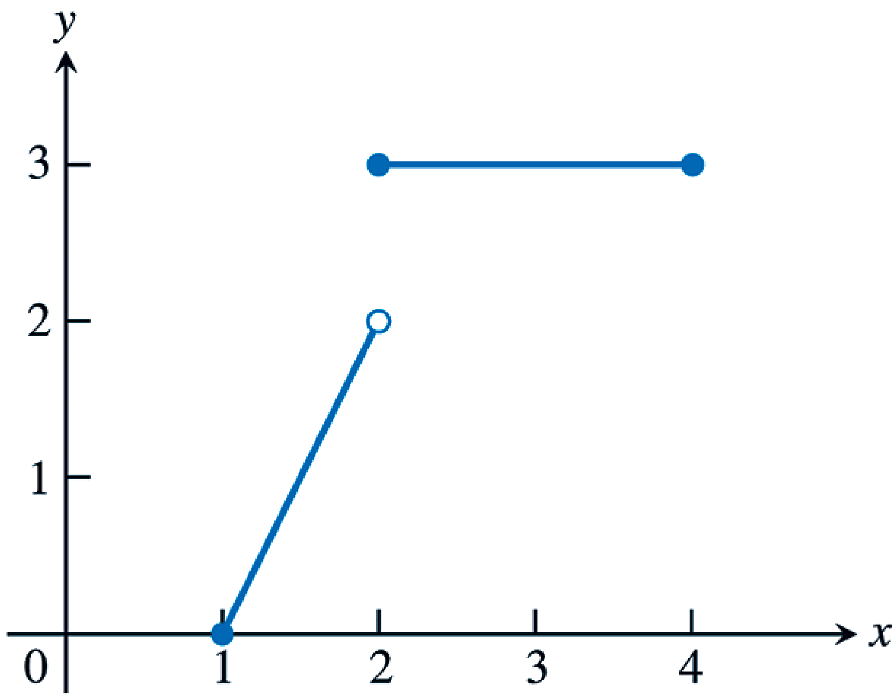


FIGURE 2.44 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

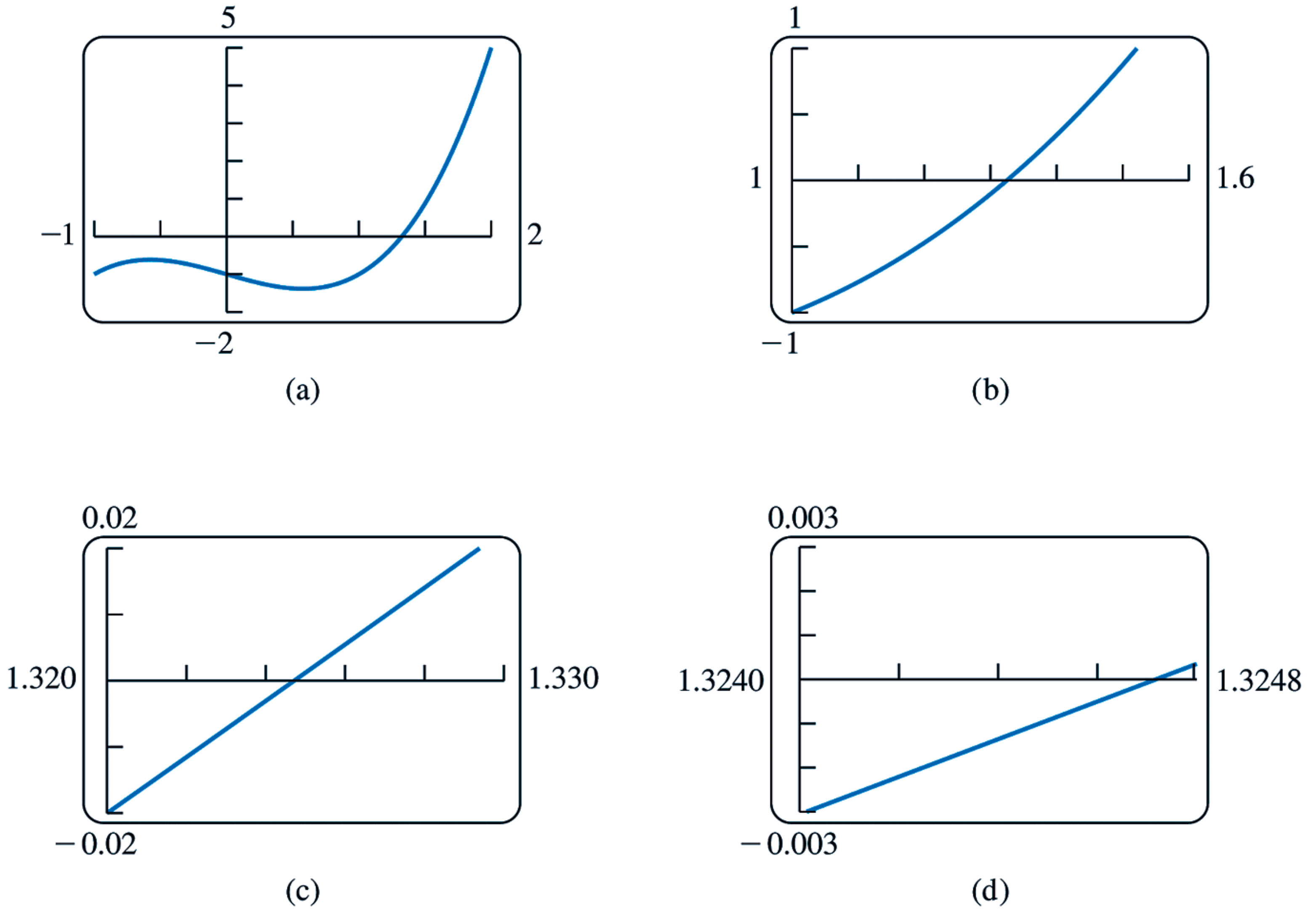


FIGURE 2.45 Zooming in on a zero of the function $f(x) = x^3 - x - 1$. The zero is near $x = 1.3247$ (Example 10).

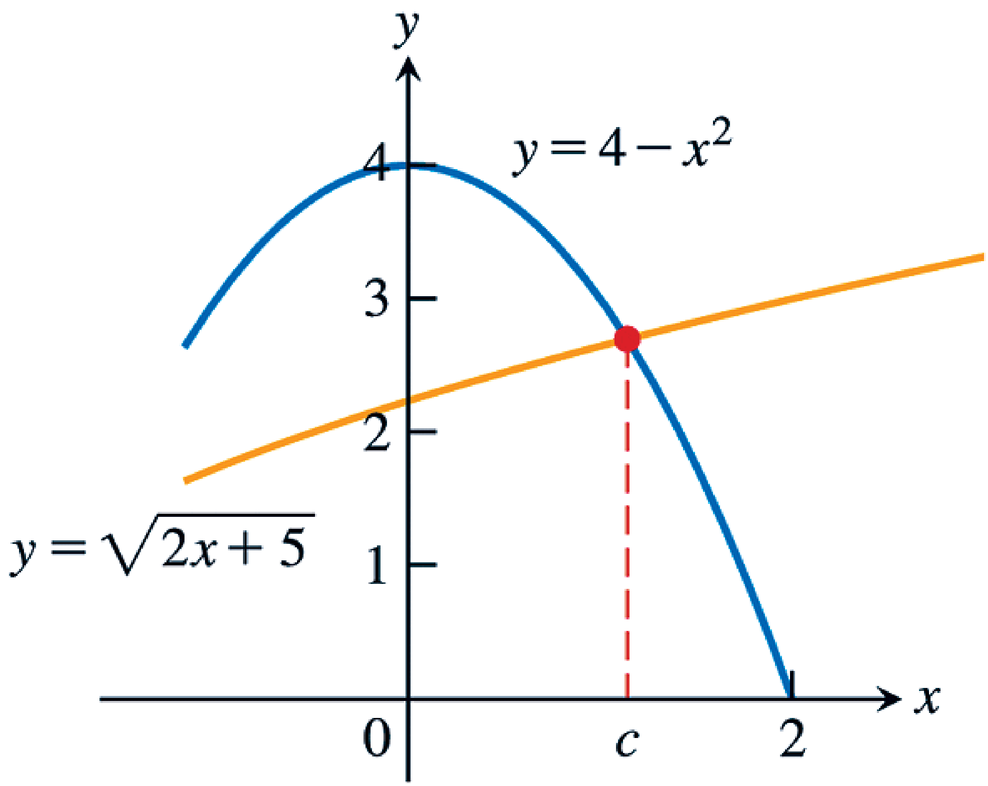


FIGURE 2.46 The curves $y = \sqrt{2x + 5}$ and $y = 4 - x^2$ have the same value at $x = c$ where $\sqrt{2x + 5} + x^2 - 4 = 0$ (Example 11).

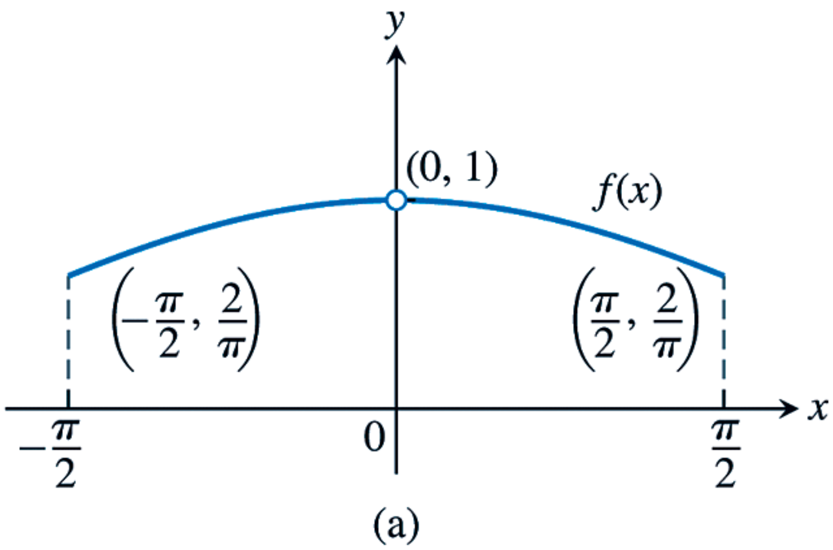


FIGURE 2.47 (a) The graph of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point (0, 1) because the function is not defined at $x = 0$. (b) We can extend the domain to include $x = 0$ by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \to 0} f(x)$ and $F(x)$ is a continuous function at $x = 0$.

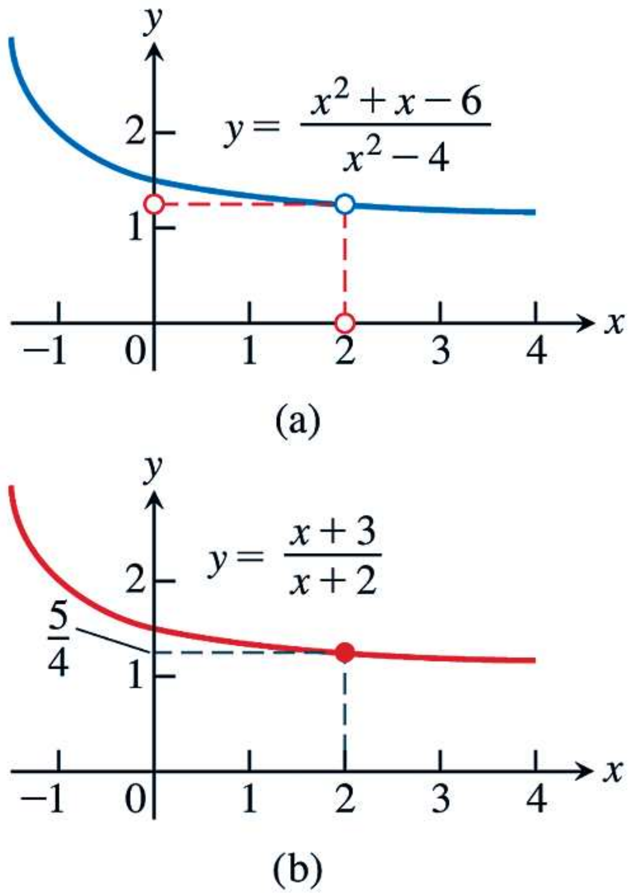


FIGURE 2.48 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 12).

Section 2.6

Limits Involving Infinity; Asymptotes of Graphs

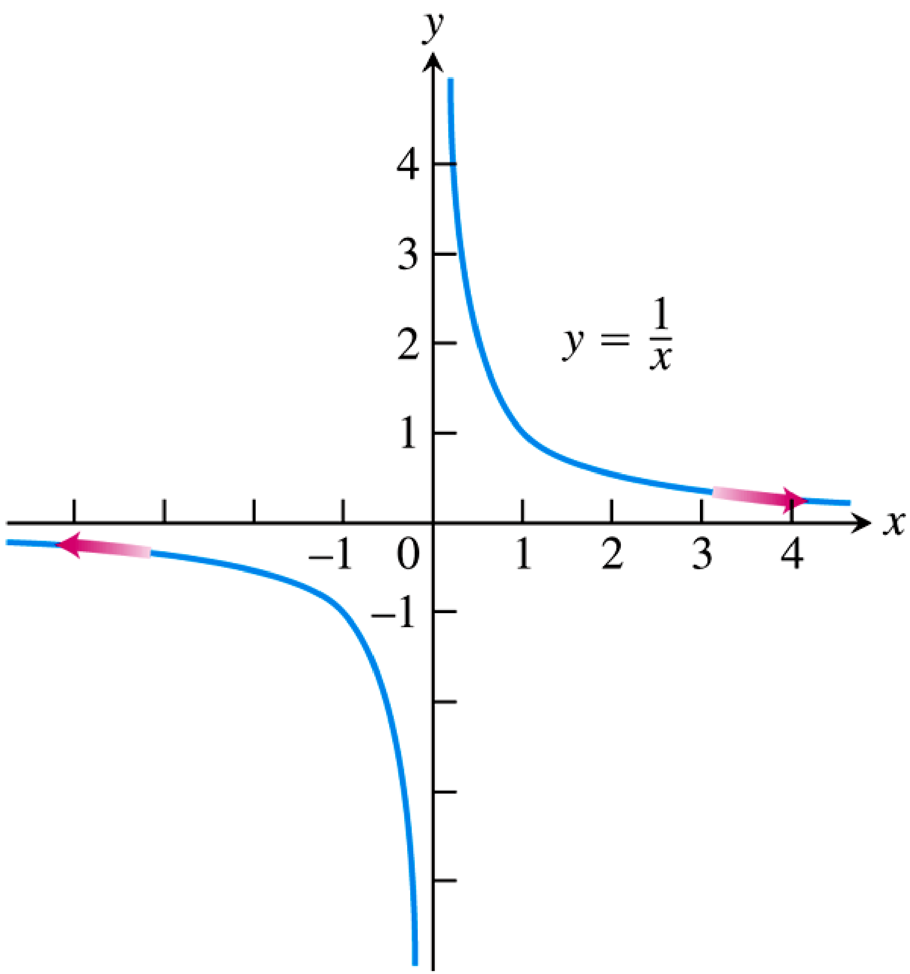


FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

2. We say that $f(x)$ has the **limit L as x approaches negative infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$

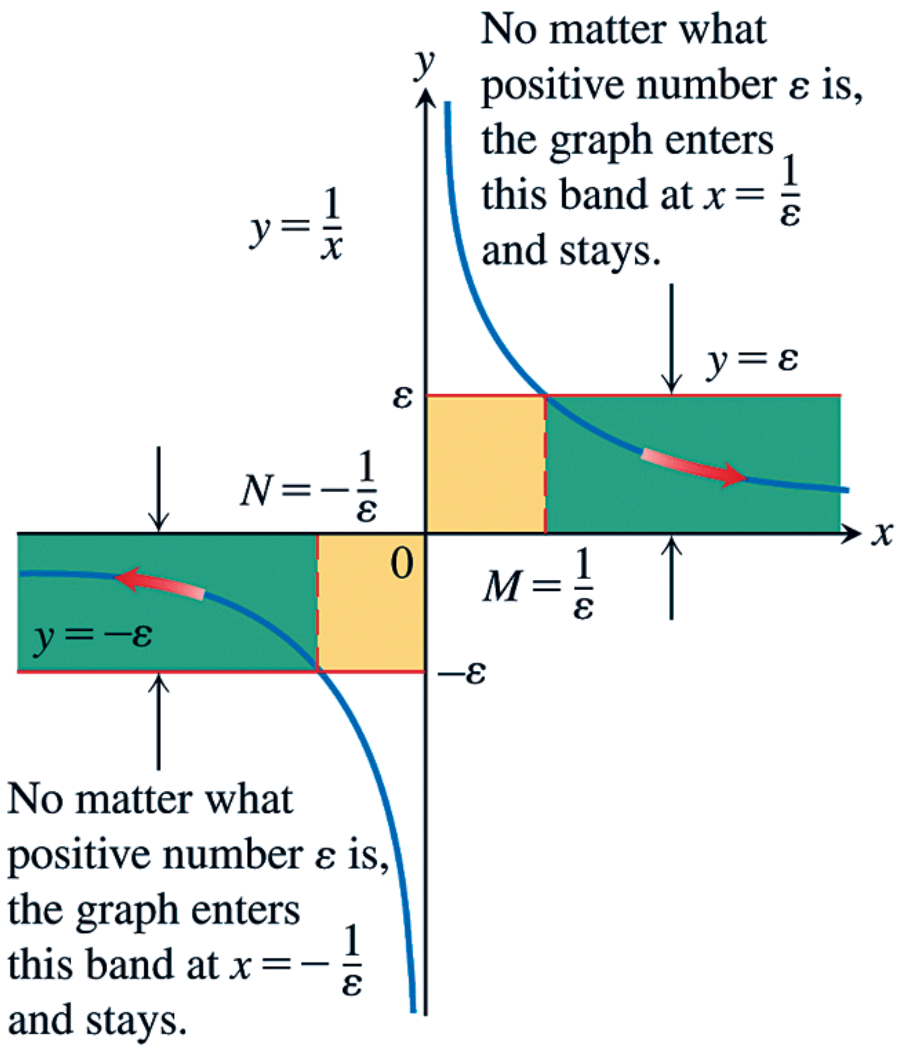


FIGURE 2.50 The geometry behind the argument in Example 1.

THEOREM 12 All the limit laws in Theorem 1 are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

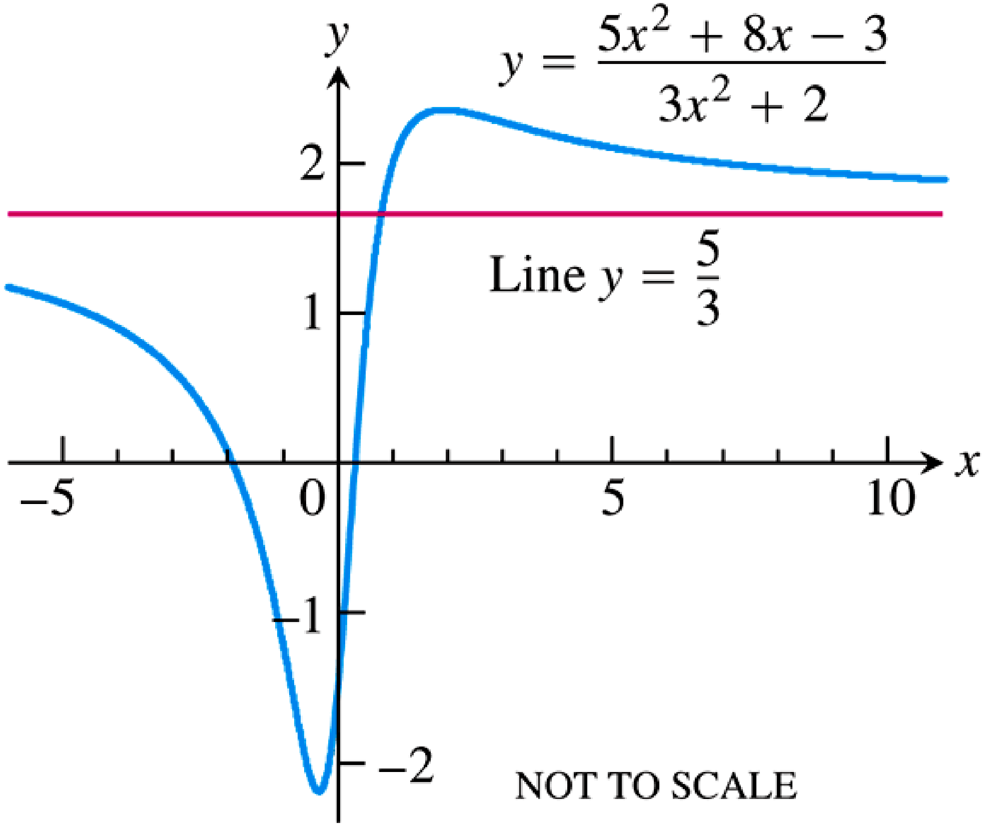


FIGURE 2.51 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

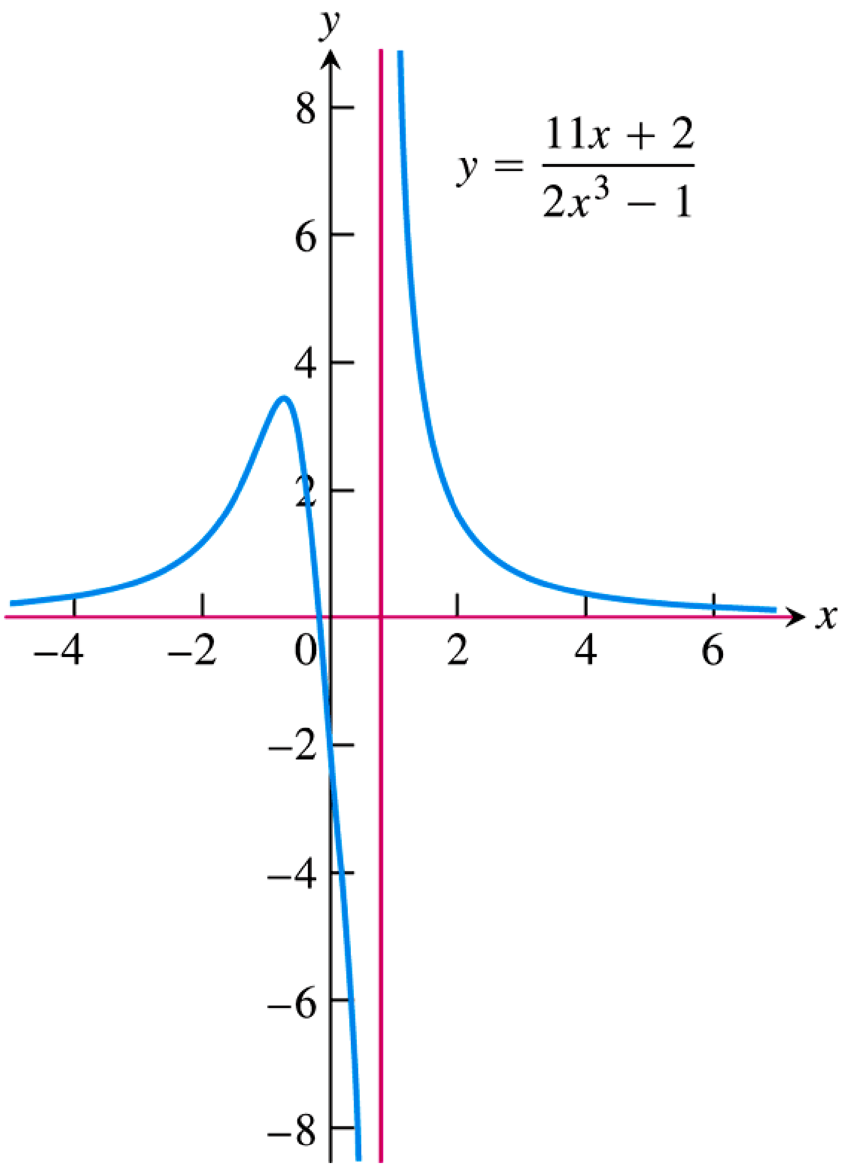


FIGURE 2.52 The graph of the function in Example 3b. The graph approaches the x-axis as $|x|$ increases.

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

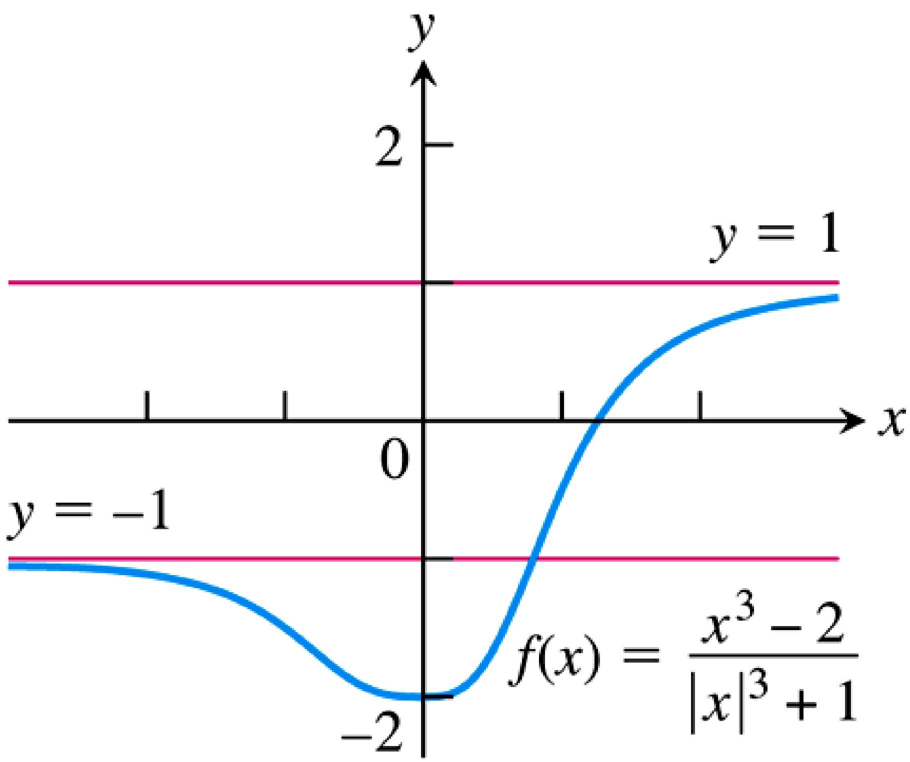


FIGURE 2.53 The graph of the function in Example 4 has two horizontal asymptotes.

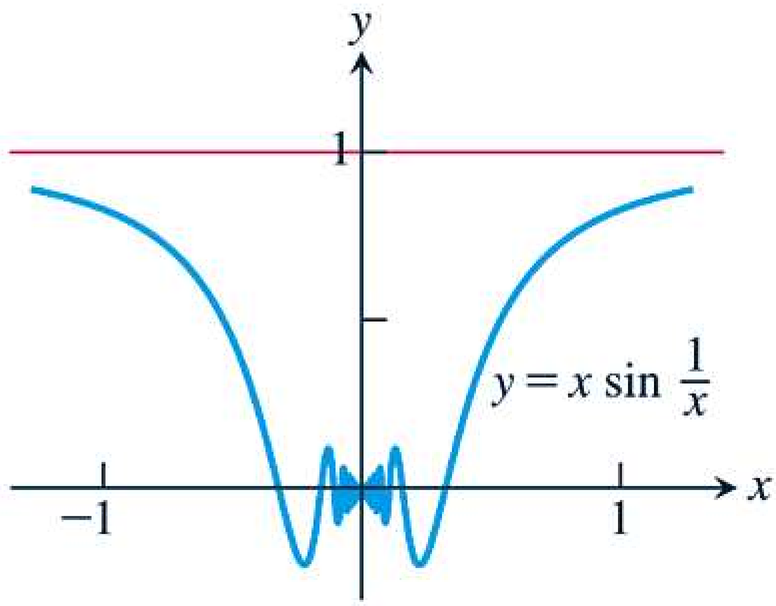


FIGURE 2.54 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 5b).

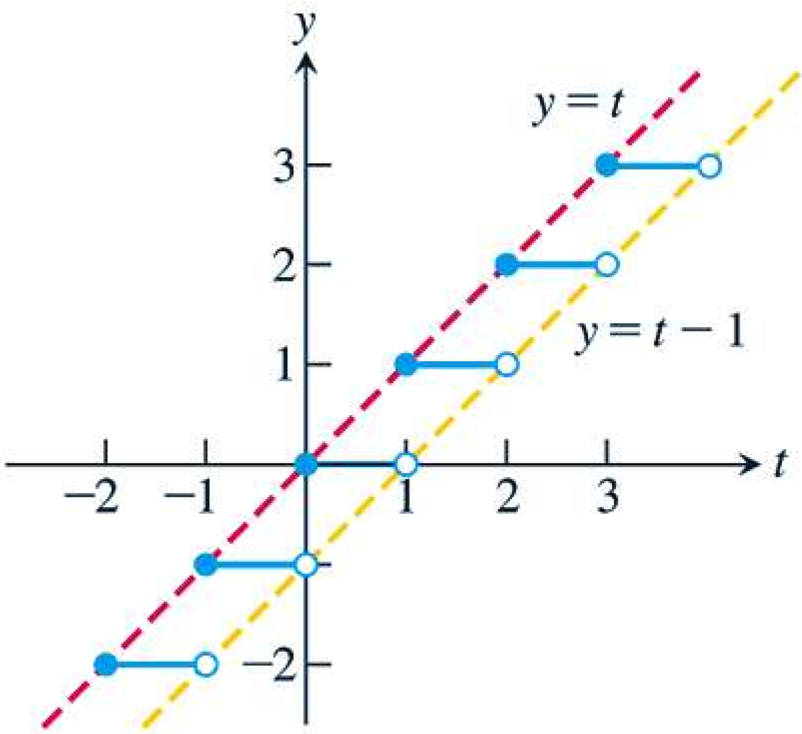


FIGURE 2.55 The graph of the greatest integer function $y = \lfloor t \rfloor$ is sandwiched between $y = t - 1$ and $y = t$.

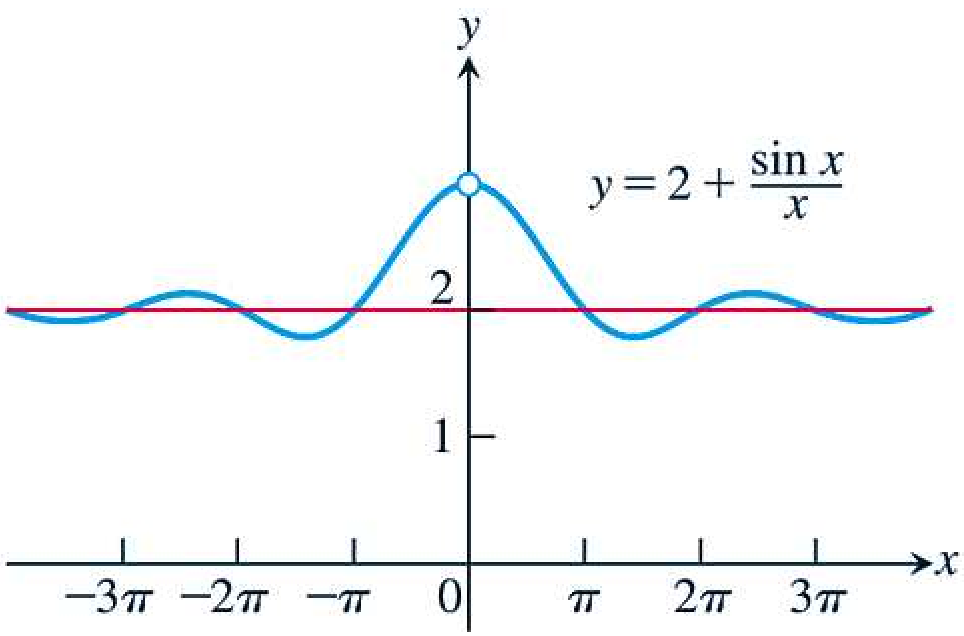


FIGURE 2.56 A curve may cross one of its asymptotes infinitely often (Example 7).

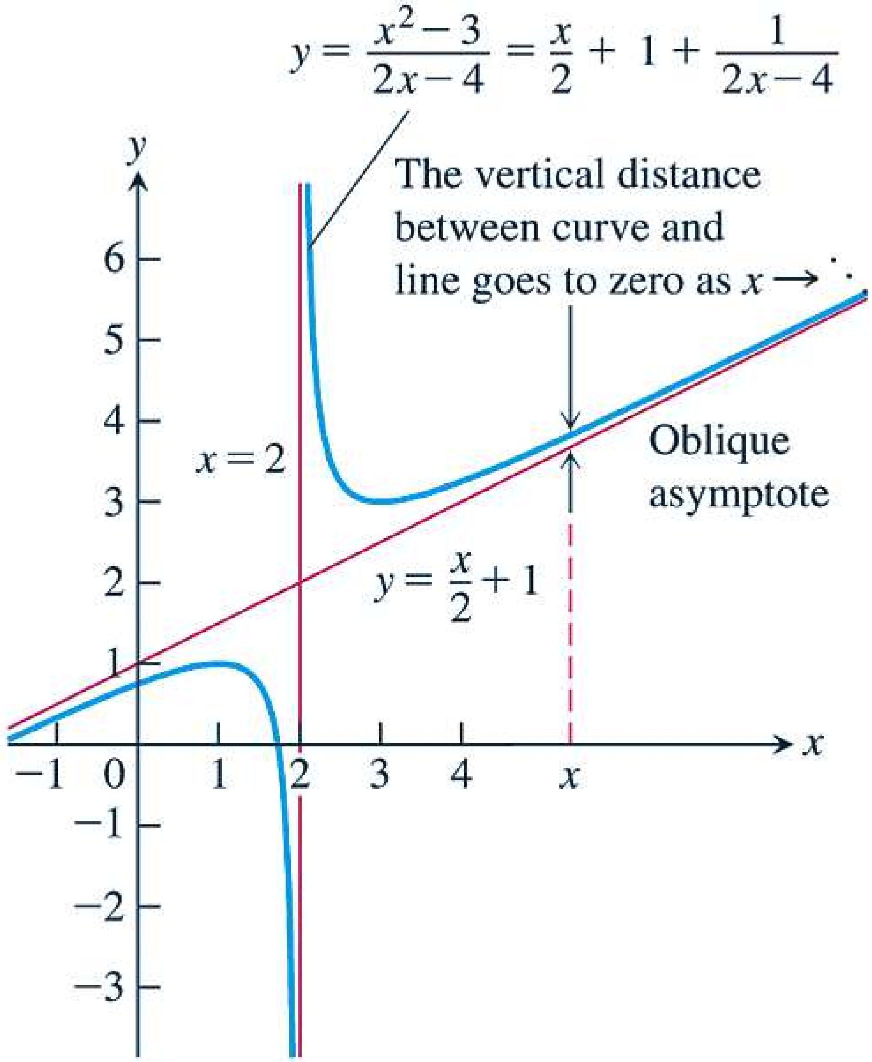


FIGURE 2.57 The graph of the function in Example 9 has an oblique asymptote.

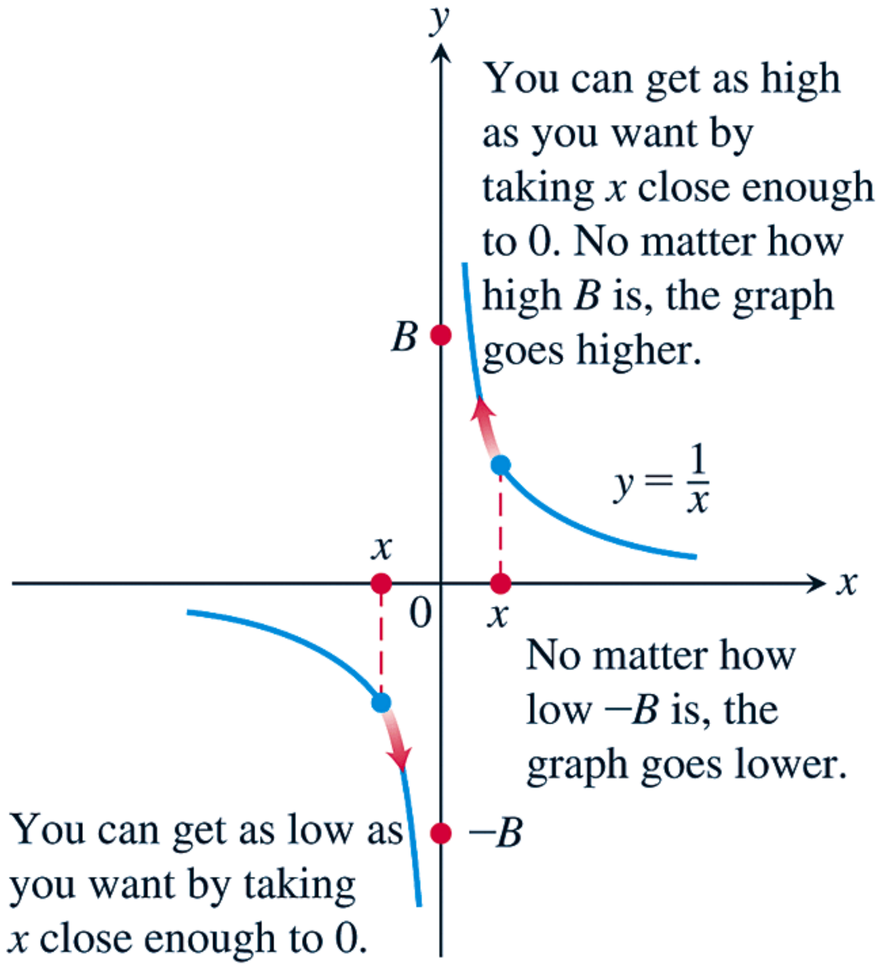


FIGURE 2.58 One-sided infinite limits:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

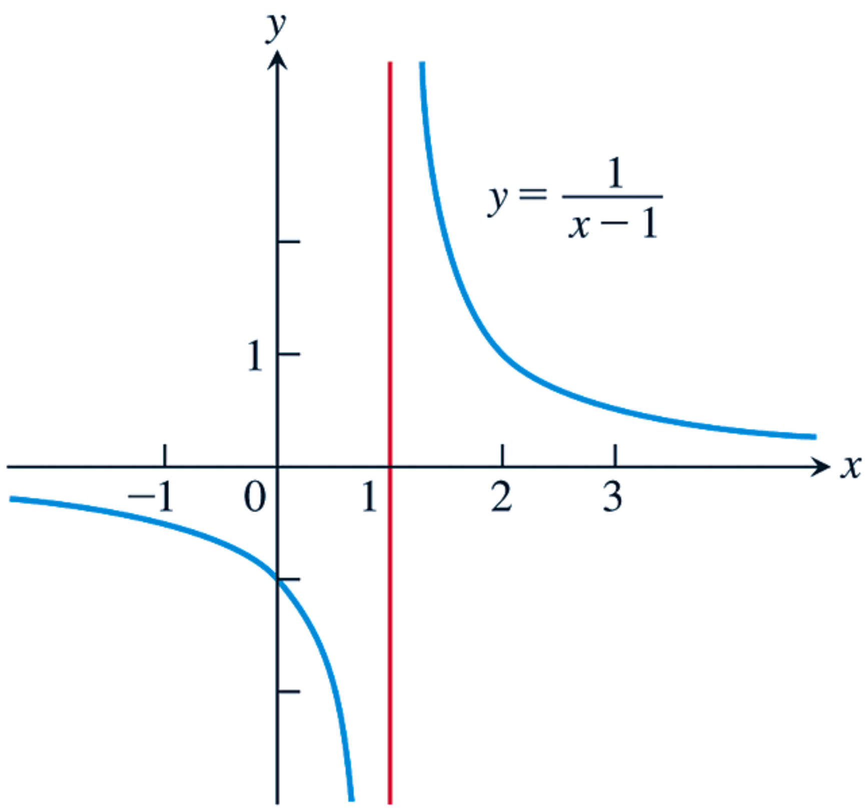


FIGURE 2.59 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 10).

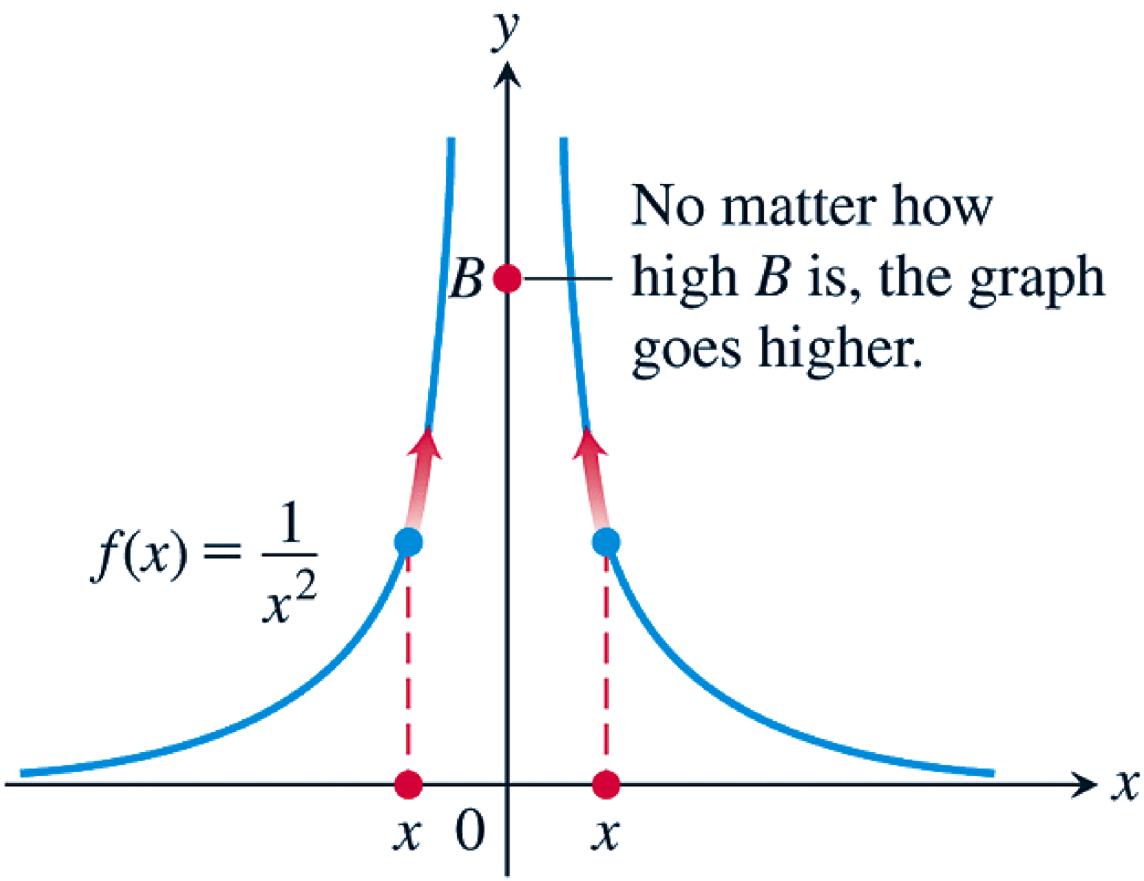


FIGURE 2.60 The graph of $f(x)$ in Example 11 approaches infinity as $x \longrightarrow 0$.

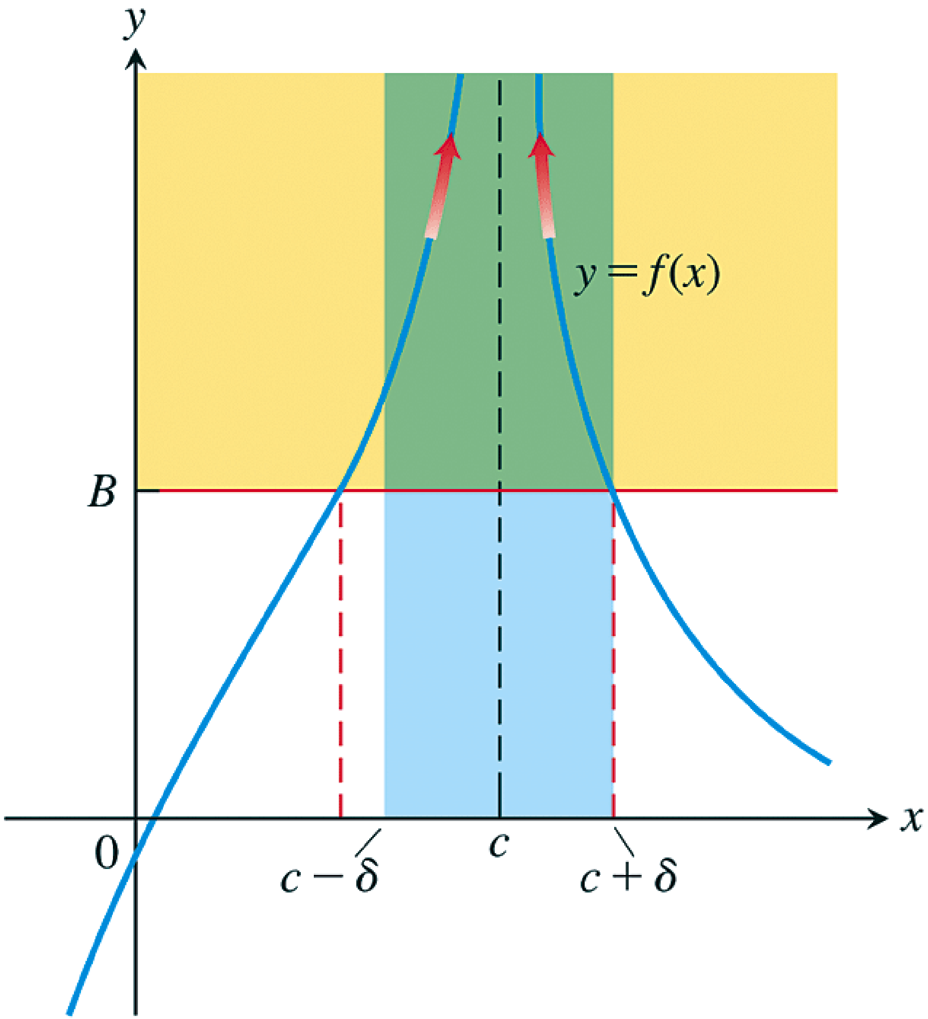


FIGURE 2.61 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.

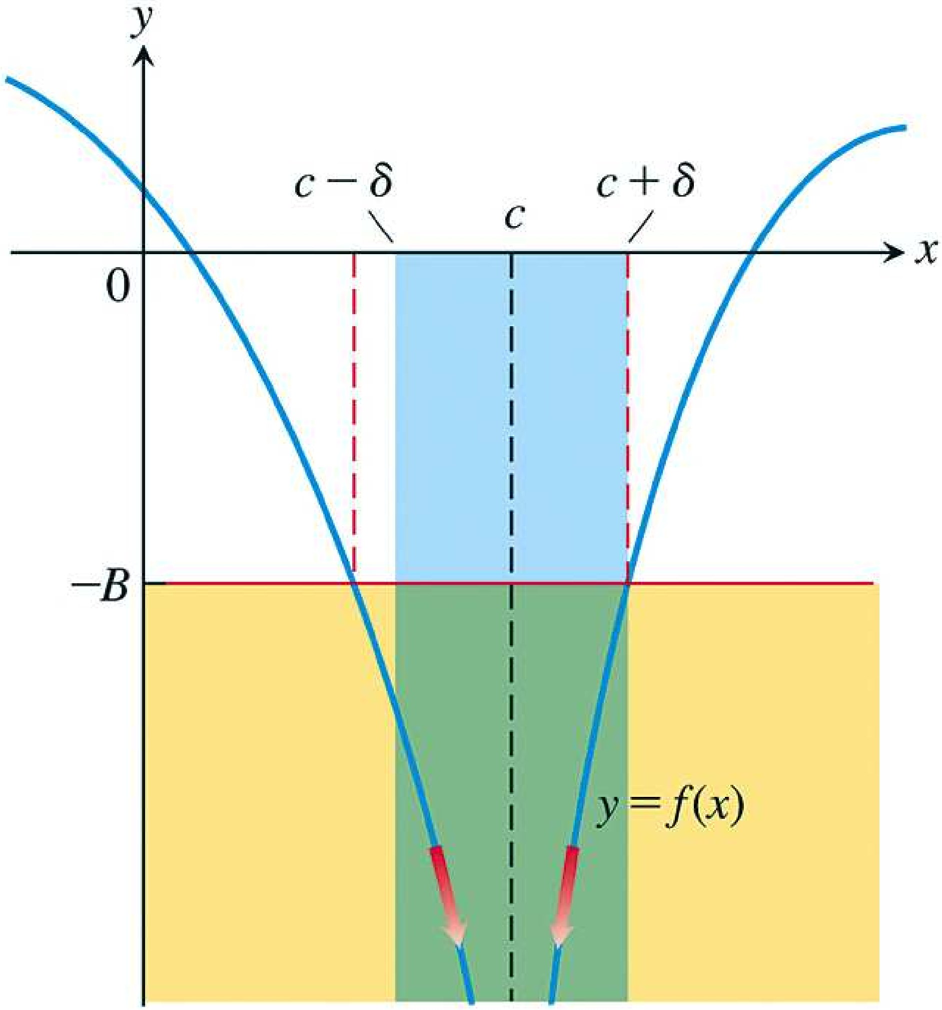


FIGURE 2.62 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

DEFINITIONS

1. We say that **$f(x)$ approaches infinity as x approaches c**, and write

$$\lim_{x \to c} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that

$$f(x) > B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

2. We say that **$f(x)$ approaches negative infinity as x approaches c**, and write

$$\lim_{x \to c} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that

$$f(x) < -B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

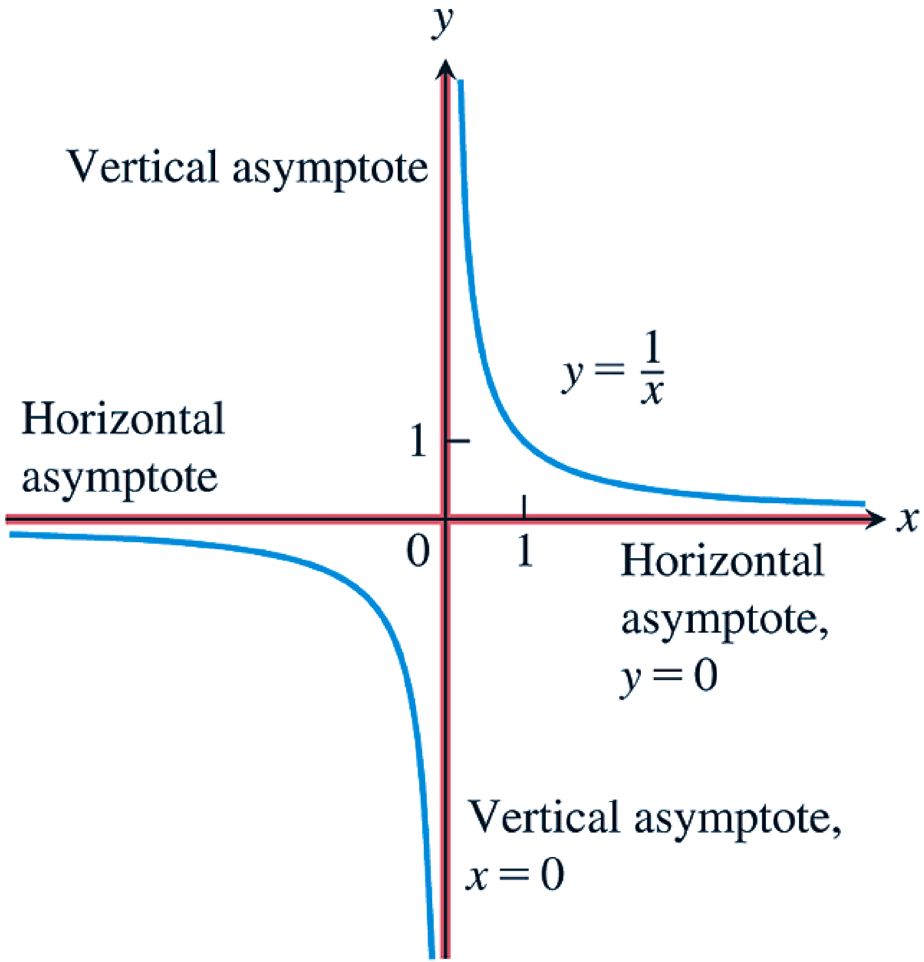


FIGURE 2.63 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

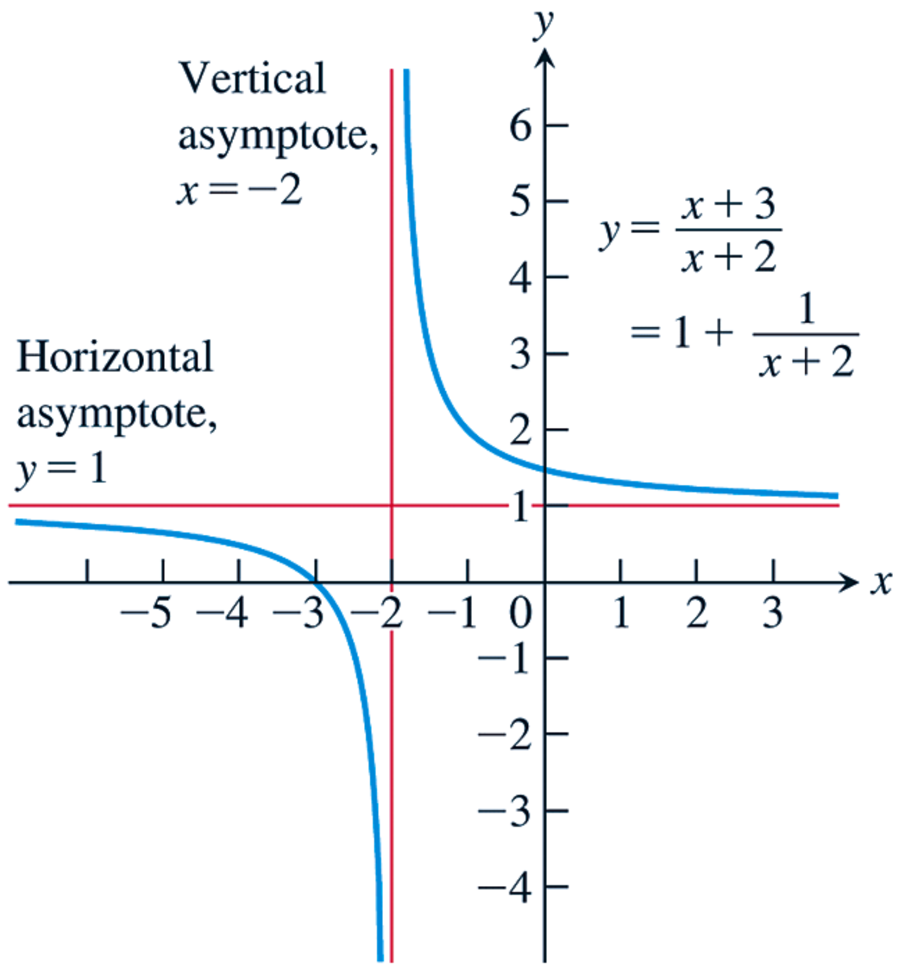


FIGURE 2.64 The lines $y = 1$ and $x = -2$ are asymptotes of the curve in Example 15.

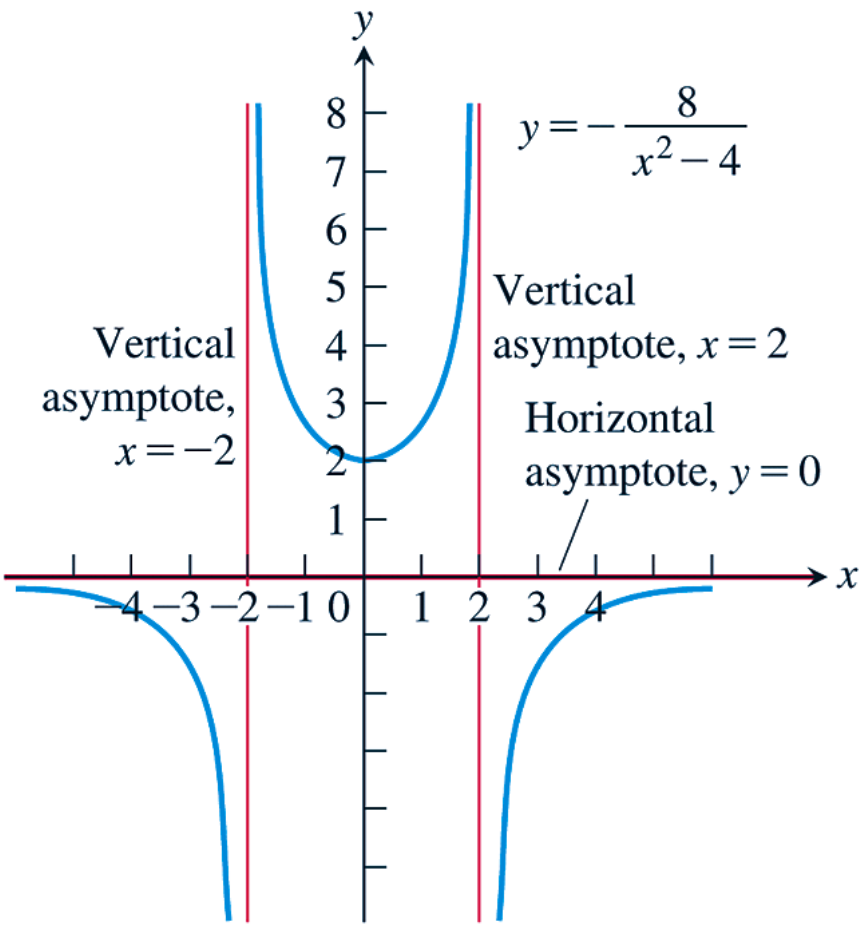


FIGURE 2.65 Graph of the function in Example 16. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided.

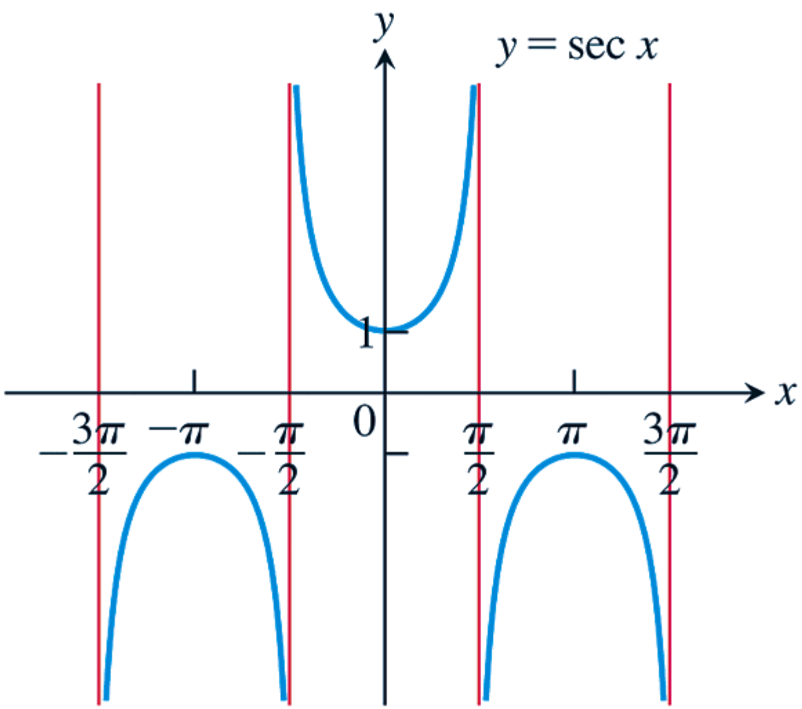
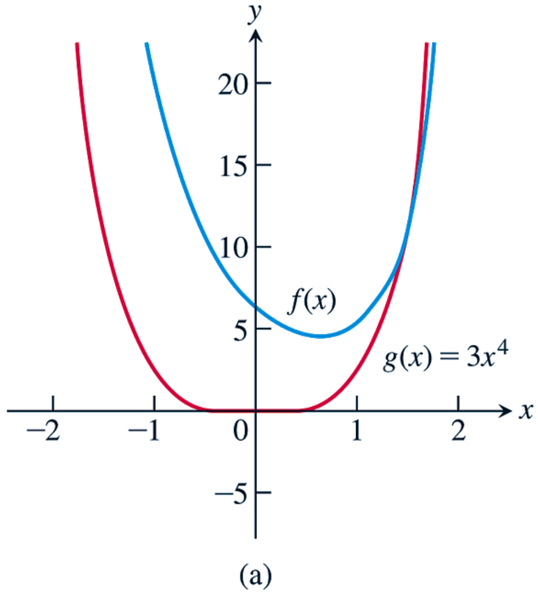


FIGURE 2.66 The graphs of sec x and tan x have infinitely many vertical asymptotes (Example 17).



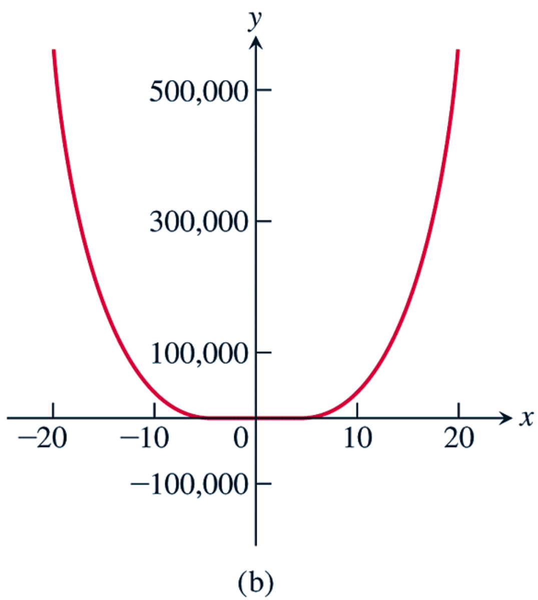


FIGURE 2.67 The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 18).